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ON THE PROBLEM OF RECOVERING THE COEFFICIENTS OF SERIES WITH RESPECT TO CHARACTERS OF ZERO-DIMENSIONAL GROUPS

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ABSTRACT. It is proved that if a series with respect to the characters of abelian compact zero-dimensional group is convergent everywhere except, possibly, a countable number of points, then the coefficients of this series can be recovered from its sum by generalized Fourier formulas in which a Henstock-Kurzweil type integral is used.

1. Introduction

In this paper some results, related to the problem of recovering coefficients of orthogonal series from its sum which were established earlier (see [17]) in the case of the Vilenkin-Price system, are extended to a more general case of series with respect to the characters of abelian compact zero-dimensional groups. It is shown that the coefficients of such a series can be recovered by generalized Fourier formulas in which so called $\mathcal{P}$-adic-path integral is used.

The problem of recovering the coefficients of orthogonal series from their sums is a generalization of the uniqueness problem for the coefficients of orthogonal series. The uniqueness can be related to pointwise convergent series or series which are summable in a certain sense, and the convergence or summability can be supposed everywhere or outside some exceptional set. (For references to literature on the rich theory of uniqueness of Walsh, Haar and Vilenkin series, including subtle theory of sets of uniqueness, see [1], [6], [12], [19], [20]. The classical trigonometric case is treated for example in [21].)

If the uniqueness theorem is proved for a certain system and so the coefficients of an orthogonal series with respect to this system are uniquely determined by

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its sum, then it is natural to expect that they may be recovered from the sum by Fourier formulas, as it takes place in the simplest cases, for example in the case of the uniform convergence. Indeed, for many known systems (trigonometric, Haar, Walsh, Vilenkin systems) it is true that every series with respect to those systems convergent everywhere to a summable function is the Fourier series of this function. But the point is that the sum of everywhere convergent orthogonal series can fail to be Lebesgue integrable. For example, it is known (see [21]) that the series

$$\sum_{k=2}^{\infty} \frac{\sin kx}{\ln k}$$

converges everywhere but fails to be Fourier-Lebesgue series. This kind of examples can be given for the other above mentioned systems as well. To integrate such a series, one needs nonabsolutely convergent integrals. In the cases where the sum is integrable in one or another known general sense, a question is whether the coefficients can be determined by Fourier formulas in which the integral is understood in the same particular sense. A complete solution to the problem of recovering the coefficients of a convergent or summable series with respect to some system is found if a general process of integration is developed so that any such a series is the Fourier series of its sum, in the sense of this generalized integral.

In the classical trigonometric case the first solution to the problem of defining an integral so powerful that the sum of any everywhere convergent series is integrable and the coefficients can be computed by the generalized Fourier formulas, is due to Denjoy. In [4] he introduced a very complicated definition of a second-order integral called a totalization T_{2s}, which recaptures a function from its second Riemann symmetric derivative. The difficulty of a T_{2s}-totalization process which involves a transfinite sequence of operations, led other authors to look for an easier solution to the coefficients problem. J. Marcinkiewicz and A. Zygmund, J. C. Burkill, R. D. James and some other authors produced a Perron type integrals reducing the problem to the one of recovering a function from its second order symmetric derivative (see [18] for details). The latest step in this direction was done by D. Preiss and B. Thomson in [11] who produced a first-order Henstock-Kurzweil type integral which integrates the approximate symmetric derivatives.

The first integral solving the coefficient problem for Haar (and Walsh) series was introduced in [13] in a descriptive form. A constructive definition of Denjoy type, based on transfinite induction, was given in [14] and later, independently, in [8]. The coefficient problem for Vilenkin series was examined in [15].

It is an advantage of application of the Henstock-Kurzweil theory (see [7], [9], [10]) that it provides a unifying approach to the above discussed coefficient problem for many orthogonal systems. The choice of the Henstock-Kurzweil type integral with respect to the particular basis is determined by the system and type of the convergence under consideration. The idea is to reduce the problem of coefficients to the one of recovering a function from its derivative with respect to an appropriate differentiation basis. While the symmetric basis is a natural choice for the classical trigonometric case, a different kind of so called $\mathcal{P}$-adic bases and respective derivatives and integrals appear in a natural way in the theory of the Vilenkin system.

