Tatra Mt. Math. Publ. **35** (2007), 25–40

TATRA MOUNTAINS
Mathematical Publications

ON $\mathcal{I}$-DIFFERENTIATION

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ABSTRACT. In this article we investigate a differentiation of some functions defined on the corps $\mathcal{K}$ with respect to some subset of $\mathcal{K}$.

Let $\mathcal{K}$ be a field of all reals (or of all complex numbers) and let $\mathcal{I} \subset \mathcal{K}$ be a nonempty set such that 0 is an accumulation point of $\mathcal{I}$. Moreover, let Y be a real (or a complex) Banach space.

1. $\mathcal{I}$-differentiability

If $f\colon \mathcal{K} \to Y$ is a function and $h \in \mathcal{K}$ then we define the $\mathcal{I}$-derivative of f at the point $x \in \mathcal{K}$ in the direction h as the limit

$$D_h^{\mathcal{I}} f(x) = \lim_{\mathcal{I} \ni r \to 0} \frac{f(x + rh) - f(x)}{r},$$

whenever the above limit exists and belongs to Y.

If $\mathcal{K} = \mathcal{I}$ and $h = 1$ then we obtain the ordinary derivative of f at the point x.

By standard reasonings we can prove that for $a, b \in \mathcal{K}$ the equality

$$D_h^{\mathcal{I}}(af + bg)(x) = aD_h^{\mathcal{I}} f(x) + bD_h^{\mathcal{I}} g(x)$$

holds, whenever $D_h^{\mathcal{I}} f(x)$ and $D_h^{\mathcal{I}} g(x)$ exist and belong to Y.

Evidently, if f is differentiable at x in the ordinary sense then for each $h \in \mathcal{K}$ and each set $\mathcal{I} \subset \mathcal{K}$ having the point 0 as an accumulation point we obtain

$$D_h^{\mathcal{I}} f(x) = hf'(x).$$

An interesting case is that one where $\mathcal{I} = Q$ (Q denoting the set of all rationals). Then for the function of Dirichlet

$$f(x) = 1 \text{ for } x \in Q \text{ and } f(x) = 0 \text{ otherwise on } \mathcal{R}$$

2000 Mathematics Subject Classification: 26A24, 26A99, 30A99.
Keywords: $\mathcal{I}$-derivative, direction, strongly $\mathcal{I}$-derivative.

and for $h \in Q$ we have

$$D_h^Q f(x) = 0 \text{ for all } x \in \mathcal{R}.$$

So there are discontinuous functions $f\colon \mathcal{R} \to \mathcal{R}$ for which the derivatives $D_h^Q f(x)$ exist and are finite at each point $x \in \mathcal{R}$ in the rational directions h.

However,

Remark 1. Suppose that the set $\mathcal{I}$ is dense in $\mathcal{K}$. If a function $f\colon \mathcal{K} \to Y$ is continuous and if for some $h \in \mathcal{K}$ there exists the derivative $D_h^{\mathcal{I}} f(x_0) \in Y$ at a point x_0 then there is $D_h^{\mathcal{K}} f(x_0) = D_h^{\mathcal{I}} f(x_0)$.

P r o o f. Take a sequence (t_n) of numbers $t_n \in \mathcal{K} \setminus \{0\}$ which converges to 0. Since the set $\mathcal{I}$ is dense in $\mathcal{K}$ and f is continuous, for each positive integer n there is an element $r_n \in \mathcal{I} \setminus \{0\}$ such that

$$|r_n - t_n| < \frac{1}{n}$$

and

$$\left\| \frac{f(x_0 + r_n h) - f(x_0)}{r_n} - \frac{f(x_0 + t_n h) - f(x_0)}{t_n} \right\| < \frac{1}{n}. \tag{1}$$

Since

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty} t_n = 0,$$

from (1) it follows that

$$\lim_{n\to\infty} \frac{f(x_0 + t_n h) - f(x_0)}{t_n} = \lim_{n\to\infty} \frac{f(x_0 + r_n h) - f(x_0)}{r_n} = D_h^{\mathcal{I}} f(x_0)$$

and the proof is finished. □

By a standard proof we obtain:

Remark 2. Let $f\colon \mathcal{K} \to Y$ be a function and let $h \in \mathcal{K}$ be a number. If the derivative $D_h^{\mathcal{I}} f(x) \in Y$ exists then

$$\lim_{\mathcal{I} \ni r \to 0} f(x + rh) = f(x).$$

Now we consider the product of two functions $f\colon \mathcal{K} \to \mathcal{K}$ and $g\colon \mathcal{K} \to Y$.

