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SOME INEQUALITIES INVOLVING INTEGRAL MEANS

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ABSTRACT. A class of generalized weighted quasi-arithmetic means in integral form $M_{[a,b],g}(p,f)$ is studied using the weighted integral form of Jensen's inequality. In particular, various inequalities and properties of the generalized weighted quasi-arithmetic means are established with respect to the properties of input functions p, f and g. Some well-known inequalities as a consequence of our results are derived.

1. Introduction

In the discrete case, a mean of a nonnegative n-tuple of real numbers $a = (a_1, \dots, a_n)$ with respect to a weight vector $p = (p_1, \dots, p_n)$ of positive real numbers, where $\sum_{i=1}^{n} p_i = 1$, $n \in \mathbb{N}$ (the set of all positive integers), is defined by the formula

$$M(p,a) = \varphi^{-1}\left(\sum_{i=1}^{n} p_i \varphi(a_i)\right), \tag{1}$$

where φ is a continuous and strictly monotone function which has its inverse function φ^{-1} satisfying the same conditions, [6]. Here φ is called the Kolmogoroff-Nagumo function associated with (1). Means of the form (1) are also refered to as *quasi-arithmetic means*. This class of means comprises the commonly used algebraic means and also other types of aggregation operators.

A great deal of literature on the concept of mean (or average) and properties of several means (e.g., arithmetic mean, geometric mean, power mean, harmonic mean) appeared in the 19th century. A. N. Kolmogoroff [10] and M. Nagumo [13] were the first who investigated the characteristic properties

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of means in general. They considered mostly the case of equal weights. The generalization to arbitrary weights and the characterization of means of the form (1) are due to Finetti [3], Jessen [8], Kitagawa [9], Aczél [1] and many others.

An integral analogue of means (1) was established by A. N. Kolmogoroff, M. Nagumo and B. de Finetti in 1930th. They showed that all types of the so-called *intrinsic means* may be expressed via

$$\mathcal{M}_F(\varphi) = \varphi^{-1}\left(\int_{\mathbb{R}} \varphi(x)\, dF(x)\right), \tag{2}$$

where $F(x)$ is a distribution and $\varphi(x)$ is a continuous real increasing function on $\mathbb{R}$ (the set of all real numbers), cf. [6]. Integral (2) is meant to be in the sense of Lebesgue–Stieltjes. This form of means coincides with the class of the so-called *integral φ-means.* It is well-known that many types of means may be rewritten according to the pattern given by the formula (2), e.g., the Stolarsky's means [19].

In this paper we study the non-symmetrical weighted version of mean (2) proposed by F. Qi in [18], considering a weight function p (i.e., a real positive Lebesgue integrable function on a finite real interval $[a,b]$) and the distribution

$$F(x) = \begin{cases} 0, & x < a, \\ x/\|p\|_{[a,b]}, & a \leqslant x \leqslant b, \\ 1, & b < x \end{cases}$$

(see Section 2 for notation). The obtained class of integral means $M_{[a,b],\,g}(p,f)$ is called the *generalized weighted quasi-arithmetic means in integral form*, cf. [5]. In Section 3, we state an analogue of Jensen's inequality for the weighted integral means as well as its conversion. This enables us to derive various inequalities for means $M_{[a,b],g}(p,f)$, $a < b$, with respect to the convexity property of functions f and g. Moreover, a comparison theorem between generalized weighted quasi-arithmetic means is established here. As a consequence, we derive the weighted integral version of Wang–Wang's geometric-harmonic mean inequality, Chebyshev's integral inequality and the well-known Pólya–Knopp's inequality.

2. Preliminaries

Let $[a,b] \subset \mathbb{R}$, $a < b$, be an interval. By $L_1([a,b])$ denote the vector space of all real Lebesgue integrable functions defined on the interval $[a,b]$ with the classical Lebesgue measure. By $L_1^+([a,b])$ denote the positive cone of $L_1([a,b])$, i.e., the vector space of all real positive Lebesgue integrable functions on $[a,b]$. In what follows $\|p\|_{[a,b]}$ denotes the finite L_1-norm of a function $p \in L_1^+([a,b])$.

Definition 2.1. Let $p \in L_1^+([a,b])$. Let $f\colon [a,b] \to [\alpha,\beta]$ be a measurable function, and $g\colon [\alpha,\beta] \to \mathbb{R}$ be a real continuous and strictly monotone function. The *generalized weighted quasi-arithmetic mean of a function f with respect to the weight function p* is a real number $M_{[a,b],\,g}(p,f)$ given by

$$M_{[a,b],\,g}(p,f) = g^{-1}\left(\frac{1}{\|p\|_{[a,b]}}\int_a^b p(x)g\big(f(x)\big)\,dx\right), \tag{3}$$

where g^{-1} denotes the inverse function to the function g.