The $\mathcal{P}$-path integral we are using here is also related to a kind of $\mathcal{P}$-adic basis, namely to the basis defined by a system of $\mathcal{P}$-adic paths. An advantage of this integral, in comparison with the one defined by the usual $\mathcal{P}$-adic basis (see [3]), is based on a fact, that it satisfies the Lusin condition (N) and due to this it is compatible with other classical non-absolute integrals of Denjoy type (see [2]).

In Section 2 of this paper we describe a group of characters of an arbitrary zero-dimensional compact abelian group G and introduce the Kurzweil-Henstock integral with respect to the system of $\mathcal{P}$-paths. In Section 3 we apply this integral to solve the coefficient problem for series with respect to characters of the group G.

2. Definitions and preliminary results

Consider a zero-dimensional compact abelian group G which satisfies the second countability axiom. It is known (see [1]) that a topology in such a group can be given by a chain of subgroups

$$G = U_0 \supset U_1 \supset \cdots \supset U_n \supset \cdots ,$$

which are therefore clopen sets with respect to this topology. As G is compact, the number of cosets generated by each U_n is finite and so the quotient group U_n/U_{n+1} is finite for each n. Let p_n be its order. We can suppose that all p_n are prime numbers. So the group G defines a sequence

$$\mathcal{P} = \{p_n\}_{n=0}^{\infty} , \qquad p_n \geqslant 2 . \tag{1}$$

We also define a sequence $m_0 = 1$ $m_s = p_0 \cdots p_{s-1}$ where p_s are elements of the sequence (1). Picking up one element $g_n \in U_n \setminus U_{n+1}$ for each n we can represent any element $g \in G$ in the form (see [1]):

$$g = \sum_{s=0}^{\infty} a_s g_s , \qquad a_s = 0, 1, \ldots , p_s - 1 . \tag{2}$$

Let X denote a group of characters of our group G. We can represent the group X as a sum of increasing sequence of subgroups (see [1]):

$$X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots ,$$

where X_n is a subgroup of the order m_n and forms the annulator of the subgroup U_n, i.e.,

$$X_n = \{\tilde{\chi} \in X\colon \tilde{\chi}(g) = 1 \text{ for all } g \in U_n\}.$$

Now, we introduce a numeration of the characters. We denote $\tilde{\chi}_0(g) = 1$. Picking up one character $\tilde{\chi} \in X_{n+1} \backslash X_n$ for each $n = 0, 1, 2, \dots$ we denote it by $\tilde{\chi}_{m_n}$. Representing each non-negative integer in the form

$$n = \sum_{k=0}^{s} a_k m_k, \qquad 0 \leqslant a_k \leqslant p_k - 1,$$

we define

$$\tilde{\chi}_n(g) = \prod_{k=0}^{s} \left(\tilde{\chi}_{m_k}(g)\right)^{a_k}. \tag{3}$$

It is shown in [1] that in that way we are getting the numeration of the whole group X.

Now, using representation (2) we can map the group to the unit interval by

$$\lambda(g) = x = \sum_{s=0}^{\infty} \frac{a_s}{m_{s+1}}. \tag{4}$$

This mapping is one-one correspondence between the group G and the interval $[0,1]$, up to a countable set. Indeed, denoting by $Q_{\mathcal{P}}$ the set of all $\mathcal{P}$-adic rational points on $[0,1]$, i.e., points of the form $\frac{r}{m_k}$, $0 \leqslant r \leqslant m_k - 1$, we note that each $x \in Q_{\mathcal{P}}$ has two $\mathcal{P}$-adic expansions in the form of a sum in (4), a finite one and an infinite one. If we exclude for such $x < 1$ one type of expansion, for example the infinite one, then the correspondence (4) is one-one.

Using the converse $g = \lambda^{-1}(x)$ of mapping (4) we can translate the functions (3) from the group G to the unit interval by putting for each $x \in [0,1]$

$$\chi_n(x) = \tilde{\chi}_n\left(\lambda^{-1}(x)\right). \tag{5}$$

These functions constitute an orthonormal system on $[0,1]$ (see [1]). All the results below are formulated for the system (5), although analogous results hold for the system (3) on the group G.

