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TANGENT CONES FOR SPACES EQUIPPED WITH DIFFERENT NORMS

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ABSTRACT. The dimension of tangent cones to graphs of mappings $f\colon X \to Y$ (where X, Y are normed spaces) is compared with the dimension of X. In the second part of the paper we study the structure of tangent cones to unit spheres in the space $C_0^1[0,1] = \{x \in C^1[0,1] : x(0) = 0\}$ with respect to different norms.

Let Z be a real normed space. By $\overline{A}$ we denote the closure of A. Throughout the paper we consider the differentiability (derivative) in the Fréchet sense ([6]).

Definition 1. Let $\emptyset \neq M \subset Z$ and $a \in \overline{M}$. The set

$$\Big\{v \in Z : \exists\ \{a_n\}_{n\in\mathbb{N}},\ a_n \in M,\ a_n \to a,\ \exists\ \{\lambda_n\}_{n\in\mathbb{N}},\ \lambda_n > 0 \colon \lim_{n\to\infty} \lambda_n(a_n - a) = v\Big\}$$

is called a tangent cone to M at the point a and is denoted by $\operatorname{Tan}(M, a)$.

The elements of $\operatorname{Tan}(M, a)$ are called vectors tangent to M at the point a ([1]–[3], [5], [6]). The set $\operatorname{Tan}(M, a)$ is also called a contingent of M at the point a.

1. Comparison of $\dim \operatorname{Tan}\big(G(f), (x_0, f(x_0))\big)$ and $\dim X$

In this section we consider mappings $f\colon X \to Y$ where X, Y are real normed spaces (of finite dimension or not). By $G(f)$ we denote the graph of f, $G(f) \subset X \times Y$, the space $X \times Y$ being equipped with the norm $\|(x, y)\| = \|x\| + \|y\|$.

First, let us formulate a basic criterion for differentiability of a mapping in terms of a tangent cone to the graph of that mapping.

By a vertical vector in $X \times Y$ we mean any vector of the form $(0_X, \beta) \in X \times Y$, $\beta \neq 0_Y$. By $p_X : X \times Y \to X$, $p_Y : X \times Y \to Y$ we denote the natural projections. The Fréchet derivative of $f\colon X \to Y$ at x_0 is denoted by $f'(x_0)$.

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Theorem 1. ([3], [7]) *Let* $\dim X < \infty$, $\dim Y < \infty$, *and let* $f\colon X \to Y$ *be continuous at* $x_0 \in X$. *Assume that* $T = \operatorname{Tan}\big(G(f), (x_0, f(x_0))\big)$ *is a linear subspace of* $X \times Y$ *without vertical vectors. Then*

(a) f *is differentiable at* x_0;

(b) $f'(x_0) = p_Y \circ (p_X|_T)^{-1}$;

(c) $\|f'(x_0)\| = \sup\big\{\|\beta\| \colon (\alpha, \beta) \in T,\ \|\alpha\| = 1\big\}$.

Let $f\colon X \to Y$ be continuous at $x_0 \in X$. Assume that $T = \operatorname{Tan}\big(G(f), (x_0, f(x_0))\big)$ is a linear subspace of $X \times Y$.

The first question we are interested in is: what are the possible relations between $\dim T$ and $\dim X$? It turns out that each of the following relations is possible

$$\dim T > \dim X\,, \quad \dim T = \dim X\,, \quad \dim T < \dim X\,. \tag{1}$$

The first relation in (1) is furnished by the function $f\colon X \to Y$, $X = Y = \mathbb{R}$,

$$f(t) = \begin{cases} \sqrt{|t|}\,\sin\frac{1}{t} & \text{if } t \neq 0\,, \\ 0 & \text{if } t = 0\,. \end{cases}$$

We have $T = \operatorname{Tan}\big(G(f), (0,0)\big) = \mathbb{R}^2$, whence $\dim T > \dim X$.

Let us observe that T (in this example) contains vertical vectors and this is the only reason the inequality $\dim T > \dim X$ is accounted for, as we will see later.