Remark 3. If for some numbers $x, h \in \mathcal{K}$ the derivatives $D_h^{\mathcal{I}} f(x) \in \mathcal{K}$ and $D_h^{\mathcal{I}} g(x) \in Y$ exist, then there exists the derivative

$$D_h^{\mathcal{I}}(fg)(x) = f(x) D_h^{\mathcal{I}} g(x) + D_h^{\mathcal{I}} f(x) g(x).$$

Proof. We repeat the standard proof. Observe that

$$\begin{aligned}\frac{f(x+rh)g(x+rh)-f(x)g(x)}{r} &= f(x+rh)\frac{g(x+rh)-g(x)}{r}+\frac{f(x+rh)-f(x)}{r}g(x)\\ &\longrightarrow f(x)D_h^{\mathcal{I}}g(x)+D_h^{\mathcal{I}}f(x)g(x)\,,\end{aligned}$$

whenever $\mathcal{I}\ni r\to 0$. □

Similarly, by a standard proof we obtain:

Remark 4. Let $f\colon \mathcal{K}\to Y$ and $g\colon \mathcal{K}\to\mathcal{K}$ be functions such that for some $h,x\in\mathcal{K}$ the derivatives $D_h^{\mathcal{I}}f(x)\in Y$ and $D_h^{\mathcal{I}}g(x)\in\mathcal{K}$ exist. If $g(x)\neq 0$ then there exist derivatives

$$D_h^{\mathcal{I}}\left(\frac{1}{g}\right)(x)=-\frac{D_h^{\mathcal{I}}g(x)}{g^2(x)}$$

and

$$D_h^{\mathcal{I}}\left(\frac{f}{g}\right)(x)=\frac{g(x)D_h^{\mathcal{I}}f(x)-D_h^{\mathcal{I}}g(x)f(x)}{g^2(x)}\,.$$

A generalization of the product is a bilinear continuous function.

Let Y_1, Y_2, Z be linear spaces over $\mathcal{K}$ with norms $||\cdot||_1$, $||\cdot||_2$ and $||\cdot||_3$, respectively and let $\phi\colon Y_1\times Y_2\to Z$ be a bilinear continuous function. For functions $f_i\colon\mathcal{K}\to Y_i$, $i=1,2$, we define

$$\psi(x)=\phi\big(f_1(x),f_2(x)\big)\,. \tag{$*$}$$

Remark 5. If the derivatives $D_h^{\mathcal{I}}f_i(x)\in Y_i$ for $i=1,2$ exist, then there exists the derivative

$$D_h^{\mathcal{I}}\psi(x)=\phi\big(D_h^{\mathcal{I}}f_1(x),f_2(x)\big)+\phi\big(f_1(x),D_h^{\mathcal{I}}f_2(x)\big)\,.$$

Proof. If

$$\mathcal{I}\setminus\{0\}\ni r_n\to 0 \ \text{ with } \ n\to\infty$$

then

$$\begin{aligned}A_n &= \frac{\psi(x+r_nh)-\psi(x)}{r_n}=\frac{\phi\big(f_1(x+r_nh),f_2(x+r_nh)\big)-\phi\big(f_1(x),f_2(x)\big)}{r_n}\\ &= \frac{\phi\big(f_1(x+r_nh),f_2(x+r_nh)\big)-\phi\big(f_1(x),f_2(x+r_nh)\big)}{r_n}\\ &\quad+\frac{\phi\big(f_1(x),f_2(x+r_nh)\big)-\phi\big(f_1(x),f_2(x)\big)}{r_n}\,.\end{aligned}$$

Since ϕ is continuous, we have

$$\lim_{n\to\infty}A_n=\phi\big(D_h^{\mathcal{I}}f_1(x),f_2(x)\big)+\phi\big(f_1(x)D_h^{\mathcal{I}}f_2(x)\big)\,,$$

and the proof is completed. □

EXAMPLE 1. Let $f\colon \mathcal{R} \to \mathcal{R}$ be the function of Dirichlet and let $g(x) = ax$, where $x \in \mathcal{R}$ and a is an irrational number. Then

$$D_h^Q f(x) = 0 \text{ for all } x \in \mathcal{R} \text{ and } h \in Q$$

and g is differentiable at each point $x \in \mathcal{R}$ and $g'(x) = a$.

However, the derivative $D_1^Q (f \circ g)(0)$ does not exist, since we have

$$\lim_{Q \ni r \to 0+} \frac{f\big(g(r)\big) - f\big(g(0)\big)}{r} = \lim_{Q \ni r \to 0+} \frac{f(ar) - f(0)}{r} = \lim_{Q \ni r \to 0+} \frac{0-1}{r} = -\infty .$$

Thus the classical theorem on ordinary differentiation of composition is not true for the derivatives $D_h^{\mathcal{I}} f$.

However, we have:

Remark 6. Let $f\colon \mathcal{K} \to Y$ and $g\colon \mathcal{K} \to \mathcal{K}$ be functions for which at a point $x \in \mathcal{K}$ the derivatives $D_h^{\mathcal{I}} f\big(g(x)\big) \in Y$ and $D_h^{\mathcal{I}} g(x) \in \mathcal{K}$ exist, $h \neq 0$. If

(a) f is differentiable at the point $g(x)$;

or

(b) $g(x + h\mathcal{I}) \subset g(x) + h\mathcal{I}$;

then there is the derivative

(c)
$$D_h^{\mathcal{I}} (f \circ g)(x) = \frac{D_h^{\mathcal{I}} g(x) D_h^{\mathcal{I}} f\big(g(x)\big)}{h} .$$