Below we shall use the fact that under mapping (4) the Haar measure on G corresponds to the Lebesgue measure on $[0,1]$. We also note that the subgroups U_k are mapped by (4) onto intervals $\left[0, \frac{1}{m_k}\right)$ and the respective cosets are mapped onto intervals

$$\triangle_r^{(k)} = \left[\frac{r}{m_k}, \frac{r+1}{m_k}\right), \qquad 0 \leqslant r \leqslant m_k - 1, \tag{6}$$

which we call *$\mathcal{P}$-adic intervals of rank k*.

This observation together with the Definition (3) of characters implies that the functions χ_n, for $0 \leqslant n \leqslant m_k - 1$, $k \geqslant 0$, are constant on each interval $\triangle_r^{(k)}$. The functions χ_n have also the following property.

Lemma 2.1. *If* $n \geqslant m_k$ *then* $\int_{\triangle_r^{(k)}} \chi_n(x)\,dx = 0$ *for each* r, $0 \leqslant r \leqslant m_k - 1$.

P r o o f. Interval (6) under mapping (4) is an image of a certain coset $U_{k,r}$ with respect to the subgroup U_k. According to definition (3) the function $\tilde{\chi}_n$, for $n \geqslant m_k$, does not belong to the annulator of the subgroup U_k, and hence there exists an element $g_0 \in U_k$ such that $\tilde{\chi}_n(g_0) \neq 1$. If $g \in U_{k,r}$ then $g + g_0 \in U_{k,r}$. Due to the invariance (with respect to translation) of the Haar measure μ we have the equality

$$\int\limits_{U_{k,r}} \tilde{\chi}_n(g)\,d\mu = \int\limits_{U_{k,r}} \tilde{\chi}_n(g+g_0)\,d\mu = \tilde{\chi}(g_0) \int\limits_{U_{k,r}} \tilde{\chi}_n(g)\,d\mu$$

which implies

$$\int\limits_{U_{k,r}} \tilde{\chi}_n(g)\,d\mu = 0\,.$$

Going back to the unit interval, by means of (4), we get the required equality. □

With each $\mathcal{P}$-adic irrational point $x \in [0,1]$, i.e., a point $x \in [0,1] \setminus Q_{\mathcal{P}}$, we associate a unique nested sequence

$$\{\triangle^{(k)}(x)\} = \big\{\big[a_k(x), b_k(x)\big)\big\}_{k=0}^{\infty} \tag{7}$$

of $\mathcal{P}$-adic intervals (6) converging to x so that

$$\{x\} = \bigcap_{k=0}^{\infty} \big[a_k(x), b_k(x)\big)\,.$$

If x is a $\mathcal{P}$-adic rational point then there exist two nested sequences (7)—the *left* one and the *right* one—for which x, starting with some k, is a common end-point.

Using the notation given by (7), we define for $\mathcal{P}$-adic irrational point x, the sequences $\mathcal{P}_x^- = \big\{a_k(x)\big\}$ and $\mathcal{P}_x^+ = \big\{b_k(x)\big\}$. If x is a $\mathcal{P}$-adic rational point, we use the same notation $\mathcal{P}_x^+$ and $\mathcal{P}_x^-$ for sequences with $a_k(x)$ being the left end-point of interval of rank k of the above mentioned left nested sequence associated with x, and $b_k(x)$ being the right end-point of interval of rank k of the right nested sequence.

Definition 2.1. The set $\mathcal{P}_x = \mathcal{P}_x^+ \cup \mathcal{P}_x^- \cup \{x\}$ is called the *$\mathcal{P}$-path leading to* x. The collection $\big\{\mathcal{P}_x \colon x \in [a,b]\big\}$ is called the *system of $\mathcal{P}$-paths* on an interval $[a,b]$.

Definition 2.2. Let $F\colon [0,1] \to \mathbb{R}$ and let $\mathcal{P} = \{\mathcal{P}_x\colon x \in [0,1]\}$ be the system of $\mathcal{P}$-paths. If

$$\lim_{\substack{y \to x \\ y \in \mathcal{P}_x}} F(y) = F(x)$$

then we say that F is *$\mathcal{P}$-path continuous* at x.