In what follows, g is always a real continuous and strictly monotone function (in accordance with Definition 2.1).

Means $M_{[a,b],g}(p,f)$ include lots of commonly used two variable integral means as particular cases when taking the suitable functions p, f and g. For instance,

(a) for $g(x) = x = I(x)$ (the identity function) we get the *weighted arithmetic mean*

$$M_{[a,b],g}(p,f) = A_{[a,b]}(p,f) = \frac{1}{\|p\|_{[a,b]}}\int_a^b p(x)f(x)\,dx\,;$$

(b) for $g(x) = x^{-1}$ we obtain the *weighted harmonic mean*

$$M_{[a,b],g}(p,f) = H_{[a,b]}(p,f) = \left(\frac{1}{\|p\|_{[a,b]}}\int_a^b \frac{p(x)}{f(x)}\,dx\right)^{-1};$$

(c) for $g(x) = x^r$ we have the *weighted power mean of order r*

$$M_{[a,b],g}(p,f) = M^{[r]}(f;p;a,b) = \begin{cases} \left(\frac{1}{\|p\|_{[a,b]}}\int_a^b p(x)f(x)^r\,dx\right)^{\frac{1}{r}}, & r \neq 0\,, \\ \exp\left(\frac{1}{\|p\|_{[a,b]}}\int_a^b p(x)\ln f(x)\,dx\right), & r = 0\,. \end{cases}$$

The case $r = 0$ corresponds to the *weighted geometric mean* and $r = 2$ is called the *weighted Euclidean quadratic mean.*

(d) if we replace $p(x)$ by $p(x)f^r(x)$ and $g(x)$ by x^{s-r} in (3), then we get the *generalized weighted mean values*

$$M_{[a,b],g}(p,f) = M_{p,f}(r,s;a,b) = \left(\frac{\int_a^b p(x)f^s(x)\,dx}{\int_a^b p(x)f^r(x)\,dx}\right)^{\frac{1}{s-r}};$$

(e) if $p(x)$ is a constant function on $[0, 2\pi]$ and $f(x)$ is of the form

$$f(x) = \begin{cases} (u^n \cos^2 x + v^n \sin^2 x)^{\frac{1}{n}}, & n \neq 0 \\ u^{\cos^2 x} v^{\sin^2 x}, & n = 0 \end{cases},$$

then we obtain the *generalization of arithmetic-geometric mean of Gauss*

$$M_{[a,b],g}(p, f) = M_{g,n}(u, v) = g^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi} g\big(f(x)\big)\,dx\right)$$

for $n \neq 0$. Choosing $n = 2$ and $g(x) = x^{-1}$ we get an original Gauss' arithmetic-geometric mean. For $n = -2$ and $g(t) = t^{-2}$ the mean may be found in [16], the case $n = 1$ and $g(t) = \ln t$ was studied in [2].

Note that means $M_{[a,b],g}(p, f)$ generalize also logarithmic means $L(a, b)$, identric means $I(a, b)$, one-parameter means $J_r(a, b)$, abstracted means $M_g(a, b)$, extended mean values $E(r, s; a, b)$, generalized logarithmic means $S_r(a, b)$, and many others. Hence, from $M_{[a,b],g}(p, f)$ we may deduce most of the two variable means. Further possible extension of means $M_{[a,b],g}(p, f)$ could be considered when f is of the form $f(\theta) = |h(re^{\imath\theta})|$, where $0 < r < 1$ and h is an analytic function in the open unit disk $D = \{z : |z| < 1\}$ of the complex plane $\mathbb{C}$. In that case, choosing $a = 0$, $b = 2\pi$, $g(x) = x^q$ for $0 < q < \infty$ and $p(x) \equiv 1$ on $[0, 2\pi]$, we get the integral mean of order q, cf. [4],

$$M_{[a,b],g}(p, f) = M_q(r, h) = \left(\frac{1}{2\pi}\int_0^{2\pi} \left|h(re^{\imath\theta})\right|^q\,d\theta\right)^{\frac{1}{q}}.$$

3. Jensen's inequality and its conversion

Many mathematical investigations deal with problems about operator of means depending on the behavior of the input functions p, f and g, resp. how functions behave under the action of means. The most known case is that of Jensen's convex functions, which originally deal with the (non-weighted integral) arithmetic mean.