P r o o f. Let (r_n) be a sequence of numbers $r_n \in \mathcal{I} \setminus \{0\}$ such that $\lim_{n\to\infty} r_n = 0$. For $n \geqslant 1$ let $u_n = \frac{g(x+r_n h) - g(x)}{h}$. If $u_n = 0$ then

$$f\big(g(x + r_n h)\big) = f\big(g(x)\big) \text{ and } \frac{f\big(g(x+r_n h)\big) - f\big(g(x)\big)}{r_n} = \frac{g(x + r_n h) - g(x)}{r_n} = 0 . \tag{1}$$

If $u_n \neq 0$ and $h \neq 0$ then

$$\frac{f\big(g(x+r_n h)\big) - f\big(g(x)\big)}{r_n} = \frac{f\big(g(x) + u_n h\big) - f\big(g(x)\big)}{u_n} \cdot \frac{g(x + r_n h) - g(x)}{r_n h} . \tag{2}$$

Since $\lim_{n\to\infty} r_n = 0$, we obtain $\lim_{n\to\infty} u_n = 0$. So in the case (a) by (2) and eventually by (1) we obtain equality (c). In the contrary case we use (b) for observing that $u_n \in \mathcal{I}$ for $n \geqslant 1$ and we repeat the reasoning. $\square$

Example 2. Let

$$f(x)=\begin{cases}\sqrt{\frac{3}{2}}x & \text{for } x\in\left\{\sqrt{2}t, t\in Q\right\},\\ \sqrt{\frac{2}{3}}x & \text{for } x\in\left\{\sqrt{3}t, t\in Q\right\},\\ x & \text{for otherwise on } \mathcal{R}.\end{cases}$$

Then there is the inverse function $f^{-1}\colon \mathcal{R}\to\mathcal{R}$ defined by

$$f^{-1}(x)=\begin{cases}\sqrt{\frac{2}{3}}x & \text{for } x\in\left\{\sqrt{2}t, t\in Q\right\},\\ \sqrt{\frac{3}{2}}x & \text{for } x\in\left\{\sqrt{3}t, t\in Q\right\},\\ x & \text{for otherwise on } \mathcal{R}.\end{cases}$$

Observe that for $h=\sqrt{2}$ there is

$$D^{Q}_{\sqrt{2}}f(0)=\sqrt{3} \text{ if } x\in\left\{\sqrt{2}t, t\in Q\right\}$$

and for $h=\sqrt{3}$

$$D^{Q}_{\sqrt{3}}f(0)=\sqrt{2} \text{ if } x\in\left\{\sqrt{3}t, t\in Q\right\}.$$

Similarly,

$$D^{Q}_{\sqrt{2}}f^{-1}(0)=\frac{2}{\sqrt{3}} \text{ if } x\in\left\{\sqrt{2}t, t\in Q\right\}$$

and

$$D^{Q}_{\sqrt{3}}f^{-1}(0)=\frac{3}{\sqrt{2}} \text{ if } x\in\left\{\sqrt{3}t, t\in Q\right\}.$$

Moreover, the inverse function f^{-1} is continuous at 0. However, if $h=\sqrt{2}$ then

$$D^{Q}_{\sqrt{2}}f^{-1}\big(f(0)\big)=D^{Q}_{\sqrt{2}}f^{-1}(0)=\frac{2}{\sqrt{3}}\neq\frac{1}{\sqrt{3}}=\frac{1}{D^{Q}_{\sqrt{2}}f(0)}$$

and if $h=\sqrt{3}$ then

$$D^{Q}_{\sqrt{3}}f^{-1}\big(f(0)\big)=D^{Q}_{\sqrt{3}}f^{-1}(0)=\frac{3}{\sqrt{2}}\neq\frac{1}{\sqrt{2}}=\frac{1}{D^{Q}_{\sqrt{3}}f(0)}.$$

Thus the classical theorem on derivatives of inverse functions is not true for the directional derivatives $D^{\mathcal{I}}_{h}f$. However, we can prove the following:

Remark 7. Let $A,B\subset\mathcal{K}$ be nonempty open sets and let $f\colon A\to B$ be an one-to-one function such that at a point $x\in\mathcal{K}$ the derivative $D^{\mathcal{I}}_{h}f(x)\in\mathcal{K}\setminus\{0\}$ in direction $h\in\mathcal{K}\setminus\{0\}$ exists. Moreover we suppose that

$$f^{-1}\big(f(x)+h\mathcal{I}\big)\subset x+h\mathcal{I}. \tag{1}$$

If the function f^{-1} is continuous at the point $f(x)$ then there exists the derivative
$$D_h^{\mathcal{I}} f^{-1}\big(f(x)\big) = \frac{h^2}{D_h^{\mathcal{I}} f(x)}\,.$$

P r o o f. Find a sequence (u_n) of elements $u_n \in \mathcal{I}\setminus\{0\}$ such that $\lim\limits_{n\to\infty} u_n = 0$ and $h \neq 0$. For $n \geqslant 1$ let
$$x_n = f^{-1}\big(f(x) + u_n h\big) \quad \text{and} \quad r_n = \frac{1}{h}(x_n - x)\,.$$
Observe that
$$r_n \in \mathcal{I} \quad \text{and} \quad \lim_{n\to\infty} r_n = 0\,.$$
Since
$$x_n - x = r_n h \quad \text{and} \quad u_n = \frac{f(x_n) - f(x)}{h}\,,$$
we have
$$\frac{f^{-1}(f(x) + u_n h) - f^{-1}\big(f(x)\big)}{u_n} = \frac{1}{\frac{u_n}{x_n - x}} = \frac{1}{\frac{f(x_n) - f(x)}{h^2 r_n}}$$
$$= \frac{1}{\frac{f(x + r_n h) - f(x)}{r_n h^2}} \longrightarrow \frac{h^2}{D_h^{\mathcal{I}} f(x)}$$
whenever $n \to \infty$. This completes the proof. □

