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# TRANSITIVITY, DENSE ORBITS AND SOME TOPOLOGIES FINER THAN THE NATURAL TOPOLOGY OF THE UNIT INTERVAL

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ABSTRACT. This paper contains some considerations connected with the following problem: What kind of conditions (weaker than the continuity) should be imposed on $f$ in order to obtain an equivalence: $f$ is a transitive function iff it is a nomadic function?

## 1. Introduction

A lot of monographs and articles contain considerations connected with transitive and nomadic functions ([1], [2], [3], [6], [9], [10], etc.). These notions have different meaning in various sources, often they are used exchangeable (see, for example, [6, p. 49] and [1, p. 258]). We shall establish the following definitions.

**Definition 1.** $f\colon X \longrightarrow X$ ($X$ is a topological space) is said to be transitive if for every pair of non-empty open sets $U$ and $V$ in $X$, there exists a positive integer $k$ such that $f^k(U) \cap V \neq \emptyset$ ([6],[2]).

**Definition 2.** $f\colon X \longrightarrow X$ ($X$ is a topological space) is said to be nomadic if there is $x \in X$ such that orbit $\{f^n(x)\colon n = 0, 1, 2, \dots\}$ is a dense set ([3]).

It is well-known that $f^n(x) = x$, if $n = 0$, and $f^n(x) = f\big(f^{n-1}(x)\big)$, if $n > 0$.

Recently, the theory of the transitivity of functions $f\colon [0,1] \longrightarrow [0,1]$ has become a very important theory. This is connected with its applications to some considerations of the chaos theory ([6], [4]). For example, some kind of chaos is implicated by the transitivity of functions $f\colon [0,1] \longrightarrow [0,1]$ ([10]).

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On the other hand it is well-known ([2]) that in the case of continuous functions $f\colon [0,1] \longrightarrow [0,1]$, $f$ is a transitive function iff it is nomadic. This statement has been considered in various sources. In particular, the following question has been considered ([9]):

> For which type of discontinuous functions are transitivity and existence of dense orbits equivalent?

The motivation of our investigations in this topic was a statement held in [9]:

> A transitive map with two points of discontinuity does not have any dense orbit in general.

This statement became some kind of challenge for us, because our observations seem to suggest that there exists a lot of functions having a "big set of discontinuity points" for which the equivalence *$f$ is transitive iff $f$ is nomadic* takes place.

The following idea is a starting point of our considerations: in the domain of considered functions we introduce a special kind of topology (finer than the usual topology of the unit interval). So, it is necessary to modify our definitions slightly.

Let $\mathcal{T}$ be some topology in $[0,1]$.

A function $f\colon [0,1] \longrightarrow [0,1]$ is said to be $\mathcal{T}$-transitive if for every pair of non-empty sets $U \in \mathcal{T}, V \in \mathcal{T}$, there exists a positive integer $k$ such that $f^k(U) \cap V \neq \emptyset$.

A function $f\colon [0,1] \longrightarrow [0,1]$ is said to be $\mathcal{T}$-nomadic if there is $x \in X$ such that orbit $\{f^n(x)\colon n = 0,1,2,\dots\}$ is a $\mathcal{T}$-dense set.

## 2. Preliminaries

We will use standard notations mostly. In particular, by the letter $\mathbb{N}$ ($\mathbb{N}_0, \mathbb{R}$) we denote the set of *all positive integers* (*nonnegative integers, real numbers*).

We restrict our considerations to the case of functions mapping the unit interval $[0,1]$ onto itself, however all theorems hold also if we replace $[0,1]$ by an arbitrary compact interval. Because of the fact, that all the considerations are connected with the unit interval $[0,1]$, in order to simplify notation, we will write $(a,b)$ ($[a,b]$, etc.) instead of $(a,b) \cap [0,1]$ ($[a,b] \cap [0,1]$, etc.).

If $\{a_n\}$ is some sequence, then the symbol $a_n \searrow a$ ($a_n \nearrow a$) will denote that $\{a_n\}$ is strictly decreasing (increasing) to $a$.

If $A$ is a subset of the domain of $f$ ($A \subset [0,1]$), then $f \restriction A$ denotes the restriction of $f$ to $A$. We will use the symbol $\Gamma(f)$ to denote the graph of $f$.

The set of all continuity (fixed) points of $f$ will be denoted by $C(f)$ ($\mathrm{Fix}(f)$). By id we will denote the identical function.

Let $f\colon [0,1] \xrightarrow{\text{onto}} [0,1]$. Then $f^0(x) = x$, and $f^n(x) = f\big(f^{n-1}(x)\big)$, if $n > 0$.