In general measure theoretical notation the Jensen's inequality theorem sounds as follows: let (Ω, A, μ) be a measurable space such that $\mu(\Omega) = 1$. If f is a real μ-integrable function and φ is a convex (concave) function on the range of f, then

$$\varphi\left(\int_\Omega f\,d\mu\right) \leqslant \int_\Omega \varphi \circ f\,d\mu,$$

$$\left(\varphi\left(\int_{\Omega} f\,d\mu\right) \geqslant \int_{\Omega} \varphi\circ f\,d\mu,\right)$$

cf. [17]. The following lemma is a direct application of this general statement to our particular situation when $\mu([a,b]) = 1$ and $d\mu = \frac{p(x)}{\|p\|_{[a,b]}}dx$.

Lemma 3.1. (**Jensen's Inequality**) *Let $p \in L_1^+([a,b])$ and $f\colon [a,b] \to [\alpha,\beta]$ be a measurable function, where $-\infty < \alpha < \beta < \infty$.*

(i) *If g is a convex function on (α,β), then*

$$g\big(A_{[a,b]}(p,f)\big) \leqslant A_{[a,b]}(p, g\circ f)\,.$$

(ii) *If g is a concave function on (α,β), then*

$$g\big(A_{[a,b]}(p,f)\big) \geqslant A_{[a,b]}(p, g\circ f)\,.$$

The following statement is a direct consequence of Jensen's inequality.

Corollary 3.2. *Let $p \in L_1^+([a,b])$ and $f\colon [a,b] \to [\alpha,\beta]$ be a measurable function, where $-\infty < \alpha < \beta < \infty$.*

(i) *If g is either a convex increasing or concave decreasing function on (α,β), then*

$$A_{[a,b]}(p,f) \leqslant M_{[a,b],\,g}(p,f)\,.$$

(ii) *If g is either a convex decreasing or concave increasing function on (α,β), then*

$$A_{[a,b]}(p,f) \geqslant M_{[a,b],\,g}(p,f)\,.$$

Proof. Let g be a convex increasing function. According to Jensen's inequality we get

$$g^{-1}\Big(g\big(A_{[a,b]}(p,f)\big)\Big) \leqslant g^{-1}\big(A_{[a,b]}(p, g\circ f)\big)\,,$$

which is equivalent to

$$A_{[a,b]}(p,f) \leqslant M_{[a,b],g}(p,f)\,.$$

The proofs of the remaining parts are similar. □

Some elementary properties of $M_{[a,b],g}(p,f)$ derived by using the weighted integral analogue of Jensen's inequality may be found in [5]. Also, some well-known inequalities among weighted means in integral form may be obtained as its direct corollaries.

The following theorem corresponds to some conversions of the Jensen's inequality for convex (concave) functions in the case of $M_{[a,b],g}(p,f)$.

THEOREM 3.3. *Let $p \in L_1^+([a,b])$. Let $f\colon [a,b] \to [\alpha,\beta]$ be a measurable function, and $g\colon [\alpha,\beta] \to \mathbb{R}$, where $-\infty < \alpha < \beta < \infty$.*

(i) *If g is convex on $[\alpha,\beta]$, then*

$$A_{[a,b]}(p, g\circ f) \leqslant \frac{g(\alpha)\bigl(\beta - A_{[a,b]}(p,f)\bigr)}{\beta-\alpha} + \frac{g(\beta)\bigl(A_{[a,b]}(p,f)-\alpha\bigr)}{\beta-\alpha}.$$

(ii) *If g is concave on $[\alpha,\beta]$, then*

$$A_{[a,b]}(p, g\circ f) \geqslant \frac{g(\alpha)\bigl(\beta - A_{[a,b]}(p,f)\bigr)}{\beta-\alpha} + \frac{g(\beta)\bigl(A_{[a,b]}(p,f)-\alpha\bigr)}{\beta-\alpha}.$$

Proof. We will prove only item (i). Item (ii) may be proved analogously.