Remark 8. Let $f\colon \mathcal{K} \to Y$ be a function such that for some $h \in \mathcal{I}$ and $x \in \mathcal{K}$ there exists the derivative $D_h^{\mathcal{I}} f(x) \in Y$. Suppose that $a \in \mathcal{K} \setminus \{0\}$ is such that $a\mathcal{I} \subset \mathcal{I}$. Then
$$D_{ah}^{\mathcal{I}} f(x) = a D_h^{\mathcal{I}} f(x)\,.$$

P r o o f. Observe that for $r \in \mathcal{I} \setminus \{0\}$ the element $ar \in \mathcal{I}$ and
$$\frac{f\big(x + r(ah)\big) - f(x)}{r} = a\frac{f\big(x + (ar)h\big) - f(x)}{ar} \longrightarrow a D_h^{\mathcal{I}} f(x)\,,$$
whenever $r \to 0$. This finishes the proof. □

Now, we consider additive functions from $\mathcal{K} \to Y$. A function $f\colon \mathcal{K} \to Y$ is additive if it satisfies Cauchy's equation
$$f(x + y) = f(x) + f(y) \ \text{ for } \ x, y \in \mathcal{K}\,. \tag{$*$}$$

It is well-known ([2]) that every additive function $f\colon \mathcal{K} \to Y$ is homogeneous over the field Q, i.e.,
$$f(rx) = rf(x) \quad \text{for } x \in \mathcal{K} \quad \text{and} \quad r \in Q\,.$$

Moreover, for each field $\mathcal{F} \supset Q$ of reals there is an additive function $f\colon \mathcal{R} \to \mathcal{R}$ such that
$$\{s \in \mathcal{R}; (\forall\, x \in \mathcal{R}) f(sx) = sf(x)\} = \mathcal{F} \ \ ([2]) .$$

Remark 9. If an additive function $f\colon \mathcal{K} \to Y$ is homogeneous over $\mathcal{I}$ then for each point $x \in \mathcal{K}$ and for each direction $h \in \mathcal{K}$ there exists the directional derivative
$$D_h^{\mathcal{I}} f(x) = f(h) .$$

P r o o f . For all $x, h \in \mathcal{K}$ and for each $r \in \mathcal{I} \setminus \{0\}$ we have
$$\frac{f(x+rh) - f(x)}{r} = \frac{f(x) + rf(h) - f(x)}{r} = f(h) ,$$
from which our remark follows. □

Corollary 1. *If $f\colon \mathcal{K} \to Y$ is an additive function then for all $x, h \in \mathcal{K}$, we have*
$$D_h^{Q} f(x) = f(h) .$$

For a function $f\colon \mathcal{K} \to Y$, which has directional derivatives $D_h^{\mathcal{I}} f(x)$ at a point x in all directions $h \in \mathcal{K} \setminus \{0\}$, consider the function
$$\mathcal{K} \setminus \{0\} \ni h \longrightarrow g_f(h) = D_h^{\mathcal{I}} f(x)$$
putting moreover $g_f(0) = D_0^{\mathcal{I}} f(x) = 0$.

There are functions $f\colon \mathcal{R} \to \mathcal{R}$ such that the function g_f is not additive. For example, if
$$f(x) = x \quad \text{for } x \in Q \quad \text{and} \quad f(x) = \sqrt{2}x \quad \text{for } x \in \mathcal{R} \setminus Q ,$$
then
$$D_1^{Q} f(0) = 1 \text{ if } h = 1 \quad \text{and} \quad D_{\sqrt{2}}^{Q} f(0) = 2 \text{ if } h = \sqrt{2}$$
and
$$2 + \sqrt{2} = D_{1+\sqrt{2}}^{Q} f(0) \neq 1 + 2 = D_1^{Q} f(0) + D_{\sqrt{2}}^{Q} f(0) .$$

However, for all additive functions $f\colon \mathcal{K} \to Y$ and for all functions $f\colon \mathcal{K} \to Y$ differentiable in an ordinary sense at x, the corresponding functions $h \to g_f(h)$ are additive.

Now we introduce the following definition:

Definition. A function $f\colon \mathcal{K} \to Y$ is said to be $\mathcal{I}$-differentiable at a point x if for each $h \in \mathcal{K}$ there is a directional derivative $D_h^{\mathcal{I}} f(x) \in Y$ and if the associated function $\mathcal{K} \ni h \longrightarrow D_h^{\mathcal{I}} f(x)$ is additive.