Definition 2.3. Let $F\colon [0,1] \to \mathbb{R}$ and let $\mathcal{P} = \{\mathcal{P}_x\colon x \in [0,1]\}$ be the system of $\mathcal{P}$-paths. If

$$\lim_{\substack{y \to x \\ y \in \mathcal{P}_x}} \frac{F(y) - F(x)}{y - x}$$

exists and is finite, then we say that F is *$\mathcal{P}$-path differentiable* at x and call this limit *$\mathcal{P}$-path derivative* of F at x denoting it by $F'_{\mathcal{P}}(x)$. We define also the *lower* and *upper $\mathcal{P}$-path derivative*, respectively, by the equality

$$\overline{F'}_{\mathcal{P}}(x) = \limsup_{\rho \to 0} \left\{ \frac{F(y) - F(x)}{y - x}\colon 0 < |y - x| < \rho\,,\ y \in \mathcal{P}_x \right\},$$
$$\underline{F'}_{\mathcal{P}}(x) = \liminf_{\rho \to 0} \left\{ \frac{F(y) - F(x)}{y - x}\colon 0 < |y - x| < \rho\,,\ y \in \mathcal{P}_x \right\}.$$

Definition 2.4. Let δ be a positive function defined on an interval $[a,b] \subset [0,1]$. A collection of interval-point pairs $\{([u_j, v_j], x_j)\}_{j=1}^{k}$ is called *δ-fine $\mathcal{P}$-partition of* $[a,b]$ if the intervals $[u_j, v_j]$ are non-overlapping, $[a,b] = \bigcup_{j=1}^{k} [u_j, v_j]$ and for each j we have: $u_j, v_j \in \mathcal{P}_x$, $u_j \leqslant x_j \leqslant v_j$ and $\max\{v_j - x_j, x_j - u_j\} < \delta(x_j)$.

It is not difficult to check that for any interval $[a,b]$, any system of $\mathcal{P}$-paths on $[a,b]$ and for any positive function δ defined on $[a,b]$ there exists a δ-fine $\mathcal{P}$-partition of $[a,b]$.

Definition 2.5. A function $f\colon [0,1] \to \mathbb{R}$ is said to be *Henstock–Kurzweil integrable with respect to the system of $\mathcal{P}$-paths* or, in brief, *$H_{\mathcal{P}}$-integrable* on $[0,1]$, with integral value A, if for every $\varepsilon > 0$, there exists a positive function δ on $[0,1]$ such that for any δ-fine $\mathcal{P}$-partition of $[0,1]$ we have

$$\left| \sum_{j=1}^{k} f(x_j)(v_j - u_j) - A \right| < \varepsilon\,.$$

We then write $(H_{\mathcal{P}}) \int_0^1 f = A$.

Remark 2.1. A definition similar to the above Definition 2.5 is given in [5] for paths satisfying the so called Intersection Condition (I.C.), i.e., for paths

$E = \{E_x \colon x \in \mathbb{R}\}$ with which it is possible to associate a positive function σ on $\mathbb{R}$ so that whenever $0 < y - x < \min(\sigma(x), \sigma(y))$ we have $E_x \cap E_y \cap [x, y] \neq \emptyset$. It is easy to see that the $\mathcal{P}$-path system does not satisfy this condition. So formally our definition is not a partial case of the definition in [5]. However, all the results in [5] depend in fact only on the existence of δ-partions for any interval. As this is true for our path system, we can state that $H_{\mathcal{P}}$-integral has all the properties which are established in [5] for the integral defined by a system of paths with I.C. In particular the indefinite $H_{\mathcal{P}}$-integral is defined and is $\mathcal{P}$-path continuous everywhere and $\mathcal{P}$-path differentiable almost everywhere on $[0,1]$.

Definition 2.5 can be extended to the case of functions defined only almost everywhere on $[0,1]$ in the following way.

Definition 2.6. A real-valued function f defined a.e. on $[0,1]$ is said to be *$H_{\mathcal{P}}$-integrable* to A if the function

$$f_1(x) = \begin{cases} f(x)\,, & \text{where } f \text{ is defined}\,, \\ 0\,, & \text{otherwise} \end{cases}$$

is $H_{\mathcal{P}}$-integrable in the sense of Definition 2.5. The value of $H_{\mathcal{P}}$-integral of f is defined then as the one of f_1.

Remark 2.2. We say that a complex-valued function f is $H_{\mathcal{P}}$-integrable (in the sense of Definition 2.5 or 2.6) if both $\operatorname{Re} f$ and $\operatorname{Im} f$ are $H_{\mathcal{P}}$-integrable with integrals A and B, respectively, and the integral of f is $A + iB$.

