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ON A PROBLEM OF TH. DAVISON

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ABSTRACT. A general solution $h\colon [0,1) \to \mathbb{R}$ of the functional equation of the form

$$h(x) = \frac{1}{a}\left[h\left(\frac{x}{a}\right) + h\left(\frac{x+1}{a}\right)\right], \qquad x \in \big[0, \min\{a-1, 1\}\big)\,,$$

where $a > 1$ is a given constant, depends on an arbitrary function. Some theorems on continuous solutions for this equation are also given.

Answering a problem of Th. Davison: [*Problem 3*, Aequationes Math. **64**, (2002), 186] we determine all functions $h\colon [0,1) \to \mathbb{R}$ satisfying the following functional equation

$$h(x) = \frac{1}{a}\left[h\left(\frac{x}{a}\right) + h\left(\frac{x+1}{a}\right)\right], \qquad x \in \big[0, \min\{1, a-1\}\big)\,,$$

where $a \in (1,\infty) \setminus \{2\}$ is a given constant. In the case $a = 2$ this equation was considered by many authors (T. Hilberding [*A functional equation for the cotangent on the open unit interval*, Aequationes Math. **61**, (2001), 179–189], Z. Kominek, J. Matkowski [*On the functional equation* $\varphi(x) = \alpha\varphi(\alpha x) + (1-\alpha)\varphi\big(1-(1-\alpha)x\big)$, Bull. Math. Soc. Sci. Math. Républ. Soc. Roum. **30(78)**, (1986), 327–334], W. Walter [*Old and new approaches to Euler's trigonometric expansions*, Amer. Math. Monthly, **89**, 1982, 225–230], for example). Generally, our equation has a large class of solutions, however, in the case $a > 2$ the zero function is a unique continuous solution for our equation.

General solutions

THEOREM 1. *Let $a \geqslant 2$ be a given constant. The functional equation of the form*

$$h(x) = \frac{1}{a}\left[h\left(\frac{x}{a}\right) + h\left(\frac{x+1}{a}\right)\right], \qquad x \in [0,1)\,, \tag{1}$$

has solutions depending on arbitrary function. More precisely, for arbitrary function $h_0\colon \left[\frac{1}{a}, 1\right) \to \mathbb{R}$ there exists a unique function $H\colon [0,1) \to \mathbb{R}$ satisfying equation (1) such that $H(x) = h_0(x)$, for every $x \in \left[\frac{1}{a}, 1\right)$.

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P r o o f . Take an arbitrary function $h_0\colon \big[\frac{1}{a}, 1\big) \to \mathbb{R}$. Let us put $I_n := \big[\frac{1}{a^{n+1}}, \frac{1}{a^n}\big)$, $n \in \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ denotes the set of all positive integers. For each $n \in \mathbb{N}$ we define

$$h_n(x) := ah_{n-1}(ax) - h_0\left(x + \frac{1}{a}\right), \qquad x \in I_n .$$

We will show that the function H given by the formula

$$H(x) := \begin{cases} \frac{1}{a-1} h_0\big(\frac{1}{a}\big) & \text{if } x = 0\,, \\ h_n(x) & \text{if } x \in I_n\,,\ n \in \mathbb{N} \cup \{0\} \end{cases}$$

fulfils our requirements. If $x = 0$ then $\frac{x}{a} = 0$ and $x + \frac{1}{a} \in I_0$, whence

$$\begin{aligned} H(0) = \frac{1}{a-1} h_0(0) &= \frac{1}{a}\left[\frac{1}{a-1} h_0\left(\frac{1}{a}\right) + h_0\left(\frac{1}{a}\right)\right] \\ &= \frac{1}{a}\left[H\left(\frac{0}{a}\right) + H\left(\frac{1}{a}\right)\right]. \end{aligned}$$

If $x \in I_0$ then $\frac{x}{a} \in I_1$ and $\frac{x+1}{a} \in I_0$. Thus

$$\begin{aligned} H(x) = h_0(x) &= \frac{1}{a}\left[ah_0(x) - h_0\left(\frac{x+1}{a}\right) + h_0\left(\frac{x+1}{a}\right)\right] \\ &= \frac{1}{a}\left[h_1\left(\frac{x}{a}\right) + h_0\left(\frac{x+1}{a}\right)\right] = \frac{1}{a}\left[H\left(\frac{x}{a}\right) + H\left(\frac{x+1}{a}\right)\right]. \end{aligned}$$