If $a, b \in \mathcal{K}$ and the functions $f, g \colon \mathcal{K} \to Y$ are $\mathcal{I}$-differentiable at x then for all $h_1, h_2 \in \mathcal{K}$ we have

$$\begin{aligned}
D^{\mathcal{I}}_{h_1+h_2}(af+bg)(x) &= aD^{\mathcal{I}}_{h_1+h_2}f(x) + bD^{\mathcal{I}}_{h_1+h_2}g(x)\\
&= a\big(D^{\mathcal{I}}_{h_1}f(x) + D^{\mathcal{I}}_{h_2}f(x)\big) + b\big(D^{\mathcal{I}}_{h_1}g(x) + D^{\mathcal{I}}_{h_2}g(x)\big)\\
&= D^{\mathcal{I}}_{h_1}(af+bg)(x) + D^{\mathcal{I}}_{h_2}(af+bg)(x)\,.
\end{aligned}$$

Therefore the set of all $\mathcal{I}$-differentiable functions from $\mathcal{K}$ to Y is a linear space.

Moreover, if the functions $f \colon \mathcal{K} \to Y$ and $g \colon \mathcal{K} \to \mathcal{K}$ are $\mathcal{I}$-differentiable at x, then for all $h_1, h_2 \in \mathcal{K}$, we have

$$\begin{aligned}
D^{\mathcal{I}}_{h_1+h_2}(gf)(x) &= D^{\mathcal{I}}_{h_1+h_2}g(x)f(x) + g(x)D^{\mathcal{I}}_{h_1+h_2}f(x)\\
&= \big(D^{\mathcal{I}}_{h_1}g(x) + D^{\mathcal{I}}_{h_2}g(x)\big)f(x) + g(x)\big(D^{\mathcal{I}}_{h_1}f(x) + D^{\mathcal{I}}_{h_2}f(x)\big)\\
&= D^{\mathcal{I}}_{h_1}(gf)(x) + D^{\mathcal{I}}_{h_2}(gf)(x)\,.
\end{aligned}$$

Thus the product of two $\mathcal{I}$-differentiable functions $g \colon \mathcal{K} \to \mathcal{K}$ and $f \colon \mathcal{K} \to Y$ is also an $\mathcal{I}$-differentiable function.

Remark 10. If a function $f \colon \mathcal{K} \to Y$ is differentiable in an ordinary sense at a point x and a function $g \colon \mathcal{K} \to \mathcal{K}$ is $\mathcal{I}$-differentiable at this point x, then the superposition $f \circ g$ is $\mathcal{I}$-differentiable at x.

P r o o f. Observe that for all $h_1, h_2 \in \mathcal{K} \setminus \{0\}, h_1 + h_2 \neq 0$ we have

$$\begin{aligned}
D^{\mathcal{I}}_{h_1+h_2}(f\circ g)(x) &= \frac{D^{\mathcal{I}}_{h_1+h_2}g(x)D^{\mathcal{I}}_{h_1+h_2}f\big(g(x)\big)}{h_1+h_2}\\
&= \frac{\big(D^{\mathcal{I}}_{h_1}g(x) + D^{\mathcal{I}}_{h_2}g(x)\big)f'\big(g(x)\big)(h_1+h_2)}{h_1+h_2}\\
&= D^{\mathcal{I}}_{h_1}g(x)f'\big(g(x)\big) + D^{\mathcal{I}}_{h_2}g(x)f'\big(g(x)\big)\\
&= \frac{D^{\mathcal{I}}_{h_1}g(x)f'\big(g(x)\big)h_1}{h_1} + \frac{D^{\mathcal{I}}_{h_2}g(x)f'\big(g(x)\big)h_2}{h_2}\\
&= \frac{D^{\mathcal{I}}_{h_1}g(x)D^{\mathcal{I}}_{h_1}f\big(g(x)\big)}{h_1} + \frac{D^{\mathcal{I}}_{h_2}g(x)D^{\mathcal{I}}_{h_2}f\big(g(x)\big)}{h_2}\\
&= D^{\mathcal{I}}_{h_1}(f\circ g)(x) + D^{\mathcal{I}}_{h_2}(f\circ g)(x)
\end{aligned}$$

and the proof is completed. □

In connection with the last remark we show that there is an additive function $g \colon \mathcal{R} \to \mathcal{R}$ and a differentiable (in the ordinary sense) function $f \colon \mathcal{R} \to \mathcal{R}$ such that the superposition $g \circ f$ does not have the directional derivative $D^{Q}_{1}(g\circ f)$ at the point 0.

EXAMPLE 3. Let H be a Hamel basis in the linear space $\mathcal{R}$ over the field Q. Then H is of cardinality continuum. Let $A = \{a_i; i \geqslant 1\}$ be an infinite countable subset of H. For each index $i \geqslant 1$ there exists a non-zero rational r_i such that

$$|r_i a_i| < \frac{1}{i^2}.$$

Let $b_i = r_i a_i$. The set

$$H' = (H \setminus A) \cup \{b_i; i \geqslant 1\}$$

is also a Hamel basis in $\mathcal{R}$ over Q. Let $g_{H'}\colon H' \to \mathcal{R}$ be defined by the formulas

$$g(b_i) = i \text{ for } i \geqslant 1 \text{ and let } g(b) = 0 \text{ for } b \in H \setminus A,$$

and let $g\colon \mathcal{R} \to \mathcal{R}$ be an additive extension of $g_{H'}$. There is a differentiable (in an ordinary sense) function $f\colon \mathcal{R} \to \mathcal{R}$ such that

$$f\left(\frac{1}{i}\right) = b_i \text{ for } i \geqslant 1.$$

Then

$$f(0) = f'(0) = 0.$$

Observe that

$$g\left(f\left(\frac{1}{i}\right)\right) = g(b_i) = i \text{ for } i \geqslant 1.$$

So, for $i \geqslant 1$ we have

$$\frac{g\left(f(\frac{1}{i})\right) - g\left(f(0)\right)}{\frac{1}{i}} = i^2 \longrightarrow \infty \text{ whenever } i \longrightarrow \infty.$$

This means that the finite directional derivative $D_1^Q(g \circ f)(0)$ does not exist.