The following theorem was proved in [17].

Theorem 2.1. *Let a function $F \colon [0,1] \to \mathbb{R}$ be $\mathcal{P}$-path continuous on $Q_{\mathcal{P}}$, $\mathcal{P}$-path differentiable a.e. on $[0,1] \setminus Q_{\mathcal{P}}$, and let the inequality $-\infty < \underline{F}'_{\mathcal{P}}(x) \leqslant \overline{F}'_{\mathcal{P}}(x) < +\infty$ hold everywhere on $[0,1] \setminus Q_{\mathcal{P}}$. Then the function*

$$f(x) = \begin{cases} F'_{\mathcal{P}}(x) & \text{if } F'_{\mathcal{P}} \text{ exists at } x \in [0,1] \setminus Q_{\mathcal{P}}\,, \\ 0 & \text{otherwise} \end{cases}$$

is $H_{\mathcal{P}}$-integrable on $[0,1]$ and F is its indefinite $H_{\mathcal{P}}$-integral.

3. Main results

Let

$$\sum_{n=0}^{\infty} c_n \chi_n \tag{8}$$

be a series in the system $\{\chi_n\}$ with partial sums $S_k = \sum\limits_{n=0}^{k-1} c_n\chi_n$. Let $F_{m_k}(x) = \int\limits_0^x S_{m_k}(t)\,dt$. By $F(x)$ we denote the limit

$$\lim_{k\to\infty} F_{m_k}(x) = \lim_{k\to\infty} \int\limits_0^x S_{m_k}(t)\,dt\,. \tag{9}$$

at points where this limit exists.

According to Lemma 2.1, $\int\limits_{\triangle_r^{(k)}} \chi_n(t)\,dt = 0$ for $n \geqslant m_k$ and $0 \leqslant r \leqslant m_k - 1$. Then at any $\mathcal{P}$-adic rational point x of rank k (i.e., point of the form $\frac{r}{m_k}$) we get $F_{m_{k+1}}(x) = F_{m_k}(x)$ and so the limit (9) reduces to

$$F(x) = F_{m_k}(x) = \int\limits_0^x S_{m_k}(t)\,dt\,. \tag{10}$$

Therefore the function $F(x)$ is defined at least at all $\mathcal{P}$-adic rational points. It is also easy to check that S_{m_k} is constant on the intervals $\triangle_r^{(k)}$ with $0 \leqslant r \leqslant m_k - 1$. So, if $x \in \triangle_r^{(k)} = \left[\frac{r}{m_k}, \frac{r+1}{m_k}\right)$ then according to (10) we have:

$$S_{m_k}(x) = \frac{1}{\left|\triangle_r^{(k)}\right|} \int\limits_{\triangle_r^{(k)}} S_{m_k}(t)\,dt = \frac{F\left(\frac{r+1}{m_k}\right) - F\left(\frac{r}{m_k}\right)}{\left|\triangle_r^{(k)}\right|}\,. \tag{11}$$

Theorem 3.1. *The series* (8) *is the Fourier series, in the sense of the $H_{\mathcal{P}}$-integral of some function f, which is $H_{\mathcal{P}}$-integrable on* $[0,1]$, *if and only if the function F associated with this series by means of expression* (10) *coincides at all $\mathcal{P}$-adic rational points with the indefinite integral* $(H_{\mathcal{P}})\int\limits_0^x f$.

P r o o f. This theorem can be easily established by an argument similar to that one used in [6, Theorem 3.1.7] for the case of Walsh system and Lebesgue integral (see also [6, Theorem 2.8.1]). □

To prove the next theorem, we need several lemmas in which we suppose that the sequence $\mathcal{P} = \{p_i\}$ is bounded.