Theorem 2. *Let* X, Y *be real normed spaces of any dimension, finite or not. Let* $f\colon X \to Y$ *be continuous at* $x_0 \in X$ *and* $T = \operatorname{Tan}\big(G(f), (x_0, f(x_0))\big)$ *a linear subspace of* $X \times Y$ *without vertical vectors. Then*

$$\dim T \leqslant \dim X\,. \tag{2}$$

Proof. Since T is a linear subspace without vertical vectors, the projection $p_X|_T\colon T \to X$ is injective whence (2) follows. □

Theorem 3. *Assume that* $f\colon X \to Y$ *satisfies the assumptions of Theorem 2 and* $\dim Y < \infty$. *Then*

$$\dim T = \dim X\,. \tag{3}$$

We omit the proof since in fact it is a part of the proof of Theorem 1, where it was shown that $p_X(T) = X$. The compactness of the sphere in Y, $\dim Y < \infty$, is important.

Note that in Theorem 3 we do not exclude the case $\dim X = \infty$.

In the next example, we show that if $\dim Y = \infty$, then there exists a Lipschitz mapping $f\colon X \to Y$ having a trivial tangent cone, i.e.,

$$\dim \operatorname{Tan}\big(G(f), (x_0, f(x_0))\big) = 0\,.$$

Example 1. Let $X = \mathbb{R}$ and $Y = l^2$ be the classical real Hilbert space with the standard inner product $\langle\ ,\ \rangle$. Let $\{e_n : n \in \mathbb{N}\}$ be a standard orthonormal system in l^2. By 0_Y we denote the neutral element of Y. Consider the mapping $f : \mathbb{R} \to l^2$ defined by

$$f(t) = \begin{cases} -2^{-1} e_1 & \text{for } t < -2^{-1}, \\ (2t + 2^{-n}) e_n - (2^{-n} + t) e_{n+1} & \text{for } t \in [-2^{-n}, -2^{-n-1}),\ n \in \mathbb{N}, \\ 0_Y & \text{for } t = 0, \\ (2t - 2^{-n}) e_n + (2^{-n} - t) e_{n+1} & \text{for } t \in (2^{-n-1}, 2^{-n}],\ n \in \mathbb{N}, \\ 2^{-1} e_1 & \text{for } t > 2^{-1}. \end{cases}$$

Observe that f is continuous and odd. We will prove that

$$\operatorname{Tan}\big(G(f), (0, 0_Y)\big) = \big\{(0, 0_Y)\big\}. \tag{4}$$

To show (4), let us take any $(\xi, v) \in \operatorname{Tan}\big(G(f), (0, 0_Y)\big)$. It follows from Definition 1 that there is a sequence $\big\{\big(x_n, f(x_n)\big)\big\}_{n\in\mathbb{N}}$, $x_n \in \mathbb{R}$, $x_n \to 0$, and there is a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$, $\lambda_n > 0$, such that

$$\lambda_n x_n \to \xi \quad \text{and} \quad \lambda_n f(x_n) \to v. \tag{5}$$

Consider the case $x_n > 0$. Without loss of generality we may assume that for each $n \in \mathbb{N}$ there exists $m_n \in \mathbb{N}$ such that $x_n \in (2^{-m_n-1}, 2^{-m_n}]$. So we have

$$\lambda_n f(x_n) = \lambda_n (2x_n - 2^{-m_n}) e_{m_n} + \lambda_n (2^{-m_n} - x_n) e_{m_n+1} = v + \varepsilon_n, \tag{6}$$

where $\varepsilon_n \to 0$ (as $n \to \infty$). Taking the inner product of (6), by e_{m_n} we get

$$\lambda_n (2x_n - 2^{-m_n}) = \langle v, e_{m_n}\rangle + \langle \varepsilon_n, e_{m_n}\rangle.$$

Since $\langle v, e_{m_n}\rangle \to 0$ and $\langle \varepsilon_n, e_{m_n}\rangle \to 0$ as $n \to \infty$, we have $\lambda_n(2x_n - 2^{-m_n}) \to 0$. As $\lambda_n x_n \to \xi$, we also get $\lambda_n 2^{-m_n} \to 2\xi$. Now, taking the inner product of (6), by e_{m_n+1} we get

$$\lambda_n (2^{-m_n} - x_n) = \langle v, e_{m_n+1}\rangle + \langle \varepsilon_n, e_{m_n+1}\rangle.$$

Similar arguments yield $\lambda_n 2^{-m_n} \to \xi$. Thus $\xi = 0$ whence $v = 0$.

The case $x_n < 0$ is discussed quite analogously.

Thus we have shown that

$$\operatorname{Tan}\big(G(f), (0, 0_Y)\big) = \big\{(0, 0_Y)\big\},$$

i.e., under the assumptions of Example 1 we have $\dim X > \dim T$.