The closure, interior and the set of all accumulation points of a set $A$ (in the natural topology of the unit interval) will be denoted by $\overline{A}$, $\mathrm{Int}(A)$ and $A^d$, respectively. By $\lambda(A)$ we will denote the Lebesgue measure of a set $A$.

Our considerations are connected with a special kind of a topology in $[0,1]$. If $\mathcal{T}$ is a topology in $[0,1]$, then we will use the notation $\mathcal{T}$-open set, $\mathcal{T}$-closed set, $\mathcal{T}$-nowhere dense set, $\mathcal{T}$-$G_\delta$-set, etc. to denote that these properties of sets are connected with the topology $\mathcal{T}$. The notations open set, closed set, nowhere dense set, etc., stand for properties in the natural topology of the unit interval. Let $g\colon [0,1] \longrightarrow [0,1]$. The notation *$g$ is $\mathcal{T}$-continuous function* means that we consider $g$ as a function mapping topological space $([0,1],\mathcal{T})$ into $[0,1]$ (with the natural topology) and $g^{-1}(a,b) \in \mathcal{T}$ for any open interval $(a,b)$.

A topology $\mathcal{T}$ (in $[0,1]$) finer than the natural topology of the unit interval $\mathcal{T}_0$ is compatible with the topology $\mathcal{T}_0$ if $\mathrm{Int}(U) \neq \emptyset$ for any $U \in \mathcal{T}$.

A Baire space is a topological space such that every nonempty open subset is of second category.

## 3. Results

Topologies compatible with the natural topology of the unit interval will represent the basis of our considerations. So, before giving main results we make some preliminary observations connected with a space endowed with such kind of topology.

**LEMMA 1.** *If $\mathcal{T}$ is a topology in $[0,1]$ compatible with the natural topology of the unit interval, then $([0,1],\mathcal{T})$ is a Baire space.*

Proof. It is easy to observe that if $A$ is a $\mathcal{T}$-nowhere dense set, then $A$ is a nowhere dense set.

(The converse implication is also true.)

Now, let $P = \bigcup_{n=1}^{\infty} A_n$, where $A_n$ are $\mathcal{T}$-nowhere dense sets (consequently $A_n$ are nowhere dense sets). If there exists a nonempty $\mathcal{T}$-open set $U \subset P$, then $\emptyset \neq \mathrm{Int}(U) \subset P$, which is impossible. $\square$

The next theorem and further considerations give a solution to the problem:

> **Main problem:** What kind of conditions (weaker than the continuity) should be imposed on $f$ in order to obtain the equivalence: $f$ is a transitive function iff it is a nomadic function?

**Theorem 2.** *Let $f\colon [0,1] \xrightarrow{\text{onto}} [0,1]$ be a function for which there exists a topology $\mathcal{T}$ in $[0,1]$ compatible with the natural topology of the unit interval such that $f^n\colon ([0,1],\mathcal{T}) \longrightarrow [0,1]$ are $\mathcal{T}$-continuous functions, if $n \in \mathbb{N}$. Then the following conditions are equivalent:*

- $(i)$ *$f$ is $\mathcal{T}$-transitive,*
- $(ii)$ *there exists a $\mathcal{T}$-dense and $\mathcal{T}$-$G_\delta$-set $Z$ such that the orbit $\{f^m(x)\colon m \in \mathbb{N}\}$ is a $\mathcal{T}$-dense set, for each $x \in Z$,*
- $(iii)$ *$f$ is $\mathcal{T}$-nomadic.*

P r o o f . We will prove the successive implications.

$(i) \Rightarrow (ii)$. Let $\{B_1, B_2, \dots\}$ be a countable base of $[0,1]$ consisting of open sets. Let us denote

$$Z = \bigcap_{n=1}^{\infty} Z_n\,,$$

where

$$Z_n = \{z \in [0,1]\colon \{f^m(z)\colon m \in \mathbb{N}\} \cap B_n \neq \emptyset\}, \quad \text{for any } n \in \mathbb{N}.$$

First, we can prove that

$$Z_n \in \mathcal{T}, \quad \text{for any } n \in \mathbb{N}. \tag{1}$$

Let $z \in Z_n$ ($n$ is a fixed positive integer). Then there exists $k_z \in \mathbb{N}$ such that $f^{k_z}(z) \in B_n \in \mathcal{T}_0 \subset \mathcal{T}$. Since $f^{k_z}$ is a $\mathcal{T}$-continuous function, there exists $\mathcal{T}$-neighbourhood $V_z$ of $z$ such that $f^{k_z}(V_z) \subset B_n$. This means that $V_z \subset Z_n$, which gives (1).