Lemma 3.1. *Suppose that the series* (8) *converges to a value $f(x_0)$ at some $\mathcal{P}$-adic-irrational point x_0. Then the limit* (9) *exists at this point, the function F, defined by this limit, has $\mathcal{P}$-path derivative at x_0 and $F'_{\mathcal{P}}(x_0) = f(x_0)$.*

P r o o f. This lemma can be obtained by repeating the proof given in [16, Lemma 2] for the case of the Vilenkin-Price system. □

Lemma 3.2. *Suppose that the series* (8) *satisfies the condition* $\limsup\limits_{n\to\infty} |S_n(x_0)| < +\infty$ *at a* $\mathcal{P}$*-adic irrational point* x_0. *Then the function* F *is defined at* x_0 *by the limit* (9) *and*

$$-\infty < \underline{(\mathrm{Re}\ F)}'_{\mathcal{P}}(x_0) \leqslant \overline{(\mathrm{Re}\ F)}'_{\mathcal{P}}(x_0) < +\infty\,, \tag{12}$$

$$-\infty < \underline{(\mathrm{Im}\ F)}'_{\mathcal{P}}(x_0) \leqslant \overline{(\mathrm{Im}\ F)}'_{\mathcal{P}}(x_0) < +\infty\,. \tag{13}$$

P r o o f . This proof is also based on a technique used in the proof of Lemma 2 in [16] with some appropriate modifications.

According to the assumption, there exists a positive number A such that

$$|S_n(x_0)| \leqslant A\,, \qquad n = 1, 2, \dots \tag{14}$$

Denoting the sequence (7) of $\mathcal{P}$-adic intervals converging to x_0 by

$$\{[a_k(x_0), b_k(x_0))\} \tag{15}$$

and using (11) we get from (14)

$$\left|\frac{F(b_k(x_0)) - F(a_k(x_0))}{b_k(x_0) - a_k(x_0)}\right| \leqslant A \tag{16}$$

for every natural k. We can represent each non-negative integer n in the following form:

$$n = sm_k + r\,,$$

with $k = 0, 1, \dots,$ $s = 0, 1, \dots, p_k - 1,$ $r = 0, 1, \dots, m_k - 1$. Then

$$\sum_{n=1}^{\infty} c_n \chi_n = c_0 \chi_0 + \sum_{k=0}^{\infty} \sum_{s=1}^{p_k - 1} \sum_{r=0}^{m_k - 1} c_{sm_k + r} \chi_{sm_k + r}\,.$$

From (14) it follows that for any natural k and for each $s = 0, 1, \dots, p_k - 1$ we have

$$\left|\sum_{r=0}^{m_k - 1} c_{sm_k + r} \chi_{sm_k + r}(x_0)\right| \leqslant 2A\,. \tag{17}$$

We now consider an arbitrary index k and the interval $\triangle^k(x_0) = [a_k(x_0), b_k(x_0))$. Note that by the Definition (3) we have, for $n = sm_k + r$, $r = 0, 1, \dots, m_k - 1$,

$$\chi_{sm_k + r} = \chi_r (\chi_{m_k})^s = \chi_r (\chi_{sm_k})$$

and so

$$\sum_{r=0}^{m_k - 1} c_{sm_k + r} \chi_{sm_k + r}(x) = \left(\sum_{r=0}^{m_k - 1} c_{sm_k + r} \chi_r(x)\right) \chi_{sm_k}(x)\,.$$

However, the functions χ_r, $r = 0, 1, \dots, m_k - 1$, are constant on $\triangle^{(k)}(x_0)$ and $|\chi_{sm_j}(x)| = 1$ for all x. Therefore, by this and by the inequality (17) we get, for all $x \in \triangle^{(j)}(x_0)$,

$$\left|\sum_{r=0}^{m_j-1} c_{sm_j+r}\chi_{sm_j+r}(x)\right| = \left|\sum_{r=0}^{m_j-1} c_{sm_j+r}\chi_{sm_j+r}(x_0)\right| \leqslant 2A\,.$$

Using this, we compute

$$\begin{aligned}\left|\int_0^{x_0} S_{m_{k+1}}(t)\,dt - \int_0^{x_0} S_{m_k}(t)\,dt\right| &\leqslant \int_0^{x_0} \sum_{s=1}^{p_k-1}\left|\sum_{r=0}^{m_k-1} c_{sm_k+r}\chi_{sm_k+r}(t)\right| dt \\ &= \sum_{s=1}^{p_k-1}\int_{a_k(x_0)}^{x_0}\left|\sum_{r=0}^{m_k-1} c_{sm_k+r}\chi_{sm_k+r}(t)\right| dt \\ &\leqslant 2Ap_k\big(x_0 - a_k(x_0)\big) \leqslant 2AC\frac{1}{m_k}\,,\end{aligned} \tag{18}$$

where $C = \limsup_{j\to\infty} p_j$. This estimate shows, that $\left\{\int_0^{x_0} S_{m_k}(t)\,dt\right\}_k$ is a Cauchy sequence and therefore F is defined by (9) at x_0.