Suppose that g is a convex function on the interval $[\alpha,\beta]$. Let us consider the integral $\int_a^b p(x)g\bigl(f(x)\bigr)\,dx$. Putting

$$\lambda(x) = \frac{f(x)-\alpha}{\beta-\alpha}, \tag{4}$$

we have $f(x) = \bigl(1-\lambda(x)\bigr)\alpha + \lambda(x)\beta$, for all $x \in [a,b]$. Since $\alpha \leqslant f(x) \leqslant \beta$ for all $x \in [a,b]$, and g is convex on $[\alpha,\beta]$,

$$\begin{aligned}\int_a^b p(x)g\bigl(f(x)\bigr)\,dx &\leqslant \int_a^b p(x)\Bigl(\bigl(1-\lambda(x)\bigr)g(\alpha) + \lambda(x)g(\beta)\Bigr)\,dx \\ &= g(\alpha)\int_a^b p(x)\bigl(1-\lambda(x)\bigr)\,dx + g(\beta)\int_a^b p(x)\lambda(x)\,dx.\end{aligned}$$

By (4) we get

$$\int_a^b p(x)\lambda(x)\,dx = \frac{1}{\beta-\alpha}\left(\int_a^b p(x)f(x)\,dx - \alpha\,\|p\|_{[a,b]}\right)$$

and therefore

$$\begin{aligned}\int_a^b p(x)g\bigl(f(x)\bigr)\,dx \leqslant{}& \frac{g(\alpha)}{\beta-\alpha}\left(\beta\,\|p\|_{[a,b]} - \int_a^b p(x)f(x)\,dx\right) \\ &+ \frac{g(\beta)}{\beta-\alpha}\left(\int_a^b p(x)f(x)\,dx - \alpha\,\|p\|_{[a,b]}\right).\end{aligned}$$

Since $\|p\|_{[a,b]}$ is positive and finite, we may write

$$
\begin{aligned}
A_{[a,b]}(p, g \circ f) &= \frac{1}{\|p\|_{[a,b]}} \int_a^b p(x) g\big(f(x)\big)\, dx \\
&\leqslant \frac{g(\alpha)\Big(\beta - \frac{1}{\|p\|_{[a,b]}} \int_a^b p(x) f(x)\, dx\Big)}{\beta-\alpha} + \frac{g(\beta)\Big(\frac{1}{\|p\|_{[a,b]}} \int_a^b p(x) f(x)\, dx - \alpha\Big)}{\beta-\alpha} \\
&= \frac{g(\alpha)\big(\beta - A_{[a,b]}(p,f)\big)}{\beta-\alpha} + \frac{g(\beta)\big(A_{[a,b]}(p,f) - \alpha\big)}{\beta-\alpha}\,.
\end{aligned}
$$

The proof is complete. □

4. Some inequalities among means

In this section we investigate some further properties of the class of generalized weighted quasi-arithmetic means expressed in the integral form. Some elementary properties of $M_{[a,b],g}(p,f)$ related to convexity (concavity) of functions f, g were obtained in [5].

Jensen's inequality provides a very important tool for a kind of comparison of means. Therefore, we state the following comparison theorem between means $M_{[a,b],g_k}(p,f)$ and $M_{[a,b],g_{k-1}}(p,f)$.

Theorem 4.1. *Let $p \in L_1^+\big([a,b]\big)$ and $f\colon [a,b] \to [\alpha,\beta]$ be a measurable function, where $-\infty < \alpha < \beta < \infty$. Let g_k, $k = 1,2,\dots,m$ be one-to-one functions defined on $[\alpha,\beta]$, and denote $G_1 = g_1$, $G_2 = g_2 \circ g_1^{-1}, \dots, G_{k+1} = g_{k+1} \circ g_k^{-1}$ for $k = 1,2,\dots,m-1$.*

(i) *If G_k are either concave increasing or convex decreasing on the range of g_k, for $k = 1,2,\dots,m$, then*
$$M_{[a,b],g_k}(p,f) \leqslant M_{[a,b],g_{k-1}}(p,f)\,, \quad k = 1,2,\dots,m\,.$$

(ii) *If G_k are either concave decreasing or convex increasing on the range of g_k, for $k = 1,2,\dots,m$, then*
$$M_{[a,b],g_k}(p,f) \geqslant M_{[a,b],g_{k-1}}(p,f)\,, \quad k = 1,2,\dots,m\,.$$

Proof. Let us suppose that $G_k = g_k \circ g_{k-1}^{-1}$ are concave increasing functions for $k = 1,2,\dots,m$. By Jensen's inequality we have

$$\frac{1}{\|p\|_{[a,b]}} \int_a^b p(x) G_k\Big(g_{k-1}\big(f(x)\big)\Big)\, dx \leqslant G_k\left(\frac{1}{\|p\|_{[a,b]}} \int_a^b p(x) g_{k-1}\big(f(x)\big)\, dx\right),$$

for $k = 1, 2, \dots, m$. Since the functions G_k are increasing, it follows that g_k are increasing too, and therefore for $k = 1, 2, \dots, m$ we get

$$g_k^{-1}\left(\frac{1}{\|p\|_{[a,b]}}\int_a^b p(x)G_k\Big(g_{k-1}\big(f(x)\big)\Big)\,dx\right)$$
$$\leqslant (g_k^{-1}\circ G_k)\left(\frac{1}{\|p\|_{[a,b]}}\int_a^b p(x)g_{k-1}\big(f(x)\big)\,dx\right).$$