Now, we shall show that

$$Z_n \text{ is a } \mathcal{T}\text{-dense set}, \quad \text{for } n \in \mathbb{N}. \tag{2}$$

Let $n \in \mathbb{N}$ and $V$ be a nonempty $\mathcal{T}$-open set. On account of $(i)$ there exists a positive integer $k_V$ such that $f^{k_V}(V) \cap B_n \neq \emptyset$. Let $t$ be an element of $V$ such that $f^{k_V}(t) \in B_n$. According to the definition of $Z_n$, we have $t \in Z_n$ and, consequently, $V \cap Z_n \neq \emptyset$.

By (1), (2), Lemma 1 and [8, Theorem 1.13] we can infer that $Z$ is $\mathcal{T}$-dense and a $\mathcal{T}$-$G_\delta$ set.

To prove this implication it is sufficient to show that for any $x \in Z$ an orbit $\{f^m(x)\colon m = 1,2,\dots\}$ is a $\mathcal{T}$-dense set. Therefore, let $x \in Z$. Then

$$\{f^m(x)\colon m = 1,2,\dots\} \cap B_n \neq \emptyset, \text{ if } n = 1,2,\dots$$

By the above, the set $\{f^m(x)\colon m = 1,2,\dots\}$ is dense and, consequently, it is $\mathcal{T}$-dense.

$(ii) \Rightarrow (iii)$. This implication is obvious.

$(iii) \Rightarrow (i)$. Let $U$, $V$ be nonempty $\mathcal{T}$-open sets. On account of the assumption $(iii)$ there exists $x \in [0,1]$ such that $\{f^m(x) \colon m = 1, 2, \dots\}$ is a $\mathcal{T}$-dense set. Then there exists a positive integer $m_0$ such that

$$f^{m_0}(x) \in U\,.$$

Since $([0,1], \mathcal{T})$ does not contain any isolated points and $([0,1], \mathcal{T})$ is a $T_1$-space, we have that $\{y\}$ is a $\mathcal{T}$-nowhere dense set, for any $y \in [0,1]$. Moreover, according to Lemma 1, $V$ is an uncountable set. As a result, we can consider a nonempty $\mathcal{T}$-open set $V_0 = V \setminus \{f^i(x) \colon i = 0, 1, \dots, m_0\}$. Repeating an application of our assumption $(iii)$, there exists a positive integer $m_1 > m_0$ such that $f^{m_1}(x) \in V_0 \subset V$. Consequently we have

$$f^{m_1 - m_0}\big(f^{m_0}(x)\big) \in V \quad \text{and} \quad f^{m_0}(x) \in U\,.$$

$\square$

**Remark 3.** Suppose that the assumptions of our theorem are fulfilled. Then, according to the above Theorem, we obtain:

$f$ is transitive $\Leftrightarrow f$ is $\mathcal{T}$-transitive $\Leftrightarrow f$ is $\mathcal{T}$-nomadic $\Leftrightarrow f$ is nomadic.

The above theorem gives a theoretical answer to our main problem. Simultaneously, it necessitates to the next question: *What kind of assumptions should we impose on $f$ in order to achieve an existence of a topology $\mathcal{T}$ having properties described in the previous theorem?* The following theorem will show that these assumptions can be connected with a Darboux property and a "special behaviour" of discontinuity points.

**THEOREM 4.** *Let $f \colon [0,1] \xrightarrow{\text{onto}} [0,1]$ be a nowhere constant Darboux function such that $f\big(D(f)\big) \subset \{0,1\} \subset \operatorname{Fix}(f) \cap \big(\operatorname{Fix}(f)\big)^d$. Then there exists a topology $\mathcal{T}$ in $[0,1]$ compatible with the natural topology of the unit interval such that $f^n$ are $\mathcal{T}$-continuous functions, for $n \in \mathbb{N}$.*

(*Consequently, $f$ is a transitive function iff $f$ is a nomadic function.*)

P r o o f. First, we note that

$$f\big(\overline{D(f)}\,\big) \subset \{0,1\}\,, \tag{3}$$

$$f^{-1}\big(\{0,1\}\big) \text{ is a nowhere dense and closed set}\,. \tag{4}$$

Condition (3) is obvious. As to the proof of the closedness of $f^{-1}\big(\{0,1\}\big)$, we suppose the contrary to our claim, that there exists $x_0 \in \overline{f^{-1}\big(\{0,1\}\big)} \setminus f^{-1}\big(\{0,1\}\big)$. Of course, $x_0 \notin \overline{D(f)}$ and so, $x_0$ is a continuity point of $f$. Let

$\{x_n\}$ be a sequence such that $\{x_n : n \in \mathbb{N}\} \subset f^{-1}(\{0,1\})$ and $x_n \longrightarrow x_0$. Thus $f(x_n) \longrightarrow f(x_0)$, and $f(x_n) \in \{0,1\}$, which gives $f(x_0) \in \{0,1\}$. This contradicts our supposition that $x_0 \notin f^{-1}(\{0,1\})$.