Now, we fix some k and choose the minimal index $i > k$, such that $a_i(x_0) \neq a_k(x_0)$ (such i exists, since x_0 is a $\mathcal{P}$-adic irrational point). Then

$$\big|x_0 - a_k(x_0)\big| > \big|a_i(x_0) - a_k(x_0)\big| \geqslant \frac{1}{m_i}\,. \tag{19}$$

By the choice of an index i we get $a_k(x_0) = \dots = a_{i-1}(x_0) \neq a_i(x_0)$. By (10), $F\big(a_k(x_0)\big) = F\big(a_{i-1}(x_0)\big) = F_{m_{i-1}}\big(a_{i-1}(x_0)\big)$, and so

$$\begin{aligned}\left|\frac{F(x_0) - F\big(a_k(x_0)\big)}{x_0 - a_k(x_0)}\right| &\leqslant \left|\frac{F(x_0) - F_{m_{i-1}}(x_0)}{x_0 - a_k(x_0)}\right| \\ &\quad + \left|\frac{F_{m_{i-1}}(x_0) - F_{m_{i-1}}\big(a_{i-1}(x_0)\big)}{x_0 - a_k(x_0)}\right|\,.\end{aligned} \tag{20}$$

As $S_{m_{i-1}}$ is constant on $\big[a_{i-1}(x_0), b_{i-1}(x_0)\big)$, the inequality (14) then implies:

$$\begin{aligned}\left|\frac{F_{m_{i-1}}(x_0) - F_{m_{i-1}}\big(a_{i-1}(x_0)\big)}{x_0 - a_k(x_0)}\right| &= \left|\frac{1}{x_0 - a_k(x_0)}\int_{a_{i-1}(x_0)}^{x_0} S_{m_{i-1}}(t)dt\right| \\ &= \big|S_{m_{i-1}}(x_0)\big| \leqslant A\,.\end{aligned} \tag{21}$$

Moreover, using (18) and (19) we estimate

$$\begin{aligned}\left|\frac{F(x_0)-F_{m_{i-1}}(x_0)}{x_0-a_k(x_0)}\right| &\leqslant \frac{1}{x_0-a_k(x_0)}\sum_{j=i-1}^{\infty}\left|\int_0^{x_0} S_{m_{j+1}}(t)\,dt-\int_0^{x_0} S_{m_j}(t)\,dt\right| \\ &\leqslant 2AC\sum_{j=i-1}^{\infty}\frac{m_i}{m_j}\leqslant 2AC\left(p_{i-1}+\sum_{j=0}^{\infty}2^{-j}\right) \\ &\leqslant 2AC(C+2)\,. \end{aligned} \tag{22}$$

Combining (20), (21) and (22) we get

$$\left|\frac{F(x_0)-F\big(a_k(x_0)\big)}{x_0-a_k(x_0)}\right|\leqslant A+2AC(C+2)\,. \tag{23}$$

In a similar way we can get the estimate

$$\left|\frac{F\big(b_k(x_0)\big)-F(x_0)}{a_k(x_0)-x_0}\right|\leqslant A+2AC(C+2)\,. \tag{24}$$

It is clear that the inequalities (23) and (24) give the same estimates for $\operatorname{Re}F$ and $\operatorname{Im}F$. As those estimates are obtained for arbitrary k, they imply (12) and (13). □

Lemma 3.3. *Suppose that the sequence $\{c_n\}$ of coefficients of the series (8) is converging to zero. Then the function F defined by (9) is $\mathcal{P}$-path continuous everywhere on $Q_{\mathcal{P}}$.*