Using the fact $g_k^{-1}\circ G_k = g_{k-1}^{-1}$ and $G_k\circ g_{k-1} = g_k$, we finally obtain the inequality

$$g_k^{-1}\left(\frac{1}{\|p\|_{[a,b]}}\int_a^b p(x)g_k\big(f(x)\big)\,dx\right) \leqslant g_{k-1}^{-1}\left(\frac{1}{\|p\|_{[a,b]}}\int_a^b p(x)g_{k-1}\big(f(x)\big)\,dx\right),$$

for $k = 1, 2, \dots, m$, which corresponds to $M_{[a,b],g_k}(p,f) \leqslant M_{[a,b],g_{k-1}}(p,f)$.

The proofs of the remaining parts are similar. □

Lemma 4.2. *Let $p \in L_1^+\big([a,b]\big)$ and $f\colon [a,b]\to\mathbb{R}$ be a continuous and integrable function with the continuous first derivative on (a,b).*

(i) *If f is a strictly monotone and convex function on $[a,b]$, then*

$$A_{[a,b]}\big(p(x), f(x)\big) \leqslant f\left(A_{[a,b]}(p(x)f'(x), x)\right).$$

(ii) *If f is a strictly monotone and concave function on $[a,b]$, then*

$$A_{[a,b]}\big(p(x), f(x)\big) \geqslant f\left(A_{[a,b]}(p(x)f'(x), x)\right).$$

Proof. Let us define θ by

$$\theta = A_{[a,b]}\big(p(x)f'(x), x\big). \tag{5}$$

Since f is strictly monotone on $[a,b]$, it follows that $\theta \in (a,b)$. The convexity of f ensures that f' is nondecreasing on (a,b) and

$$f(x) + f'(x)(\theta - x) \leqslant f(\theta),$$

cf. [14]. Multiplying both sides of the above inequality by $\frac{p(x)}{\|p\|_{[a,b]}}$ and integrating over the interval $[a,b]$ with respect to x, we get

$$A_{[a,b]}\big(p(x), f(x)\big) + \theta\cdot A_{[a,b]}\big(p(x), f'(x)\big) - A_{[a,b]}\big(p(x), xf'(x)\big) \leqslant f(\theta).$$

Since

$$
\begin{aligned}
&A_{[a,b]}\big(p(x)f'(x),x\big)\cdot A_{[a,b]}\big(p(x),f'(x)\big)-A_{[a,b]}\big(p(x),xf'(x)\big)\\
&=A_{[a,b]}\big(p(x),xf'(x)\big)-A_{[a,b]}\big(p(x),xf'(x)\big)=0\,,
\end{aligned}
$$

replacing θ by (5) we obtain

$$
A_{[a,b]}\big(p(x),f(x)\big)\leqslant f\Big(A_{[a,b]}\big(p(x)f'(x),x\big)\Big)\,.
$$

The proof of item (ii) is similar. □

EXAMPLE 4.3. Let $[a,b]=(0,1/2]$ and suppose $p\in L_1^+\big((0,1/2]\big)$. Let $f(x)=\ln\frac{1-x}{x}$ on $(0,1/2]$. It is easy to verify that the function $f(x)$ is strictly decreasing and convex on $(0,1/2]$ and $f'(x)=\frac{1}{x(x-1)}$, i.e., the assumptions from Lemma (4.2) (i) are satisfied. Thus,

$$
\begin{aligned}
\frac{1}{\|p\|_{(0,1/2]}}\int_0^{\frac12}p(x)\ln\frac{1-x}{x}\,dx&\leqslant\ln\left(\frac{\int_0^{\frac12}\frac{p(x)}{x(x-1)}\,dx-\int_0^{\frac12}\frac{p(x)}{x-1}\,dx}{\int_0^{\frac12}\frac{p(x)}{x-1}\,dx}\right)\\
&=\ln\left(\frac{\int_0^{\frac12}\frac{p(x)}{x}\,dx}{\int_0^{\frac12}\frac{p(x)}{1-x}\,dx}\right).
\end{aligned}
$$