It remains to show that f is Lipschitz. By our construction the mapping f is affine on each interval $[2^{-n-1}, 2^{-n}]$, $n \in \mathbb{N}$, and we have

$$\left\| \frac{f(2^{-n}) - f(2^{-n-1})}{2^{-n} - 2^{-n-1}} \right\| = \left\| \frac{2^{-n} e_n - 2^{-n-1} e_{n+1}}{2^{-n-1}} \right\| \leqslant \frac{\|e_n\|}{2^{-1}} + \|e_{n+1}\| = 3.$$

It follows that on each interval $[\varepsilon, 2^{-1}]$, $0<\varepsilon<2^{-1}$, the continuous mapping f is piecewise affine and Lipschitz with constant $L=3$. This immediately implies that f is Lipschitz with constant $L=3$ on $[0,2^{-1}]$. Since f is odd and constant on $[2^{-1},\infty)$, we may conclude that f is Lipschitz with constant $L=3$ on the whole of $\mathbb{R}$.

EXAMPLE 2. Let X be a real normed space, $\dim X=\infty$. Fix a sequence $\{e_n\}_{n\in\mathbb{N}}$, $e_n\in X$, such that $\|e_n\|=1$, $\|e_n-e_k\|>1$, $n\neq k$. (It is well-known that such sequences $\{e_n\}_{n\in\mathbb{N}}$ exist since $\dim X=\infty$.) Let $B_n=B(e_n,10^{-1})$, $n\in\mathbb{N}$, be an open ball with center e_n and radius 10^{-1}. We have[1] $\operatorname{dist}(B_n,B_k)\geqslant\frac{4}{5}$, $n\neq k$.

Consider the sequence of balls $\{b_n\}_{n\in\mathbb{N}}$ where $b_n=\left\{x\in X\colon \left\|x-\frac{e_n}{n}\right\|<\frac{1}{10n}\right\}$ such that $b_n\cap b_m=\emptyset$ for $n\neq m$. Note that each b_n is homothetic with B_n and lies in the cone $C_n=\{tx\colon t\geqslant 0,\, x\in B_n\}$. We have $C_n\cap C_k=\{0_X\}$, $n\neq k$. By 0_X we denote the neutral element of X.

Define the mapping $f\colon X\to\mathbb{R}$ letting

$$f(x)=\begin{cases}0 & \text{if } x\notin\bigcup_{n=1}^{\infty} b_n,\\ 10\big(2+(-1)^n\big)\operatorname{dist}\big(\{x\},\partial b_n\big) & \text{if } x\in b_n,\ n\in\mathbb{N},\end{cases}\tag{7}$$

where ∂b_n denotes the boundary of the ball b_n.

We will show that $f\colon X\to\mathbb{R}$ has the following properties:

(a) f is Lipschitz;

(b) $T=\operatorname{Tan}\big(G(f),(0_X,0)\big)=X\times\{0\}$ and hence $\dim T=\infty$;

(c) f is not Fréchet differentiable at $x=0_X$;

(d) for each $x\in X$: $\lim\limits_{t\to 0}\frac{f(tx)}{t}=0$, moreover

(e) $\big(\forall\, x\neq 0_X\ \exists\, B(x,\rho)\ \exists\,\delta>0\ \forall\, t\in\mathbb{R}\ \forall\, y\in B(x,\rho)\colon\ [|t|<\delta\Rightarrow f(ty)=0]\big)$.

Proof of (a). Since each sphere ∂b_n is a closed set, it is well-known that the mapping $X\ni x\mapsto\operatorname{dist}\big(\{x\},\partial b_n\big)$ is Lipschitz with constant $L=1$:

$$\forall\, x',x''\in X\colon\ \big|\operatorname{dist}\big(\{x'\},\partial b_n\big)-\operatorname{dist}\big(\{x''\},\partial b_n\big)\big|\leqslant d(x',x'').\tag{8}$$

Furthermore, as b_n are pairwise disjoint it easily follows that f is Lipschitz with constant $L=30$.

Proof of (d), (e). Note that $f(0_X)=0$ and $\operatorname{dist}\big(\{0_X\},\partial b_n\big)\geqslant\frac{9}{10n}>0$. Since each half-line from 0_X can meet only one b_n at most, it follows that each Gâteaux derivative vanishes at $x=0_X$ (i.e., we have (d)). Moreover, taking into consideration the location of $\{b_n\}$, it is also easy to see that the relation (e) is fulfilled.