Now, we shall show that $f^{-1}(\{0,1\})$ is a nowhere dense set. According to the closedness of this set, it is sufficient to show that there is no open interval $(a,b)$, such that $(a,b) \subset f^{-1}(\{0,1\})$.

Let $(a,b)$ be an arbitrary open interval. Since $f$ is nowhere constant, then neither $(a,b) \subset f^{-1}(0)$ nor $(a,b) \subset f^{-1}(1)$. Thus $f((a,b)) = \{0,1\}$, which is impossible, because of the Darboux property of $f$.

A basic part of the proof starts with the construction of suitable families of sets which will be a neighbourhood system of the required topology. We shall make the construction in a few steps.

Now, we define a family $B(x)$ of sets containing $x$, for $x \in [0,1]$.

According to our assumptions, there exists a strictly decreasing (increasing) sequence $\{\alpha_n\}$ ($\{\beta_n\}$) of elements of $\mathrm{Fix}(f)$ such that $\alpha_n \searrow 0$ and $\alpha_1 < \frac{1}{2}$ ($\beta_n \nearrow 1$ and $\beta_1 > \frac{1}{2}$).

The family $B(0)$ consists of all sets $U_n^{\{k_i\}}(0)$ of the form:

$$U_n^{\{k_i\}}(0) = \{0\} \cup \bigcup_{i=n}^{\infty} \left(\alpha_i - \frac{1}{k_i}, \alpha_i + \frac{1}{k_i}\right),$$

where $n \in \mathbb{N}$ and $\{k_i\} = \{k_i\}_{i=n}^{\infty}$ is a sequence of positive integers.

In a similar way we can define a family $B(1)$ consisting of all sets $U_t^{\{m_i\}}(1)$ of the form:

$$U_t^{\{m_i\}}(1) = \{1\} \cup \bigcup_{i=t}^{\infty} \left(\beta_i - \frac{1}{m_i}, \beta_i + \frac{1}{m_i}\right),$$

where $t \in \mathbb{N}$ and $\{m_i\} = \{m_i\}_{i=t}^{\infty}$ is a sequence of positive integers.

Now, let $x_0 \in (0,1)$.

If $f^k(x_0) \notin \{0,1\}$, for each $k \in \mathbb{N}$ then, by (3),

$$f^n(x_0) \in C(f), \quad \text{for } n \in \mathbb{N}_0 .$$

In this case, let us put

$$B(x_0) = \left\{ \left(x_0 - \frac{1}{m}, x_0 + \frac{1}{m}\right) : m = 1, 2, \dots \right\}.$$

Finally, we have to consider the case if there exists a positive integer $k \in \mathbb{N}$ such that $f^k(x_0) \in \{0,1\}$.

Let

$$k_{x_0} = \max\left\{k \in \mathbb{N}_0 : f^k(x_0) \notin \{0,1\}\right\}.$$

Assume, for instance, that $f^{k_{x_0}+1}(x_0)=0$.

Since $f$ is a nowhere constant Darboux function, there exists a sequence $\left\{p_m^{k_{x_0}}\right\}_{m=1}^{\infty}$ such that $p_m^{k_{x_0}} \searrow f^{k_{x_0}}(x_0)$, $\left\{f\left(p_m^{k_{x_0}}\right)\right\}_{m=1}^{\infty}$ is a subsequence of $\{\alpha_n\}$ and there exists a sequence $\left\{q_m^{k_{x_0}}\right\}_{m=1}^{\infty}$ such that $q_m^{k_{x_0}} \nearrow f^{k_{x_0}}(x_0)$, $\left\{f\left(q_m^{k_{x_0}}\right)\right\}_{m=1}^{\infty}$ is a subsequence of $\{\alpha_n\}$.

Let us consider the following cases.

If $k_{x_0}=0$ then we obtain $p_m^0$ and $q_m^0$ $(m=1,2,\dots)$.