2. Strong $\mathcal{I}$-differentiability

Let $f\colon \mathcal{K} \to Y$ be a function and let $x_0 \in \mathcal{K}$ be a point. If the limit

$$\lim_{\mathcal{K}\times\mathcal{I}\ni(x,r)\to(x_0,0)} \frac{f(x+rh) - f(x)}{r} = Ds_h^{\mathcal{I}} f(x_0)$$

exists and belongs to Y, then it is called the strong $\mathcal{I}$-derivative of f at x_0 in direction $h \in \mathcal{K}$.

We shall say that f is strongly $\mathcal{I}$-differentiable if $Ds_h^{\mathcal{I}} f(x_0) \in Y$ exists for all $h, x_0 \in \mathcal{K}$.

Observe that if the functions $f, g\colon \mathcal{K} \to Y$ are strongly $\mathcal{I}$-differentiable at a point x in the direction $h \in \mathcal{K}$ and if $a, b \in \mathcal{K}$, then the function $af + bg$ is strongly $\mathcal{I}$-differentiable at x in the direction h and

$$Ds_h^{\mathcal{I}}(af + bg)(x) = aDs_h^{\mathcal{I}} f(x) + bDs_h^{\mathcal{I}} g(x) .$$

The definition of a strong Q-differentiability of functions $f\colon \mathcal{R} \to \mathcal{R}$ was introduced by Z. Boros in [1], where he shows that a function $f\colon \mathcal{R} \to \mathcal{R}$ is strongly Q-differentiable if and only if there exists a continuously differentiable function $g\colon \mathcal{R} \to \mathcal{R}$ and an additive function $\phi\colon \mathcal{R} \to \mathcal{R}$ such that

$$f(x) = g(x) + \phi(x) \ \text{ for } \ x \in \mathcal{R} .$$

Remark 11. If a function $f\colon \mathcal{K} \to Y$ is strongly $\mathcal{I}$-differentiable at a point $x \in \mathcal{K}$ in the direction h, $a \in \mathcal{K}$ with $a \neq 0$ is an element such that $a\mathcal{I} \subset \mathcal{I}$, then f is strongly $\mathcal{I}$-differentiable at x in direction ah and

$$Ds_{ah}^{\mathcal{I}} f(x) = aDs_h^{\mathcal{I}} f(x) .$$

P r o o f. For $a \neq 0$ we have

$$\frac{f\big(x + r(ah)\big) - f(x)}{r} = a\frac{f\big(x + (ra)h\big) - f(x)}{ra} \longrightarrow aDs_h^{\mathcal{I}} f(x) ,$$

whenever $\mathcal{I} \ni r \to 0$.

In this part we show that the results of Boros are also true for functions from $\mathcal{K} \to Y$. In the proofs we apply the methods of Boros.

Since for all additive functions $f\colon \mathcal{K} \to Y$ and for all $h, x \in \mathcal{K}$ and $r \in Q\setminus\{0\}$ we have

$$\frac{f(x + rh) - f(x)}{r} = f(h) ,$$

each additive function from $\mathcal{K}$ to Y is strongly Q-differentiable. $\square$

Remark 12. Each continuously differentiable function $f\colon \mathcal{K} \to Y$ is strongly $\mathcal{I}$-differentiable. Moreover, for all $h, x_0 \in \mathcal{K}$ the equality $Ds_h^{\mathcal{I}} f(x_0) = f'(x_0)h$ is true.

P r o o f. Fix elements $h, x_0 \in \mathcal{K}$. For each $x \in \mathcal{K}$ and $r \in \mathcal{I}$ we have

$$f(x + rh) = f(x) + f'(x)rh + rh\eta(x + rh), \ \text{ where } \ \lim_{r\to 0} \eta(x + rh) = 0 .$$

Thus,

$$\frac{f(x + rh) - f(x)}{r} = f'(x)h + h\eta(x + rh) \longrightarrow f'(x_0)h ,$$

whenever $x \to x_0$ and $r \to 0$. $\square$

Remark 13. If $f\colon \mathcal{K} \to Y$ is strongly $\mathcal{I}$-differentiable then it is $\mathcal{I}$-differentiable at each point, and $D_h^{\mathcal{I}} f(x_0) = Ds_h^{\mathcal{I}} f(x_0)$ for $h \in \mathcal{K}$.