[1] By $\operatorname{dist}(A,B)$ we mean $\inf\big\{\|a-b\|\colon a\in A,\ b\in B\big\}$.

Next, we will show (b). Let $(\xi, y) \in T$. Then by Definition 1, there exist sequences: $\{x_m\}_{m\in\mathbb{N}}$, $x_m \in X$, $x_m \to 0_X$, and $\{\lambda_m\}_{m\in\mathbb{N}}$, $\lambda_m > 0$, such that

$$\lambda_m x_m \to \xi \quad \text{and} \quad \lambda_m f(x_m) \to y \qquad \text{as } m \to \infty .$$

From the construction of f it follows that ξ can hit only one b_n at most. Therefore it is easy to see that there is $m_0 \in \mathbb{N}$ such that $x_m \notin \bigcup_{n=1}^{\infty} b_n$ for $m > m_0$. Then by (7) we have $\lambda_m f(x_m) = 0$ for $m > m_0$, whence $y = \lim_{m\to\infty} \lambda_m f(x_m) = 0$. Thus we have shown that

$$(\xi, y) \in T \implies y = 0 .$$

Now, take any vector $(v, 0) \in X \times \{0\}$. Since there is at most one b_n which may contain v, there exists $m_1 \in \mathbb{N}$ such that $\frac{v}{m} \notin \bigcup_{n=1}^{\infty} b_n$ for $m \geqslant m_1$. Putting $x_m = \frac{v}{m}$ for $m \geqslant m_1$, $x_m = 0_X$ for $m < m_1$, and $\lambda_m = m$ for $m \in \mathbb{N}$, we obtain $\lambda_m x_m \to v$ and $\lambda_m f(x_m) \to 0$. It follows that $(v, 0) \in T$. Thus we have proved equality (b).

The equality $\dim T = \infty$ is of course trivial.

Proof of (c) is also immediate because

$$\lim_{n\to\infty} \frac{f\left(\frac{e_n}{n}\right)}{\left\|\frac{e_n}{n}\right\|} = \lim_{n\to\infty} \left(2 + (-1)^n\right)$$

does not exist, which shows that $f\colon X \to \mathbb{R}$ is not Fréchet differentiable at 0_X.

2. Tangent cones for different norms

In this section we will see in examples how the change of norms influences the tangent cones.

Let X be a normed space ($\dim X \leqslant \infty$) equipped with two norms $\|\cdot\|_1$, $\|\cdot\|_2$ such that $\|\cdot\|_1 \leqslant \|\cdot\|_2$. Let $\emptyset \neq M \subset X$. By $T_1(M, x_0)$, $T_2(M, x_0)$ we denote the tangent cones of M at x_0 corresponding to norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. From Definition 1 it is immediate that

$$\forall\, x_0 \in \overline{M}\colon\ T_2(M, x_0) \subset T_1(M, x_0) . \tag{9}$$

Remark 1. If $\dim X < \infty$ then obviously $T_2(M, x_0) = T_1(M, x_0)$, because in a finite dimensional space all norms are equivalent.

The question arises whether the inclusion (9) is proper. We will show that in (9) both a strict inclusion and an equality are possible.

To show this, consider the linear space $C_0^1[0,1]$ of C^1-functions $x\colon [0,1]\to\mathbb{R}$ vanishing at $t=0$ (i.e., $x(0)=0$). Introduce two norms

$$\|x\| = \sup_{t\in[0,1]} \big|x(t)\big| \quad \text{and} \quad \|x\|_1 = \sup_{t\in[0,1]} \big|x'(t)\big| . \tag{10}$$

Clearly $\|\cdot\| \leqslant \|\cdot\|_1$ and these norms are not equivalent.

In the following lemma we use a regularization operator. Let $f\colon \mathbb{R}\to\mathbb{R}$ be continuous. Consider $\varphi_h\colon \mathbb{R}\to\mathbb{R}$ defined by

$$\varphi_h(t) = \begin{cases} \frac{1}{2h} & \text{for } |t|<h , \\ 0 & \text{for } |t|\geqslant h . \end{cases}$$

Then the convolution $f*\varphi_h\colon \mathbb{R}\to\mathbb{R}$ defined by

$$(f*\varphi_h)(t) = \int_{-\infty}^{\infty} f(\tau)\varphi_h(t-\tau)\,d\tau = \frac{1}{2h}\int_{t-h}^{t+h} f(\tau)\,d\tau$$

is known for having the following properties ([4]):

- $f*\varphi_h \in C^1(\mathbb{R})$;
- $f*\varphi_h|_K \to f|_K$ uniformly on each compact set $K\subset\mathbb{R}$, as $h\to 0$.