If $k_{x_0}>0$ then we apply our assumptions that $f$ is a nowhere constant Darboux function again, with $f^{k_{x_0}}(x_0)$ replaced by $f^{k_{x_0}-1}(x_0)$. So, we can deduce that there exist sequences $\left\{p_m^{k_{x_0}-1}\right\}$, $\left\{q_m^{k_{x_0}-1}\right\}$ such that

$$p_m^{k_{x_0}-1} \searrow f^{k_{x_0}-1}(x_0)\,,\ q_m^{k_{x_0}-1} \nearrow f^{k_{x_0}-1}(x_0)$$

and

$$\begin{aligned}\left\{p_m^{k_{x_0}-1}\colon m\in\mathbb{N}\right\} &\subset f^{-1}\Big(\left\{p_m^{k_{x_0}}\colon m\in\mathbb{N}\right\}\cup\left\{q_m^{k_{x_0}}\colon m\in\mathbb{N}\right\}\Big)\,,\\ \left\{q_m^{k_{x_0}-1}\colon m\in\mathbb{N}\right\} &\subset f^{-1}\Big(\left\{p_m^{k_{x_0}}\colon m\in\mathbb{N}\right\}\cup\left\{q_m^{k_{x_0}}\colon m\in\mathbb{N}\right\}\Big)\,.\end{aligned}$$

Thus (note that $f^{k_{x_0}-1}(x_0)\in C(f)$)

$$\lim_{m\to\infty} f\left(p_m^{k_{x_0}-1}\right)=f^{k_{x_0}}(x_0)=\lim_{m\to\infty} f\left(q_m^{k_{x_0}-1}\right).$$

Let us note that

$$\lim_{m\to\infty} f^2\left(p_m^{k_{x_0}-1}\right)=0=\lim_{m\to\infty} f^2\left(q_m^{k_{x_0}-1}\right).$$

Let us continue this procedure.

If $k_{x_0}-1=0$, then we obtain $p_m^0$ and $q_m^0$.

If $k_{x_0}-1>0$, then we apply the above procedure again. We finish these considerations after obtaining the sequences $\{p_m^0\}$ and $\{q_m^0\}$. Then the following conditions are fulfilled:

$$q_m^0 \nearrow x_0 \swarrow p_m^0. \tag{5}$$

$$\begin{aligned}\left\{p_m^0\colon m\in\mathbb{N}\right\},\left\{q_m^0\colon m\in\mathbb{N}\right\} &\subset f^{-1}\Big(\left\{p_m^1\colon m\in\mathbb{N}\right\}\cup\left\{q_m^1\colon m\in\mathbb{N}\right\}\Big)\,,\\ \left\{p_m^1\colon m\in\mathbb{N}\right\},\left\{q_m^1\colon m\in\mathbb{N}\right\} &\subset f^{-1}\Big(\left\{p_m^2\colon m\in\mathbb{N}\right\}\cup\left\{q_m^2\colon m\in\mathbb{N}\right\}\Big)\,,\\ &\dots\\ \left\{p_m^{k_{x_0}}\colon m\in\mathbb{N}\right\},\left\{q_m^{k_{x_0}}\colon m\in\mathbb{N}\right\} &\subset f^{-1}\Big(\left\{\alpha_n\colon n\in\mathbb{N}\right\}\Big)\,.\end{aligned} \tag{6}$$

$$\lim_{m\to\infty} f^s(p_m^0) = f^s(x_0) = \lim_{m\to\infty} f^s(q_m^0), \text{ if } s = 0, 1, \dots, k_{x_0}, k_{x_0}+1. \qquad (7)$$

Let $B(x_0)$ consist of all sets $U_{m,n}^{\{k_i\},\{t_i\}}(x_0)$ of the form

$$U_{m,n}^{\{k_i\},\{t_i\}}(x_0) = \{x_0\} \cup \bigcup_{i=m}^{\infty}\left(p_i^0 - \frac{1}{k_i}, p_i^0 - \frac{1}{k_i}\right) \cup \bigcup_{i=n}^{\infty}\left(q_i^0 - \frac{1}{t_i}, q_i^0 - \frac{1}{t_i}\right),$$

where $m \in \mathbb{N}, n \in \mathbb{N}$, and $\{k_i\} = \{k_i\}_{i=m}^{\infty}$, $\{t_i\} = \{t_i\}_{i=n}^{\infty}$ are sequences of positive integers.

For the sake of the legibility of further parts of the proof, we can establish the relationship between $x_0$ and the suitable terms of sequences $\{p_i^0\}$ and $\{q_i^0\}$. Therefore we shall write $p_i^{x_0}$ instead of $p_i^0$ and $q_i^{x_0}$ instead of $q_i^0$.