The above inequality may be rewritten as follows

$$
\exp\left(\frac{1}{\|p\|_{(0,1/2]}}\int_0^{\frac12}p(x)\ln\frac{1-x}{x}\,dx\right)\leqslant\frac{\int_0^{\frac12}\frac{p(x)}{x}\,dx}{\int_0^{\frac12}\frac{p(x)}{1-x}\,dx}\,,
$$

which is equivalent to

$$
\frac{\exp\left(\frac{1}{\|p\|_{(0,1/2]}}\int_0^{\frac12}p(x)\ln(1-x)\,dx\right)}{\exp\left(\frac{1}{\|p\|_{(0,1/2]}}\int_0^{\frac12}p(x)\ln x\,dx\right)}\leqslant\frac{\int_0^{\frac12}\frac{p(x)}{x}\,dx}{\int_0^{\frac12}\frac{p(x)}{1-x}\,dx}\,. \tag{6}
$$

Using the notation

$$
G_{(0,1/2]}\big(p(x),g(x)\big)=\exp\left(\frac{1}{\|p\|_{(0,1/2]}}\int_0^{\frac12}p(x)\ln g(x)\,dx\right)\,,
$$

for the weighted geometric mean and

$$H_{(0,1/2]}(p(x), g(x)) = \left(\frac{1}{\|p\|_{(0,1/2]}} \int_0^{\frac{1}{2}} \frac{p(x)}{g(x)}\, dx \right)^{-1},$$

for the weighted harmonic mean, we may rewrite inequality (6) as follows

$$\frac{G_{(0,1/2]}(p(x), 1-x)}{G_{(0,1/2]}(p(x), x)} \leqslant \frac{H_{(0,1/2]}(p(x), 1-x)}{H_{(0,1/2]}(p(x), x)},$$

which is equivalent to the weighted integral inequality of Wang–Wang, cf. [11], in the form

$$\frac{H_{(0,1/2]}(p(x), x)}{H_{(0,1/2]}(p(x), 1-x)} \leqslant \frac{G_{(0,1/2]}(p(x), x)}{G_{(0,1/2]}(p(x), 1-x)}.$$

The following theorem is an easy corollary of Lemma 4.2.

THEOREM 4.4. *Let $p \in L_1^+([a,b])$, and suppose that $f\colon [a,b] \to [\alpha,\beta]$ is a continuous and integrable function on $[a,b]$ with the continuous first derivative on (a,b), and $g\colon [\alpha,\beta] \to \mathbb{R}$.*

(i) *If g is either a convex increasing or concave decreasing function on $[\alpha,\beta]$, and f is a strictly monotone and concave function on $[a,b]$, then*

$$f\Big(A_{[a,b]}(p(x)f'(x), x)\Big) \leqslant M_{[a,b],g}(p(x), f(x)).$$

(ii) *If g is either a concave increasing or convex decreasing function on $[\alpha,\beta]$, and f is a strictly monotone and convex function on $[a,b]$, then*

$$f\Big(A_{[a,b]}(p(x)f'(x), x)\Big) \geqslant M_{[a,b],g}(p(x), f(x)).$$

Proof. Let g be a concave decreasing function on $[\alpha,\beta]$ and f be a strictly monotone and concave function on $[a,b]$. From Corollary 3.2 and Lemma 4.2 we immediately get the inequalities

$$M_{[a,b],g}(p(x), f(x)) \geqslant A_{[a,b]}(p(x), f(x)) \geqslant f\Big(A_{[a,b]}(p(x)f'(x), x)\Big).$$

The remaining parts may be proved analogously. $\square$

Theorem 4.5. *Let* $(p,f) \in \Big(L_1^+([a,b]) \times L_1^+([a,b))\Big)$, *where* $f\colon [a,b] \to [\alpha,\beta]$ *is a continuous function with the continuous first derivative on* (a,b). *Let* $g\colon [\alpha,\beta] \to \mathbb{R}$.