Assume that H satisfies (1) for each $x \in \bigcup\limits_{k=0}^{n-1} I_k$, $n \geqslant 1$. Take $z \in I_n$. Then $\frac{z}{a} \in I_{n+1}$ and $\frac{z+1}{a} \in I_0$. Therefore

$$\begin{aligned} H(z) - \frac{1}{a}H\left(\frac{z}{a}\right) - \frac{1}{a}H\left(\frac{z+1}{a}\right) &= h_n(z) - \frac{1}{a}h_{n+1}\left(\frac{z}{a}\right) - \frac{1}{a}h_0\left(\frac{z+1}{a}\right) \\ &= h_n(z) - \frac{1}{a}\left[ah_n(z) - h_0\left(\frac{z+1}{a}\right)\right] - \frac{1}{a}h_0\left(\frac{z+1}{a}\right) = 0\,. \end{aligned}$$

Since $\bigcup\limits_{k=0}^{\infty} I_k = (0,1)$, the proof of Theorem 1 is complete. $\square$

THEOREM 2. *Let $a \in (1,2)$ be a fixed real number. Every function $H\colon [0,1) \to \mathbb{R}$ satisfying the functional equation*

$$h(x) = \frac{1}{a}\left[h\left(\frac{x}{a}\right) + h\left(\frac{x+1}{a}\right)\right], \qquad x \in [0, a-1)\,, \tag{2}$$

is uniquely determined by a function $h_0\colon \big[\frac{a-1}{a}, 1\big) \to \mathbb{R}$. More precisely, for every function $h_0\colon \big[\frac{a-1}{a}, 1\big) \to \mathbb{R}$ there exists a unique function $H\colon [0,1) \to \mathbb{R}$ satisfying equation (2), such that $H(x) = h_0(x)$, for every $x \in \big[\frac{a-1}{a}, 1\big)$.

P r o o f . Let us put $I_0 := \big[\frac{a-1}{a}, 1\big)$ and $I_n := \big[\frac{a-1}{a^{n+1}}, \frac{a-1}{a^n}\big)$, $n \in \mathbb{N}$. Now, we can repeat the argumentation used in the proof of Theorem 1. $\square$

Solutions which are continuous in the interval $(0,1)$

Remark 1. Let $a > 2$ be a fixed real number. If $H\colon [0,1) \to \mathbb{R}$ is a solution for equation (1) continuous in the interval $(0,1)$ then there exists a left-hand side limit

$$\lim_{x\to 1-0} H(x)\,.$$

P r o o f. Observe that $\frac{2}{a} < 1$. It follows from the continuity of H at the points $\frac{1}{a}$ and $\frac{2}{a}$ and by (1) that

$$\lim_{x\to 1-0} H(x) = \frac{1}{a}\left[H\left(\frac{1}{a}\right) + H\left(\frac{2}{a}\right)\right].$$

□

Theorem 3. *Let $a > 2$ be a given constant and let $h_0\colon \left[\frac{1}{a}, 1\right) \to \mathbb{R}$ be an arbitrary continuous function satisfying the following conditions:*

$$\textit{there exists a finite limit} \quad c := \lim_{x\to 1-} h_0(x) \tag{3}$$

and

$$ac = h_0\left(\frac{1}{a}\right) + h_0\left(\frac{2}{a}\right). \tag{4}$$

Then there exists a unique solution $H\colon [0,1) \to \mathbb{R}$ of equation (1) continuous in the interval $(0,1)$ such that $H(x) = h_0(x)$ for every $x \in \left[\frac{1}{a}, 1\right)$.

P r o o f. Let H, h_n and I_n, $n \in \mathbb{N}$, have the same meaning as in the proof of Theorem 1. It is enough to prove that H is continuous in $(0,1)$. If $x \in I_1$ then $H(x) = h_1(x) = ah_0(ax) - h_0\left(x + \frac{1}{a}\right)$. Hence and by (3) we obtain

$$\lim_{x\to \frac{1}{a}-} H(x) = ac - h_0\left(\frac{2}{a}\right) = H\left(\frac{1}{a}\right).$$