We can immediately see that

$$U \setminus \{x\} \text{ is a nonempty open set, for each } x \in [0,1] \text{ and } U \in B(x). \qquad (8)$$

$$\text{For each } x \in [0,1] \text{ and for an open set } V \text{ containing } x \text{ there exists } U \in B(x) \text{ such that } U \subset V. \qquad (9)$$

Now, we shall show that the family $\{B(x)\colon x \in [0,1]\}$ fulfills conditions (BP1), (BP2), (BP3) of a neighbourhood system [7] (p. 28). Condition (BP1):
For every $x \in [0,1]$, $B(x) \neq \emptyset$ and for every $U \in B(x)$, $x \in U$
is obvious.

Now, we will consider condition (BP2):
If $x \in U \in B(y)$ then there exists a set $V \in B(x)$ such that $V \subset U$.

Obviously, it is sufficient to consider only the case $x \neq y$. From (8) it follows that $x$ belongs to an open set $U \setminus \{y\}$. According to (9) there exists a set $V \in B(x)$ such that $V \subset U \setminus \{y\}$.

Finally, we shall show that our family $\{B(x)\colon x \in [0,1]\}$ fulfills condition (BP3):
For each $U_1 \in B(x)$, $U_2 \in B(x)$, there is a set $U \in B(x)$ such that $U \subset U_1 \cap U_2$.

If either $x \in \{0,1\}$ or $x \in (0,1)$, and $f^k(x) \notin \{0,1\}$, for each $k \in \mathbb{N}$, condition (BP3) is fulfilled in an obvious way.

We only need to consider the case

$$x \in (0,1) \text{ and } f^k(x) \in \{0,1\}, \text{ for some positive integer } k.$$

Let $U_1 = U_{m,n}^{\{k_i\},\{t_i\}}(x)$ and $U_2 = U_{j,l}^{\{c_i\},\{d_i\}}(x)$. Put $\gamma = \max(m,j)$, $\xi = \max(n,l)$ and

$$h_i = \max(k_i, c_i)(\text{for } i \geqslant \gamma);\ l_i = \max(t_i, d_i)(\text{for } i \geqslant \xi).$$

Then we can define the set $U$ in the following way:

$$U = U_{\gamma,\xi}^{\{h_i\},\{l_i\}}(x) \in B(x)\,.$$

Obviously, $U \subset U_1 \cap U_2$.

Let $\mathcal{T}$ be the topology generated by the neighbourhood system $\{B(x)\colon x \in [0,1]\}$ ([7], Proposition 1.2.3).

From (9) it follows that $\mathcal{T}$ is a finer topology than the natural topology of the unit interval. Consequently, according to (8) we infer that $\mathcal{T}$ is compatible with this topology.

It remains to prove that

$$f^n \text{ is a } \mathcal{T}\text{-continuous function } (n = 1, 2, \ldots)\,.$$

Let us fix $n_0 \in \mathbb{N}$ and $z_0 \in [0,1]$. We will prove that

$$z_0 \text{ is a } \mathcal{T}\text{-continuity point of } f^{n_0}\,. \tag{10}$$

Let $\varepsilon_{n_0}$ be a positive real number. We assume that if $f^{n_0}(z_0) < 1\big(f^{n_0}(z_0) > 0\big)$, then $f^{n_0}(z_0) + \varepsilon_{n_0} < 1$ ($f^{n_0}(z_0) - \varepsilon_{n_0} > 0$).

The proof of (10) will be divided into two cases.

$1^o$ $f^{n_0}(z_0) \notin \{0,1\}$. Thus $f^k(z_0) \notin \{0,1\}$, for $k \in \{0,1,\ldots,n_0\}$. With respect to the assumptions of our theorem we have that $f^{n_0-1}(z_0)$ is a continuity point of $f$. Then there exists $\varepsilon_{n_0-1} > 0$ such that

$$\big(f^{n_0-1}(z_0) - \varepsilon_{n_0-1}, f^{n_0-1}(z_0) + \varepsilon_{n_0-1}\big) \subset (0,1)$$

and

$$f\Big(\big(f^{n_0-1}(z_0) - \varepsilon_{n_0-1}, f^{n_0-1}(z_0) + \varepsilon_{n_0-1}\big)\Big) \subset \big(f^{n_0}(z_0) - \varepsilon_{n_0}, f^{n_0}(z_0) + \varepsilon_{n_0}\big)\,.$$