(i) *If* g *is either convex increasing or concave decreasing on* $[\alpha,\beta]$, *then*

$$A_{[a,b]}(p,f) \leqslant f(a) + M_{[a,b],g}\left(p(x), \int_a^x f'(t)\,dt\right).$$

(ii) *If* g *is either concave increasing or convex decreasing on* $[\alpha,\beta]$, *then*

$$A_{[a,b]}(p,f) \geqslant f(a) + M_{[a,b],g}\left(p(x), \int_a^x f'(t)\,dt\right).$$

Proof. Consider the case when g is convex increasing on $[\alpha,\beta]$. The direct calculation yields

$$M_{[a,b],g}\left(p(x), \int_a^x f'(t)\,dt\right) \geqslant A_{[a,b]}\left(p(x), \int_a^x f'(t)\,dt\right)$$
$$= \frac{1}{\|p\|_{[a,b]}} \int_a^b p(x)f(x)\,dx - \frac{f(a)}{\|p\|_{[a,b]}} \int_a^b p(x)\,dx = A_{[a,b]}(p,f) - f(a).$$

□

Similar results may be obtained when considering integrals $\int_x^b f'(t)\,dt$. An easy corollary for the generalized weighted quasi-arithmetic mean of product of two functions follows from the Chebyshev's integral inequality in the following form, cf. [12].

Lemma 4.6. (**Chebyshev's Inequality**) *Let* $p \in L_1^+([a,b])$ *and let* $h,k\colon [a,b] \to \mathbb{R}$ *be two integrable functions, both increasing or both decreasing on* $[a,b]$. *Then*

$$A_{[a,b]}(p,h) \cdot A_{[a,b]}(p,k) \leqslant A_{[a,b]}(p,hk). \tag{7}$$

If one of the functions h *or* k *is nonincreasing and the other nondecreasing, then the inequality in* (7) *is reversed.*

Theorem 4.7. *Let* $p \in L_1^+([a,b])$, *let* $h,k\colon [a,b] \to [\alpha,\beta]$ *be two integrable functions and* $f\colon [\alpha,\beta] \to \mathbb{R}$. *Let* g *be a real continuous monotone function defined on the range of* hk.

(i) *If* g *is convex increasing or concave decreasing,* f *is concave increasing or convex decreasing and* h,k *are either both increasing or both decreasing* *functions, then*

$$M_{[a,b],f}(p,h)\cdot M_{[a,b],f}(p,k)\leqslant M_{[a,b],g}(p,hk)\,.$$

(ii) *If g is concave increasing or convex decreasing, f is convex increasing or concave decreasing and one of the functions h,k is nonincreasing and the other one nondecreasing, then*

$$M_{[a,b],f}(p,h)\cdot M_{[a,b],f}(p,k)\geqslant M_{[a,b],g}(p,hk)\,.$$

Proof. Let us prove item (i). Suppose that h,k are both increasing functions, g is convex increasing and f is concave increasing function. From Corollary 3.2 it follows that

$$M_{[a,b],f}(p,h)\leqslant A_{[a,b]}(p,h)\quad\text{and}\quad M_{[a,b],f}(p,k)\leqslant A_{[a,b]}(p,k)\,.$$

Since h,k are both increasing functions, then

$$M_{[a,b],f}(p,h)\cdot M_{[a,b],f}(p,k)\leqslant A_{[a,b]}(p,h)\cdot A_{[a,b]}(p,k)\,.$$

Applying Lemma 4.6 and Corollary 3.2 we get

$$M_{[a,b],f}(p,h)\cdot M_{[a,b],f}(p,k)\leqslant A_{[a,b]}(p,hk)\leqslant M_{[a,b],g}(p,hk)\,.$$

Similarly we may prove the remaining parts. □

5. Some applications

In this section we deal with the geometric mean operator $\mathbf{G}\colon L_1^+\big([0,\infty)\big)\to L_1^+\big([0,\infty)\big)$ defined as follows: If $f\in L_1^+\big([0,\infty)\big)$, then

$$[\mathbf{G}f](x)=\exp\left(\frac{1}{x}\int\limits_0^x \ln f(t)\,dt\right),\quad x\in(0,\infty)\,, \tag{8}$$

i.e., we consider $g^{-1}(\cdot)=\exp(\cdot)$ and the weight function $p(t)\equiv 1$.

There exist many inequalities involving the geometric mean operator (8), cf. [15]. We will prove the following generalization of the inequality considered in [7] and give an easy corollary related to Pólya–Knopp's inequality and its weighted form.