Proof. Fix $h, x_0 \in \mathcal{K}$. For $r \in \mathcal{I} \setminus \{0\}$ we have

$$\frac{f(x_0 + rh) - f(x_0)}{r} \longrightarrow Ds_h^{\mathcal{I}} f(x_0)\,,$$

whenever $r \to 0$. Hence

$$D_h^{\mathcal{I}} f(x_0) = Ds_h^{\mathcal{I}} f(x_0)\,.$$

Now, fix elements $h_1, h_2 \in \mathcal{K}$. Then

$$\begin{aligned}
D_{h_1+h_2}^{\mathcal{I}} f(x_0) &= Ds_{h_1+h_2}^{\mathcal{I}} f(x_0) \\
&= \lim_{\mathcal{K}\times\mathcal{I}\ni(x,r)\to(x_0,0)} \frac{f\big(x + r(h_1 + h_2)\big) - f(x)}{r} \\
&= \lim_{\mathcal{K}\times\mathcal{I}\ni(x,r)\to(x_0,0)} \frac{f(x + rh_1 + rh_2) - f(x + rh_1)}{r} \\
&\quad + \lim_{\mathcal{K}\times\mathcal{I}\ni(x,r)\to(x_0,0)} \frac{f(x + rh_1) - f(x)}{r} \\
&= Ds_{h_2}^{\mathcal{I}} f(x_0) + Ds_{h_1}^{\mathcal{I}} f(x_0) = D_{h_2}^{\mathcal{I}} f(x_0) + D_{h_1}^{\mathcal{I}} f(x_0)\,.
\end{aligned}$$

Now the associated function $h \longrightarrow D_h^{\mathcal{I}} f(x_0)$ is additive and the proof is finished. □

Remark 14. Let $f\colon \mathcal{K} \to Y$ be strongly $\mathcal{I}$-differentiable, $h_0 \in \mathcal{K}$ and for $x \in \mathcal{K}$ let $\psi(x) = Ds_{h_0}^{\mathcal{I}} f(x)$. Then the function $\psi\colon \mathcal{K} \to Y$ is continuous.

Proof. Fix a real $\eta > 0$. According to the definition, for each $x \in \mathcal{K}$ there is $\delta(x) > 0$ such that

$$\left\| \frac{f(y + rh_0) - f(y)}{r} - Ds_{h_0}^{\mathcal{I}} f(x) \right\| < \frac{\eta}{2} \tag{1}$$

for all y with $|y - x| < \delta(x)$ and $r \in \big((-\delta(x), \delta(x)) \cap \mathcal{I}\big) \setminus \{0\}$. For a fixed $x_0 \in \mathcal{K}$ take

$$x \in \Big(x_0 - \frac{\delta(x_0)}{2}, x_0 + \frac{\delta(x_0)}{2}\Big)\,,$$

and choose $y \in \mathcal{K}$ and $r \in \mathcal{I} \setminus \{0\}$ such that

$$|y - x| < \min\left(\delta(x), \frac{\delta(x_0)}{2}\right) \text{ and } |r| < \min\big(\delta(x), \delta(x_0)\big)\,.$$

Then

$$|y - x_0| \leqslant |y - x| + |x - x_0| < 2\frac{\delta(x_0)}{2} = \delta(x_0)\,.$$

Therefore (1) holds for r, y. From this it follows that

$$\begin{aligned}\|\psi(x)-\psi(x_0)\| &= \|Ds^{\mathcal{I}}_{h_0}f(x)-Ds^{\mathcal{I}}_{h_0}f(x_0)\|\\ &\leqslant \left\|Ds^{\mathcal{I}}_{h}f(x)-\frac{f(y+rh_0)-f(y)}{r}\right\|\\ &\quad+\left\|\frac{f(y+rh_0)-f(y)}{r}-Ds^{\mathcal{I}}_{h_0}f(x_0)\right\|<\frac{\eta}{2}+\frac{\eta}{2}=\eta\,.\end{aligned}$$

This finishes the proof. □

Lemma 1. *Suppose that $\mathcal{I}\subset\mathcal{R}$ such that $\mathcal{I}$ is dense in $\mathcal{R}$ and $\mathcal{I}-\mathcal{I}\subset\mathcal{I}$ (i.e., $\mathcal{I}$ is a dense subgroup of $(\mathcal{R},+)$). Let $f\colon\mathcal{K}\to Y$ be a strongly $\mathcal{I}$-differentiable function. Moreover, suppose that there are $a,b\in\mathcal{R}$ and $h\in\mathcal{K}$ such that $a<b$ and $h\neq 0$, and there is a real constant $M\geqslant 0$ for which*

$$\|Ds^{\mathcal{I}}_{h}f(x)\|\leqslant M \quad \text{for each } x\in[a,b]h=\{th;t\in[a,b]\}\,.$$

If for $u,v\in[a,b]$ the relation $v-u\in\mathcal{I}$ holds, then

$$\|f(uh)-f(vh)\|\leqslant M|u-v|\,.$$

P r o o f . Fix a real $\eta>0$. For each $x\in[a,b]h$ there exists a real $\delta(x)>0$ such that

$$\left\|\frac{f(y+rh)-f(y)}{r}-Ds^{\mathcal{I}}_{h}f(x)\right\|<\eta$$

for all

$$y\in U\bigl(x,\delta(x)\bigr)=\bigl\{z\in\mathcal{K};|z-x|<\delta(x)\bigr\}$$

and

$$r\in\mathcal{I}\cap U\left(0,\frac{\delta(x)}{|h|}\right) \text{ with } r\neq 0\,.$$