Lemma 4. *Let $w\in C_0^1[0,1]$ and $w'(t)>0$ for $t\in[0,1]$. Then for each $t_0\in(0,1)$ and each $\varepsilon>0$ there exists $u\in C_0^1[0,1]$ such that $u'\geqslant 0$, $u'(t_0)=0$ and $\|w-u\|<\varepsilon$.*

P r o o f. Let $w\in C_0^1[0,1]$ be given, $w'>0$. Fix $t_0\in(0,1)$ and $\varepsilon>0$. Since w is continuous, there exists $\delta>0$, $\delta<\frac{1}{2}\min\{t_0, 1-t_0\}$, such that $\|w-w_\delta\|<\frac{\varepsilon}{3}$ where $w_\delta\colon[0,1]\to\mathbb{R}$ is defined as follows

$$w_\delta(t) = \begin{cases} w(t) & \text{for } t\in[0,t_0-2\delta]\cup[t_0+2\delta,1] , \\ \frac{w(t_0-2\delta)-w(t_0)}{-\delta}(t-t_0+\delta)+w(t_0) & \text{for } t\in(t_0-2\delta, t_0-\delta) , \\ w(t_0) & \text{for } t\in[t_0-\delta, t_0+\delta] , \\ \frac{w(t_0+2\delta)-w(t_0)}{\delta}(t-t_0-\delta)+w(t_0) & \text{for } t\in(t_0+\delta, t_0+2\delta) . \end{cases}$$

Define the extension $\overline{w}_\delta\colon\mathbb{R}\to\mathbb{R}$ of w_δ putting

$$\overline{w}_\delta(t) = \begin{cases} w_\delta(t) & \text{for } t\in[0,1] , \\ 0 & \text{for } t<0 , \\ w_\delta(1) & \text{for } t>1 . \end{cases}$$

Consider the regularization $\overline{w}_\delta*\varphi_h\colon\mathbb{R}\to\mathbb{R}$ defined by

$$(\overline{w}_\delta*\varphi_h)(t) = \int_{-\infty}^{\infty}\overline{w}_\delta(\tau)\,\varphi_h(t-\tau)\,d\tau = \frac{1}{2h}\int_{t-h}^{t+h}\overline{w}_\delta(\tau)\,d\tau \qquad \text{for } 0<h<\frac{\delta}{4} .$$

Finally, define $u\colon [0,1] \to \mathbb{R}$ letting $u(t) = (\overline{w}_\delta * \varphi_h)\big|_{[0,1]}(t) - (\overline{w}_\delta * \varphi_h)(0)$. Obviously $u(0) = 0$ and $u \in C_0^1[0,1]$. Since u is constant on $[t_0 - h, t_0 + h]$, $h < \frac{\delta}{4}$, we have $u'(t_0) = 0$.

Since $(\overline{w}_\delta * \varphi_h)\big|_{[0,1]} \to \overline{w}_\delta\big|_{[0,1]}$ uniformly as $h \to 0$, there exists $h > 0$, $h < \frac{\delta}{4}$, such that $\Big\|(\overline{w}_\delta * \varphi_h - \overline{w}_\delta)\big|_{[0,1]}\Big\| < \frac{\varepsilon}{3}$ and $\big\|(\overline{w}_\delta * \varphi_h)(0)\big\| < \frac{\varepsilon}{3}$.

Then for $t \in [0,1]$ we get

$$\begin{aligned}\big\|u(t) - w(t)\big\| &= \big\|(\overline{w}_\delta * \varphi_h)(t) - (\overline{w}_\delta * \varphi_h)(0) - w(t)\big\| \\ &\leqslant \big\|(\overline{w}_\delta * \varphi_h)(t) - \overline{w}_\delta(t)\big\| + \big\|(\overline{w}_\delta * \varphi_h)(0)\big\| + \big\|\overline{w}_\delta(t) - w(t)\big\| < \varepsilon .\end{aligned}$$