THEOREM 5.1. *Let n be a natural number and r,s,q be real numbers satisfying $2r^n-1>q-sr^n$. If*

$$\int\limits_0^\infty t^{q-sr^n}|f(t)|^{r^n}\,dt<\infty\,,$$

then

$$\int_0^\infty x^q \exp\left[r^{2n} x^{-r^n} \int_0^x t^{r^n-1} \ln|x^{-s} f(t)|\, dt\right] dx$$
$$\leqslant Ce \int_0^\infty t^{q-sr^n} \big|f(t)\big|^{r^n}\, dt\,, \tag{9}$$

where

$$C = \frac{r^n}{2r^n + r^n s - q - 1}\,.$$

P r o o f. Using the substitution $t = xy$, we may rewrite the left side of (9) as follows

$$\int_0^\infty x^q \exp\left[r^{2n} \int_0^1 y^{r^n-1} \ln\big|x^{-s} f(xy)\big|\, dy\right] dx\,.$$

Since $e = \exp\left[-r^{2n} \int\limits_0^1 y^{r^n-1} \ln y\, dy\right]$, we have that (9) has the form

$$\int_0^\infty x^q \exp\left[r^{2n} \int_0^1 y^{r^n-1} \ln\big|x^{-s} f(xy)\big|\, dy\right] dx$$
$$\leqslant C \cdot \exp\left[-r^{2n} \int_0^1 y^{r^n-1} \ln y\, dy\right] \cdot \int_0^\infty t^{q-sr^n} |f(t)|^{r^n}\, dt\,.$$

The direct calculation yields

$$C \int_0^\infty t^{q-sr^n} \big|f(t)\big|^{r^n}\, dt \geqslant \exp\left[r^{2n} \int_0^1 y^{r^n-1} \ln y\, dy\right]$$
$$\times \int_0^\infty x^q \exp\left[r^{2n} \int_0^1 y^{r^n-1} \ln\big|x^{-s} f(xy)\big|\, dy\right] dx$$
$$= \int_0^\infty x^q \exp\left[r^{2n} \int_0^1 y^{r^n-1} \ln\big|x^{-s} y f(xy)\big|\, dy\right] dx\,.$$

Therefore, we get the inequality

$$\int_0^\infty x^q \exp\left[r^n \int_0^1 y^{r^n-1} \ln\big|x^{-s} y f(xy)\big|^{r^n}\, dy\right] dx \leqslant C \int_0^\infty t^{q-sr^n} \big|f(t)\big|^{r^n}\, dt\,.$$

By Jensen's inequality, the left side is dominated by

$$r^n \int_0^\infty x^q \cdot \left(\int_0^1 y^{r^n-1} \left| x^{-s} y f(xy) \right|^{r^n} dy \right) dx \,. \tag{10}$$

Applying Fubini's theorem to (10) and using the substitution $t = xy$, we get

$$\begin{aligned} & r^n \int_0^1 y^{r^n-1} \cdot \left(\int_0^\infty x^{q-sr^n} \left| f(xy) \right|^{r^n} dx \right) dy \\ &= r^n \int_0^1 y^{2r^n+sr^n-q-2} \cdot \left(\int_0^\infty t^{q-sr^n} \left| f(t) \right|^{r^n} dt \right) dy \\ &= \frac{r^n}{2r^n + sr^n - q - 1} \cdot \int_0^\infty t^{q-sr^n} |f(t)|^{r^n} \, dt \,. \end{aligned}$$

□

Note that from the inequality (9) we may obtain some well-known inequalities as direct consequences. For instance, the following Pólya–Knopp's inequality.

Corollary 5.2. *Let $r = 1, q = s = n = 0$ and $f(t) \geqslant 0$. Then inequality* (9) *reduces to*

$$\int_0^\infty \exp \left[\frac{1}{x} \int_0^x \ln f(t) \, dt \right] dx \leqslant e \int_0^\infty f(x) \, dx \,, \tag{11}$$

or equivalently,

$$\int_0^\infty [\mathbf{G} f](x) \, dx \leqslant e \int_0^\infty f(x) \, dx \,.$$

Some weighted versions of Pólya–Knopp's inequality (11) may be also derived directly from (9), for example:

$$\int_0^\infty x^q \exp \left[\frac{1}{x} \int_0^x \ln f(t) \, dt \right] dx \leqslant \frac{e}{1-q} \int_0^\infty x^q f(x) \, dx \,,$$

for every $q < 1$, which is more general than (11).

Using Theorem 5.1 we may obtain some other interesting integral inequalities. One of them is the Cochran–Lee's type inequality.

COROLLARY 5.3. *Let $s = 0$, let $r^n = \lambda > 0$ and q be real numbers such that $2\lambda - 1 > q$. Then* (9) *has the form*

$$\int_0^\infty x^q \exp\left[\frac{\lambda^2}{x^\lambda}\int_0^x t^{\lambda-1}\ln\left|f(t)\right|\,dt\right]dx \leqslant Ce\int_0^\infty x^q\left|f(x)\right|^\lambda\,dx\,,$$

where

$$C = \frac{\lambda}{2\lambda - q - 1}\,.$$

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