If $n_0 - 1 > 0$ then we apply this procedure again: $f^{n_0-2}(z_0)$ is a continuity point of $f$ and there exists $\varepsilon_{n_0-2} > 0$ such that

$$\big(f^{n_0-2}(z_0) - \varepsilon_{n_0-2}, f^{n_0-2}(z_0) + \varepsilon_{n_0-2}\big) \subset (0,1)$$

and

$$f\Big(\big(f^{n_0-2}(z_0) - \varepsilon_{n_0-2}, f^{n_0-2}(z_0) + \varepsilon_{n_0-2}\big)\Big) \subset \big(f^{n_0-1}(z_0) - \varepsilon_{n_0-1}, f^{n_0-1}(z_0) + \varepsilon_{n_0-1}\big)\,.$$

We continue the procedure in this fashion obtaining a finite sequence of positive numbers $\{\varepsilon_k\}_{k=0}^{n_0}$ such that

$$f\Big(\big(f^k(z_0) - \varepsilon_k, f^k(z_0) + \varepsilon_k\big)\Big) \subset \big(f^{k+1}(z_0) - \varepsilon_{k+1}, f^{k+1}(z_0) + \varepsilon_{k+1}\big)\,,$$
$$\text{for } k \in \{0,1,\ldots,n_0-1\}\,.$$

Consequently,

$$f^{n_0}\big((z_0-\varepsilon_0, z_0+\varepsilon_0)\big) \subset \big(f^{n_0}(z_0)-\varepsilon_{n_0}, f^{n_0}(z_0)+\varepsilon_{n_0}\big).$$

On account of the above inclusion and the arbitrariness of $\varepsilon_{n_0}$, $z_0$ is a continuity point of $f^{n_0}$ and so it is a $\mathcal{T}$-continuity point of $f^{n_0}$.

$2^o$ $f^{n_0}(z_0) \in \{0,1\}$. Suppose, for instance, that $f^{n_0}(z_0)=0$. Let us put $M=\min\{m \in \mathbb{N}\colon \alpha_m \in [0,\varepsilon_{n_0})\}$. For any $m \geqslant M$ let $k_m$ be a number such that

$$\left(\alpha_m - \frac{1}{k_m}, \alpha_m + \frac{1}{k_m}\right) \subset [0,\varepsilon_{n_0}). \tag{11}$$

First, we will consider the case when $z_0=0$. Since $f^{n_0}(\alpha_m)=\alpha_m \notin \{0,1\}$ ($m \geqslant M$) then, according to considerations contained in $1^o$, there exist $k'_m$ ($m \geqslant M$) such that

$$f^{n_0}\left(\left(\alpha_m - \frac{1}{k'_m}, \alpha_m + \frac{1}{k'_m}\right)\right) \subset \left(\alpha_m - \frac{1}{k_m}, \alpha_m + \frac{1}{k_m}\right), \text{ for } m \geqslant M.$$

So, we put $W=\{0\} \cup \bigcup\limits_{m=M}^{\infty}\left(\alpha_m - \frac{1}{k'_m}, \alpha_m + \frac{1}{k'_m}\right)$ and, by (11), we have $f^{n_0}(W) \subset [0,\varepsilon_{n_0})$.

Now, it is sufficient to consider a case $z_0 \in (0,1)$ (note that by the assumption $f^{n_0}(z_0) \neq 1$, we have $z_0 \neq 1$).

Let $k_{z_0} = \max\{k \in \{0,1,\dots,n_0-1\}\colon f^k(z_0) \in (0,1)\}$. Similarly as above, we can find positive integers $k''_m$ ($m \geqslant M$) such that

$$f^{n_0-k_{z_0}-1}\left(\left(\alpha_m - \frac{1}{k''_m}, \alpha_m + \frac{1}{k''_m}\right)\right) \subset \left(\alpha_m - \frac{1}{k_m}, \alpha_m + \frac{1}{k_m}\right), \text{ for } m \geqslant M. \tag{12}$$

Let us recall the sequences $\{p_m^{z_0}\}_{m=1}^{\infty}$, $\{q_m^{z_0}\}_{m=1}^{\infty}$, connected with the local base $B(z_0)$. According to (6)

$$f^{k_{z_0}+1}(p_m^{z_0}), f^{k_{z_0}+1}(q_m^{z_0}) \in \{\alpha_n\colon n \in \mathbb{N}\}, \quad \text{for each } m.$$

Next, condition (7) gives

there exists $M^\star$ such that $f^{k_{z_0}+1}(p_m^{z_0}), f^{k_{z_0}+1}(q_m^{z_0}) \in [0,\varepsilon_{n_0})$, for $m \geqslant M^\star$.