Therefore H is continuous at the point $\frac{1}{a}$. Assume the continuity of H in the interval $\left[\frac{1}{a^n}, \frac{1}{a^{n-1}}\right)$, for some $n \geqslant 1$. Then

$$\begin{aligned}\lim_{x\to \frac{1}{a^n}-} H(x) &= \lim_{x\to \frac{1}{a^n}-}\left[aH(ax) - H\left(x + \frac{1}{a}\right)\right]\\ &= aH\left(\frac{1}{a^{n-1}}\right) - H\left(\frac{1}{a^n} + \frac{1}{a}\right) = H\left(\frac{1}{a^n}\right),\end{aligned}$$

which proves the continuity of H at the point $\frac{1}{a^n}$. On account of induction principle we infer that H is continuous in the interval $(0,1)$. This finishes the proof. □

Immediately from the Theorem 3 and Remark 1 it follows

Remark 2. Let $a > 2$ be a fixed real number. Every solution $H\colon [0,1) \to \mathbb{R}$ of equation (1) continuous in the interval $(0,1)$ is determined by a continuous function $h_0\colon \left[\frac{1}{a}, 1\right) \to \mathbb{R}$ fulfilling conditions (3) and (4).

In a similar way we can prove the following

Remark 3. Let $a \in (1,2)$ be a fixed real number. If $H\colon [0,1) \to \mathbb{R}$ is a solution for the equation (2) continuous in the interval $(0,1)$ then there exists a left-hand side limit

$$c := \lim_{x \to 1-0} H(x)\,, \tag{5}$$

and, moreover,

$$c = aH(a-1) - H\left(\frac{a-1}{a}\right). \tag{6}$$

Theorem 4. *Let $a \in (1,2)$ be a fixed real number. Every solution $H\colon [0,1) \to \mathbb{R}$ to equation (2) continuous in the interval $(0,1)$ is uniquely determined by a continuous function $h_0\colon \left[\frac{a-1}{a}, 1\right) \to \mathbb{R}$ satisfying the conditions (5) and (6) (with h_0 instead of H).*

Continuous solutions

Lemma. *Let $a > 1$ be a fixed number. If $H\colon [0,1) \to \mathbb{R}$ is locally bounded in a neighbourhood of zero and it fulfils equation (1) (or (2), if $a \in (0,1)$) then*

$$H(x) = \sum_{k=1}^{\infty} \frac{1}{a^k}\, H\left(\frac{1}{a} + \frac{x}{a^k}\right), \qquad x \in \left[0, \min\{a-1, 1\}\right). \tag{7}$$

P r o o f. Let us fix $x \in \left[0, \min\{a-1, 1\}\right)$. It follows from (1) (or (2)) that the equality

$$H(x) = \frac{1}{a^n} H\left(\frac{x}{a^n}\right) + \sum_{k=1}^{n} \frac{1}{a^k} H\left(\frac{1}{a} + \frac{x}{a^k}\right) \tag{8}$$

holds for $n = 1$. Assume (8) for a positive integer n. Setting in (1) (or (2)) $\frac{x}{a^n}$ instead of x we obtain

$$H\left(\frac{x}{a^n}\right) = \frac{1}{a} H\left(\frac{x}{a^{n+1}}\right) + \frac{1}{a} H\left(\frac{1}{a} + \frac{x}{a^{n+1}}\right)$$

which on account of induction hypotheses (8) implies

$$\begin{aligned} H(x) &= \frac{1}{a^{n+1}} H\left(\frac{x}{a^{n+1}}\right) + \frac{1}{a^{n+1}} H\left(\frac{1}{a} + \frac{x}{a^{n+1}}\right) + \sum_{k=1}^{n} \frac{1}{a^k} H\left(\frac{1}{a} + \frac{x}{a^k}\right) \\ &= \frac{1}{a^{n+1}} H\left(\frac{x}{a^{n+1}}\right) + \sum_{k=1}^{n+1} \frac{1}{a^k} H\left(\frac{1}{a} + \frac{x}{a^k}\right). \end{aligned}$$

This finishes the proof of (8) from which our assertion easily follows. □

Theorem 5. *Let $a \geqslant 2$ be a fixed number. If $H\colon [0,1) \to \mathbb{R}$ is a continuous solution to equation (1) then H is constant. If $a > 2$ then $H \equiv 0$.*

Proof. Assume H is a continuous solution to equation (1). According to our Lemma we have

$$H(x) = \sum_{k=1}^{\infty} \frac{1}{a^k} H\left(\frac{1}{a} + \frac{x}{a^k}\right), \qquad x \in [0,1).$$