It follows that $u\colon [0,1] \to \mathbb{R}$ has the required properties. □

Example 3. Next we will show that tangent cones to the same set at the same point could be different for different norms. Namely, consider in $C_0^1[0,1]$ the unit sphere S_1 defined by the norm $\|\cdot\|_1$:

$$S_1 = \big\{x \in C_0^1[0,1] \colon \|x - \mathrm{id}\,\|_1 = 1\big\} = \big\{x \in C_0^1[0,1] \colon \sup_{t\in[0,1]} \big|x'(t) - 1\big| = 1\big\}.$$

Clearly, $x = 0 \in S_1$. Let us find out how the tangent cones to S_1 at $x = 0$ with respect to the norms $\|\cdot\|$, $\|\cdot\|_1$ look like. Let $T(S_1, 0)$ denote the tangent cone to S_1 at $x = 0$ with respect to the norm $\|\cdot\|$, while $T_1(S_1, 0)$ will denote the tangent cone to S_1 at $x = 0$ with respect to the norm $\|\cdot\|_1$.

We are going to show that

$$T(S_1, 0) = \big\{w \in C_0^1[0,1] \colon w' \geqslant 0\big\} \tag{11}$$

and

$$T_1(S_1, 0) = \big\{w \in C_0^1[0,1] \colon w' \geqslant 0,\ \exists\, t_0 \in [0,1] \colon w'(t_0) = 0\big\}. \tag{12}$$

First, we will prove (11). Let $v \in T(S_1, 0)$. By Definition 1 we have that there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \in S_1$, $\|x_n\| \to 0$, and there is a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$, $\lambda_n > 0$, such that $\|\lambda_n x_n - v\| \to 0$. Since $x_n \in S_1$, we have $0 \leqslant x_n' \leqslant 2$ for each $n \in \mathbb{N}$. Therefore each function x_n, and each $\lambda_n x_n$, respectively, is nondecreasing for each $n \in \mathbb{N}$. This implies that $v' \geqslant 0$. Since $v \in T(S_1, 0)$ was arbitrary, we may conclude that

$$T(S_1, 0) \subset \big\{w \in C_0^1[0,1] \colon w' \geqslant 0\big\}. \tag{13}$$

To get the inverse inclusion, take any $w \in C_0^1[0,1]$, $w' \geqslant 0$. If there is a point $t_0 \in [0,1]$ such that $w'(t_0) = 0$ then, letting $w_n = \frac{w}{n}$ and $\lambda_n = n$ for $n \in \mathbb{N}$, we obtain $w = \lim\limits_{n\to\infty} \lambda_n w_n$, $w \in T(S_1, 0)$, because $w_n \in S_1$.

Now, consider the case when $w'(t) > 0$ for each $t \in [0,1]$. Then by Lemma 4 there exists a point $t_0 \in [0,1]$ and a sequence $\{w_n\}_{n\in\mathbb{N}}$ such that $w_n \in C_0^1[0,1]$, $w_n \to w$, $w_n'(t_0) = 0$.

It is clear that there is $n_o \in \mathbb{N}$ such that $\frac{w_n'}{n} \leqslant 2$ for $n > n_o$. Put $v_n = \frac{w_n}{n}$ for $n > n_o$, $v_n = 0$ for $n \leqslant n_o$ and $\lambda_n = n$ for $n \in \mathbb{N}$. This yields $v_n \in S_1$ since $v_n'(t_0) = 0$ for each $n \in \mathbb{N}$. It follows that $\lim\limits_{n\to\infty} \lambda_n v_n = w$ and $w \in T(S_1, 0)$. Since w was arbitrary, we get

$$\{w \in C_0^1[0,1] \colon w' \geqslant 0\} \subset T(S_1, 0). \tag{14}$$

Now (11) follows from (13) and (14).

Now, let us prove (12). Let $w \in T_1(S_1, 0)$. By Definition 1, there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$, $y_n \in S_1$, $\|y_n\|_1 \to 0$, and there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$, $\lambda_n > 0$, such that

$$\|\lambda_n y_n - w\|_1 \to 0.$$

Since $y_n \in S_1$ we have $0 \leqslant y_n' \leqslant 2$, and there is $t_n \in [0,1]$ such that either $y_n'(t_n) = 0$ or $y_n'(t_n) = 2$ for each $n \in \mathbb{N}$. Since $\|\lambda_n y_n - w\|_1 \to 0$, we have

$$\|\lambda_n y_n' - w'\| \to 0. \tag{15}$$

Then the function sequence $\{y_n'\}_{n\in\mathbb{N}}$, $y_n' = \dfrac{1}{\lambda_n}\lambda_n y_n'$, is uniformly convergent to 0, since $\frac{1}{\lambda_n} \to 0$ and the sequence $\{\lambda_n y_n'\}_{n\in\mathbb{N}}$ is bounded. It follows that there is a number $N \in \mathbb{N}$ such that

$$0 \leqslant y_n'(t) < 2 \qquad \text{for each } n > N, \text{ for each } t \in [0,1]. \tag{16}$$