P r o o f. Fix a point $x_0\in Q_{\mathcal{P}}$. Let $\big\{\big[a_k(x_0),b_k(x_0)\big)\big\}$ be one of the two nested sequences (7) for which x_0, starting with some k, is a common endpoint. Fix $\varepsilon>0$. Since $\lim\limits_{n\to\infty}c_n=0$ then there exists k such that for any $n\geqslant m_k$ we get $|c_n|<\varepsilon$. By (10) we have for $i>k$:

$$\begin{aligned}F\big(b_i(x_0)\big)-F\big(a_i(x_0)\big)&=F_{m_i}\big(b_i(x_0)\big)-F_{m_i}\big(a_i(x_0)\big) \\ &=\sum_{n=0}^{m_i-1}\int_{a_i(x_0)}^{b_i(x_0)} c_n\chi_n(t)\,dt \qquad (25) \\ &=\int_{a_i(x_0)}^{b_i(x_0)} S_{m_k}(t)\,dt+\sum_{l=k}^{i-1}\int_{a_i(x_0)}^{b_i(x_0)}\sum_{n=m_l}^{m_{l+1}-1}c_n\chi_n(t)\,dt\,.\end{aligned}$$

Since S_{m_k} is constant on $\triangle^{(k)}(x_0)$, we can choose $i_0>k$ so that for each $i\geqslant i_0$

$$\left|\int_{a_i(x_0)}^{b_i(x_0)} S_{m_k}(t)\,dt\right|<\varepsilon\,. \tag{26}$$

For $l \geqslant k$ we also have the estimate

$$\sum_{n=m_l}^{m_{l+1}-1} \left|c_n \chi_n(t)\right| < m_{l+1} \cdot \varepsilon\,,$$

which implies

$$\begin{aligned}\sum_{l=k}^{i-1} \int\limits_{a_i(x_0)}^{b_i(x_0)} \sum_{n=m_l}^{m_{l+1}-1} \left|c_n \chi_n(t)\right| dt &< \sum_{l=k}^{i-1} \frac{m_{l+1} \cdot \varepsilon}{m_i} \\ &= \varepsilon \left(1 + \frac{1}{p_{i-1}} + \frac{1}{p_{i-2}p_{i-1}} + \cdots + \frac{1}{p_{k+1} \cdots p_{i-1}}\right) \\ &\leqslant \varepsilon \left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{i-k-1}}\right) \leqslant 2\varepsilon\,.\end{aligned} \tag{27}$$

Combining (25), (26) and (27) we obtain that $\left|F\big(b_i(x_0)\big) - F\big(a_i(x_0)\big)\right| < 3\varepsilon$ if $i \geqslant i_0$. □

THEOREM 3.2. *Suppose that the sequence p_i is bounded, the series* (8) *converges a.e. to a function f and everywhere on $[0,1] \setminus Q_{\mathcal{P}}$ we have*

$$\limsup_{n\to\infty} \left|S_n(x)\right| < +\infty\,. \tag{28}$$

Then f is $H_{\mathcal{P}}$-integrable on $[0,1]$ (in the sense of Definition 2.6) and (8) *is $H_{\mathcal{P}}$-Fourier series of f.*

P r o o f. If x is a $\mathcal{P}$-adic irrational point at which the series (8) converges to $f(x)$ then, according to Lemma 3.1, the series (9) converges to $F(x)$ at this point and the function F is $\mathcal{P}$-path differentiable at x with $F'_{\mathcal{P}}(x) = f(x)$. So, F is $\mathcal{P}$-path differentiable a.e. on $[0,1]$.

By assumption (28) and by Lemma 3.2, the inequalities (12) and (13) hold everywhere on the set of $\mathcal{P}$-adic irrational points. Since $|\chi(x_n)| = 1$ everywhere on $[0,1]$, and since the series (8) converges at least at one $\mathcal{P}$-adic irrational point, it follows that $\lim\limits_{n\to\infty} c_n = 0$. Then by Lemma 3.3 F is $\mathcal{P}$-path continuous everywhere on $Q_{\mathcal{P}}$. This implies that $\operatorname{Re} F$ and $\operatorname{Im} F$ are also $\mathcal{P}$-path continuous on $Q_{\mathcal{P}}$. Thus by Theorem 2.1, $\operatorname{Re} F$ and $\operatorname{Im} F$ are $H_{\mathcal{P}}$-integrable and $\operatorname{Re} F$ and $\operatorname{Im} F$ are their indefinite $H_{\mathcal{P}}$-integrals. Hence f is $H_{\mathcal{P}}$-integrable (see Remark 2.2) with F being its indefinite $H_{\mathcal{P}}$-integral. Now application of Theorem 3.1 completes the proof. □

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