Therefore

$$f^{k_{z_0}+1}(p_m^{z_0}) \in \{\alpha_M, \alpha_{M+1}, \alpha_{M+2}, \dots\},$$
$$f^{k_{z_0}+1}(q_m^{z_0}) \in \{\alpha_M, \alpha_{M+1}, \alpha_{M+2}, \dots\},$$

for $m \geqslant M^\star$.

Obviously, $f^{k_{z_0}+1}(p_m^{z_0}), f^{k_{z_0}+1}(q_m^{z_0}) \notin \{0,1\}$, for $m \geqslant M^\star$. Then, according to the proof of the case $1^o$, for each $m \geqslant M^\star$, there exist positive integers $w_m$, $v_m$ such that

$$f^{k_{z_0}+1}\left(\left(p_m^{z_0} - \frac{1}{w_m}, p_m^{z_0} + \frac{1}{w_m}\right)\right) \subset \left(\alpha_j - \frac{1}{k_j''}, \alpha_j + \frac{1}{k_j''}\right)$$

and

$$f^{k_{z_0}+1}\left(\left(q_m^{z_0} - \frac{1}{v_m}, q_m^{z_0} + \frac{1}{v_m}\right)\right) \subset \left(\alpha_l - \frac{1}{k_l''}, \alpha_l + \frac{1}{k_l''}\right),$$

for $j \geqslant M, l \geqslant M$.

Combining the above inclusions with (11) and (12), we obtain (for $m \geqslant M^\star$):

$$f^{n_0}\left(\left(p_m^{z_0} - \frac{1}{w_m}, p_m^{z_0} + \frac{1}{w_m}\right)\right) \subset [0, \varepsilon_{n_0})$$

and

$$f^{n_0}\left(\left(q_m^{z_0} - \frac{1}{v_m}, q_m^{z_0} + \frac{1}{v_m}\right)\right) \subset [0, \varepsilon_{n_0}).$$

Let us consider a set $U_{M^\star,M^\star}^{\{v_i\},\{w_i\}}(z_0) \in B(z_0)$. It is easy to see that $f^{n_0}\left(U_{M^\star,M^\star}^{\{v_i\},\{w_i\}}(z_0)\right) \subset [0, \varepsilon_{n_0})$. □

**Corollary 5.** *Let $C \subset [0,1]$ be a Cantor-like set such that $0,1 \in C$ and let $f\colon [0,1] \longrightarrow [0,1]$ be a function such that $f(0)=0$, $f(1)=1$, $f(C) \subset \{0,1\}$, $f \restriction [a,b]$ is a nowhere constant continuous function and $0,1 \in f([a,b])$, for any component $(a,b)$ of $[0,1] \setminus C$. Then $f$ is a transitive function iff it is a nomadic function.*

P r o o f . Obviously, $f$ is a Darboux function. According to Theorem 4, the proof is completed by showing that $0 \in (\mathrm{Fix}(f))^d$ and $1 \in (\mathrm{Fix}(f))^d$. We restrict our considerations to a point $0$ (in the case of $1$, the proof is similar).

Let $\varepsilon > 0$ and let $(a,b)$ be a component of $[0,1] \setminus C$ such that $(a,b) \subset [0,\varepsilon)$. Note that $\Gamma(f \restriction [a,b]) \cap \Gamma(\mathrm{id} \restriction [a,b]) \neq \emptyset$. So, there exists point $x_0$ in $\mathrm{Fix}(f)$ such that $x_0 \in (a,b) \subset [0,\varepsilon)$. □

Example. Let $C$ be a Cantor-like set such that $\lambda(C) > 0$ and $0,1 \in C$. Moreover, let $C^\star$ be a nonmeasurable subset of $C$ such that $0 \notin C^\star$, $1 \in C^\star$ and $a,b \notin C^\star$, for any component $(a,b)$ of $[0,1] \setminus C$. Let us denote by $d_{a,b}$ the middle-point of $[a,b]$. Now, we define a function $f$ in the following way:

$$f(x) = \begin{cases} 0 & \text{if } x \in C \setminus C^\star; \\ 1 & \text{if } x \in C^\star \text{ or } x = d_{a,b}, \text{ for any component } (a,b) \text{ of } [0,1] \setminus C; \\ \text{linear} & \text{in } [a, d_{a,b}], [d_{a,b}, b], \text{ for component } (a,b) \text{ of } [0,1] \setminus C. \end{cases}$$

Then $f$ is a nonmeasurable function fulfilling the assumptions of Corollary 5. Note that the set of all discontinuity points of $f$ has the positive Lebesgue measure.

It is easy to see that the assumptions of Corollary 5 can be modified.

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