Therefore we have $y_n'(t_n) = 0$ for all $n > N$. Moreover, $w' \geqslant 0$ since $\lambda_n > 0$ and $y_n' \geqslant 0$ for each $n \in \mathbb{N}$.

The sequence $\{t_n\}_{n\in\mathbb{N}}$, $t_n \in [0,1]$, is bounded, hence it contains a subsequence $\{t_{n_k}\}_{k\in\mathbb{N}}$ convergent to a certain $t_0 \in [0,1]$. Then from (15) we get

$$\forall\, k \in \mathbb{N} \colon \left|\lambda_{n_k} y_{n_k}'(t_{n_k}) - w'(t_{n_k})\right| \leqslant \left\|\lambda_{n_k} y_{n_k}' - w'\right\|. \tag{17}$$

Since $y_{n_k}'(t_{n_k}) = 0$, from (17) it follows that

$$\forall\, k \in \mathbb{N} \colon \left|w'(t_{n_k})\right| \leqslant \left\|\lambda_{n_k} y_{n_k}' - w'\right\|. \tag{18}$$

Let k tends to ∞ in (18). Then we see that $\lim\limits_{k\to\infty} w'(t_{n_k}) = 0$. Since $w \in C_0^1[0,1]$, we have $w'(t_0) = 0$. Being w arbitrary, we get the inclusion

$$T_1(S_1, 0) \subset \{w \in C_0^1[0,1] \colon w' \geqslant 0,\ \exists\, t_0 \in [0,1] \colon w'(t_0) = 0\}. \tag{19}$$

Now, take an arbitrary function $w \in C_0^1[0,1]$ such that $w' \geqslant 0$ and there is $t_0 \in [0,1]$ at which $w'(t_0) = 0$. Put $y_n = \frac{w}{n}$ for $n \in \mathbb{N}$. Then obviously $y_n'(t_0) = 0$. Since $w \in C^1$, there is $N \in \mathbb{N}$ such that $y_n \in S_1$ for $n > N$.

Letting $y_n = \frac{w}{n}$ for $n > N$, $y_n = 0$ for $n \leqslant N$ and $\lambda_n = n$ for all $n \in \mathbb{N}$, we conclude that $w \in T_1(S_1, 0)$. Thus we have proved that

$$\{w \in C_0^1[0,1] \colon w' \geqslant 0,\ \exists\, t_0 \in [0,1] \colon w'(t_0) = 0\} \subset T_1(S_1, 0). \tag{20}$$

Then (19) and (20) imply (12).

It follows from (9) that $T_1(S_1, 0) \subset T(S_1, 0)$, however, $T(S_1, 0) \setminus T_1(S_1, 0) \neq \emptyset$. Indeed, for instance for $\mathrm{id} \colon [0,1] \to [0,1]$, we have $\mathrm{id} \in T(S_1, 0)$ but obviously $\mathrm{id} \notin T_1(S_1, 0)$.

Example 4. Now, consider the sphere S in $C_0^1[0,1]$ defined by

$$S = \{x \in C_0^1[0,1] \colon \|x - \mathrm{id}\| = 1\}.$$

We will show that the tangent cones to S at $x = 0$ with respect to norms $\|\cdot\|$, $\|\cdot\|_1$ are the same. Let $T(S,0)$ denote the tangent cone to S at $x = 0$ with respect to the norm $\|\cdot\|$, while $T_1(S,0)$ is the tangent cone to S at $x = 0$ with respect to $\|\cdot\|_1$. We are going to prove that

$$T(S,0) = T_1(S,0) = \{x \in C_0^1[0,1] \colon x(1) = 0\}. \tag{21}$$

First, observe that $\|\cdot\| \leqslant \|\cdot\|_1$ implies $T_1(S,0) \subset T(S,0)$.