Let us fix a $b \in (0,1)$. Then $\frac{1+b}{a} < 1$ and H is uniformly continuous in $\left[0, \frac{1+b}{a}\right]$. Take an arbitrary $\varepsilon > 0$ and choose a $\delta > 0$ such that

$$x, y \in \left[0, \frac{1+b}{a}\right] \text{ and } |x-y| < \delta \quad \Longrightarrow \quad |H(x) - H(y)| < \varepsilon.$$

For all $x, y \in [0,b]$ such that $|x-y| < a\delta$ we have

$$\bigl|H(x) - H(y)\bigr| = \left| \sum_{k=1}^{\infty} \left(H\left(\frac{1}{a} + \frac{x}{a^k}\right) - H\left(\frac{1}{a} + \frac{y}{a^k}\right)\right)\right| < \sum_{k=1}^{\infty} \frac{1}{a^k} \varepsilon = \frac{1}{a-1} \varepsilon \leqslant \varepsilon.$$

Repeating this procedure r-times, where $r \in \mathbb{N}$ is chosen such that $(r-1)\delta \leqslant b < r\delta$ we infer that

$$\bigl|H(x) - H(y)\bigr| \leqslant \varepsilon,$$

for all $x, y \in [0,b]$, which implies that H is a constant function in $[0,b]$. Due to arbitrariness $b \in [0,1)$ we get $H = \text{const}$ in the interval $[0,1)$. Clearly, $H \equiv 0$ if $a > 2$. □

Remark 4. Let $a > 1$ be a fixed real number. Every continuous function $H\colon [0,1) \to \mathbb{R}$ fulfilling equation (2) satisfies the condition

$$\lim_{x \to \frac{1}{a} - 0} H(x) = (a-1) H(0). \tag{9}$$

THEOREM 6. *Let $a \in (1,2)$ be a fixed real number. Every continuous solution $H\colon [0,1) \to \mathbb{R}$ to equation (2) is uniquely determined by a continuous function $h_0\colon \left[0,\frac{1}{a}\right) \to \mathbb{R}$ satisfying condition (9) (with h_0 instead of H).*

Proof. Let $h_0\colon \left[0,\frac{1}{a}\right) \to \mathbb{R}$ be a continuous function satisfying condition (9) (with h_0 instead of H). Let $n_0 \in \mathbb{N}$ be chosen such that

$$\sum_{k=1}^{n_0} \frac{1}{a^k} < 1 \leqslant \sum_{k=1}^{n_0+1} \frac{1}{a^k}\,.$$

Put $I_0 := \left[0,\frac{1}{a}\right)$, $I_n := \left[\sum\limits_{k=1}^{n} \frac{1}{a^k}, \sum\limits_{k=1}^{n+1} \frac{1}{a^k}\right)$, $n = 1,\dots,n_0-1$, $I_{n_0} := \left[\sum\limits_{k=1}^{n_0} \frac{1}{a^k}, 1\right)$ and define the functions h_n and H in the following manner:

$$h_n(u) := ah_{n-1}(au-1) - h_0\left(u - \frac{1}{a}\right), \qquad u \in I_n\,,\ n = 1,\dots,n_0\,,$$

and

$$H(u) := h_n(u)\,, \qquad u \in I_n\,,\ n = 0,1,\dots,n_0\,.$$

It is easy to check that H fulfils equation (2). We will prove that H is continuous in $[0,1)$. It suffices to prove the continuity of H at each point of the form $\sum\limits_{k=1}^{n} \frac{1}{a^k}$, $n = 1,\dots,n_0$. If $u \in I_1$, $u \to \frac{1}{a}+0$ then $au-1$, $u-\frac{1}{a} \in I_0$, whence by (9) we obtain

$$\lim_{u\to\frac{1}{a}+0} H(u) = \lim_{u\to\frac{1}{a}+0}\left[ah_0(au-1) - h_0\left(u-\frac{1}{a}\right)\right] = (a-1)h_0(0)\,.$$

Moreover, $H(\frac{1}{a}) = (a-1)h_0(0)$. Therefore H is continuous at the point $\frac{1}{a}$. If we assume the continuity of H at the point $\sum\limits_{k=1}^{m} \frac{1}{a^k}$, $m < n_0$, then in a similar way one can prove the continuity of H at the point $\sum\limits_{k=1}^{m+1} \frac{1}{a^k}$. □

Theorem 6 and Lemma imply the following corollary.

COROLLARY. *Let $a \in (1,2)$ be a fixed number. Functional equation (7) has continuous solutions depending on arbitrary functions.*

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