Thus to prove (21) we only have to show that

$$T(S,0) \subset \{x \in C_0^1[0,1] \colon x(1) = 0\} \subset T_1(S,0). \tag{22}$$

Consider $T(S,0)$. Let $v \in T(S,0)$, $v \neq 0$. There is a sequence $\{x_n\}_{n\in\mathbb{N}}$, $x_n \in S$, $\|x_n\| \to 0$, and there is a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$, $\lambda_n > 0$, such that $\|\lambda_n x_n - v\| \to 0$. As $v \neq 0$, we have $\frac{x_n}{\|x_n\|} \to \frac{v}{\|v\|}$. Since $x_n \in S$ we get

$$\forall\, t \in [0,1] \colon t - 1 \leqslant x_n(t) \leqslant t + 1 \quad \text{and} \quad \exists\, t_n \in [0,1] \colon \left|x_n(t_n) - t_n\right| = 1.$$

Therefore for each $n \in \mathbb{N}$ the two cases are possible:

either (1) $x_n(t_n) - t_n = 1$

or (2) $x_n(t_n) - t_n = -1$.

Let us show that case (1) is impossible. Indeed, otherwise we would have $x_n(t_n) = 1 + t_n$ and $\frac{x_n(t_n)}{\|x_n\|} = \frac{1+t_n}{\|x_n\|} \to \infty$, since $\|x_n\| \to 0$. A contradiction, since the sequence $\left\{\frac{x_n(t_n)}{\|x_n\|}\right\}_{n\in\mathbb{N}}$ is bounded. Thus we only have case (2). We then get $x_n(t_n) = t_n - 1 \leqslant 0$. It follows that $t_n \to 1$, as $\|x_n\| \to 0$. Since $x_n(t_n) \leqslant 0$ we have

$$\frac{x_n(t_n)}{\|x_n\|} = \frac{t_n - 1}{\|x_n\|} \ \to\ \frac{v(1)}{\|v\|} \leqslant 0 \qquad \text{as } n \to \infty.$$

Therefore

$$v(1) \leqslant 0. \tag{23}$$

On the other hand, since $x_n(t) \geqslant t - 1$ for all $t \in [0,1]$, we have $x_n(1) \geqslant 0$ for each $n \in \mathbb{N}$. Hence $\frac{x_n(1)}{\|x_n\|} \to \frac{v(1)}{\|v\|} \geqslant 0$. It follows, taking (23) into account, that $v(1) = 0$.

We have thus shown that

$$T(S,0) \subset \{x \in C_0^1[0,1] \colon x(1) = 0\}. \tag{24}$$

Now, take $x \in C_0^1[0,1]$, $x \neq 0$, $x(1) = 0$. Since the derivative x' is bounded on $[0,1]$, there exists a number $N \in \mathbb{N}$ such that for all $t \in [0,1]$ and for each $n \geqslant N$ we have $\dfrac{x'(t)}{n} < 1$. Then

$$\forall\, n \geqslant N \;\; \forall\, t \in [0,1]\colon \frac{x(t)}{n} - (t-1) = \int_1^t \left(\frac{x'(\tau)}{n} - 1 \right) d\tau \geqslant 0 . \tag{25}$$

Being the function x bounded, there is $N_1 \in \mathbb{N}$ such that

$$\forall\, n \geqslant N_1 \;\forall\, t \in [0,1]\colon \frac{x(t)}{n} \leqslant t + 1 . \tag{26}$$

Then (25) and (26) imply that for each $n \geqslant N_o$, $N_o = \max\{N, N_1\}$, we have

$$\forall\, t \in [0,1]\colon t - 1 \leqslant \frac{x(t)}{n} \leqslant t + 1 .$$

This means that

$$\forall\, n \geqslant N_o\colon \left\| \frac{x}{n} - \mathrm{id} \right\| \leqslant 1 .$$

Since $x(1) = 0$, the latter relation implies that $\frac{x}{n} \in S$ for each $n \geqslant N_o$. Letting $x_n = \frac{x}{n}$ for $n \geqslant N_o$, $x_n = 0$ for $n < N_o$ and $\lambda_n = n$ for $n \in \mathbb{N}$ we get

$$\|\lambda_n x_n - x\|_1 \to 0 \qquad \text{as } n \to \infty .$$

Hence $x \in T(S,0)$.

Thus we have shown that $T_1(S,0) \subset T(S,0)$, when, combined with (22), yields (21), what was to be shown.

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