Observe that

$$[a,b]h\subset\bigcup_{x\in[a,b]}U\bigl(x,\delta(x)\bigr)\,.$$

Since the segment $[a,b]$ h is compact, there are points $x_1,\dots,x_k\in[a,b]h$ such that

$$[a,b]h\subset\bigcup_{j=1}^{k}U\bigl(x_j,\delta(x_j)\bigr)\,.$$

Let $\delta=\min\bigl\{\delta(x_j);\ j=1,2,\dots,k\bigr\}$. Evidently, $\delta>0$. Find elements $x,y\in[a,b]h$ such that

$$\frac{y-x}{h}\in\mathcal{I} \quad\text{and}\quad |y-x|<\delta\,.$$

There exists an index $m \leqslant k$ with $x \in U\big(x_m, \delta(x_m)\big)$. Then

$$\frac{y-x}{h} \in \mathcal{I} \cap \left(0, \frac{\delta(x_m)}{|h|}\right) \quad \text{and} \quad y = x + \frac{y-x}{h}h\,.$$

From this it follows that

$$\left\| \frac{f(y)-f(x)}{\frac{y-x}{h}} - Ds_h^{\mathcal{I}} f(x_m) \right\| < \eta\,,$$

and consequently

$$\big\|f(y)-f(x)\big\| < \big(\big\|Ds_h^{\mathcal{I}} f(x_m)\big\| + \eta\big)\frac{|y-x|}{|h|} \leqslant (M+\eta)\frac{|y-x|}{|h|}\,.$$

Let us consider $u, v \in [a,b]$ such that $v > u$ and $v - u \in \mathcal{I}$. There are reals $u = y_0 < y_1 < \dots < y_n = v$ belonging to $\mathcal{R}$ such that

$$y_{j+1} - y_j \in \mathcal{I} \quad \text{and} \quad y_{j+1} - y_j < \frac{\delta}{|h|} \quad \text{for } j = 0, 1, \dots, n-1\,.$$

Observe that

$$\begin{aligned}\big\|f(vh) - f(uh)\big\| &= \left\| \sum_{j=1}^{n} \big(f(y_j h) - f(y_{j-1}h)\big) \right\| \leqslant \sum_{j=1}^{n} \big\|f(y_j h) - f(y_{j-1}h)\big\| \\ &\leqslant \sum_{j=1}^{n} (M+\eta)(y_j - y_{j-1}) = (M+\eta)(v-u)\,.\end{aligned}$$

Letting η tend to 0^+ we obtain our statement. □

Corollary 2. *Suppose that $\mathcal{I} \subset \mathcal{R}$ such that $\mathcal{I}$ is dense in $\mathcal{R}$ and $\mathcal{I} - \mathcal{I} \subset \mathcal{I}$ (i.e., $\mathcal{I}$ is a dense subgroup of $(\mathcal{R},+)$). Let $f\colon \mathcal{K} \to Y$ be a strongly $\mathcal{I}$-differentiable function and $h \in \mathcal{K} \setminus \{0\}$ such that $Ds_h^{\mathcal{I}} f(x) = 0$ for each $x \in \mathcal{K}$. If $u, v \in \mathcal{R}$ satisfy $u - v \in \mathcal{I}$ then $f(uh) = f(vh)$.*

Proof. If $u = v$ then evidently $f(uh) = f(vh)$. Take $u, v \in \mathcal{I}$ such that $u < v$ and choose $a, b \in \mathcal{R}$ with $a < u < v < b$. Now, applying Lemma 1 we obtain our corollary. □

Remark 15. Suppose that $\mathcal{I} \subset \mathcal{R}$ such that $\mathcal{I}$ is dense in $\mathcal{R}$ and $\mathcal{I} - \mathcal{I} \subset \mathcal{I}$ (i.e., $\mathcal{I}$ is a dense subgroup of $(\mathcal{R},+)$). If $f\colon \mathcal{R} \to Y$ is a strongly $\mathcal{I}$-differentiable function, then there exists a continuously differentiable function $g\colon \mathcal{R} \to Y$ and an additive function $\phi\colon \mathcal{R} \to Y$ such that

$$f(x) = g(x) + \phi(x) \quad \text{for all } x \in \mathcal{R}\,.$$

P r o o f . Let us assume that a function $f\colon \mathcal{R} \to Y$ is strongly $\mathcal{I}$-differentiable. Since the function

$$\mathcal{R} \ni x \to Ds_1^{\mathcal{I}} f(x)$$

is continuous, we may define

$$g(x) = f(0) + \int_0^x Ds_1^{\mathcal{I}} f(s)ds\,.$$

Then $g\colon \mathcal{R} \to Y$ is a continuously differentiable function and

$$g'(x) = Ds_1^{\mathcal{I}} f(x) \quad \text{for all } x \in \mathcal{R}\,.$$

Let $\phi = f - g$. Then the function $\phi\colon \mathcal{R} \to Y$ is strongly $\mathcal{I}$-differentiable and for all $x, h \in \mathcal{R}$, we obtain

$$\begin{aligned} Ds_h^{\mathcal{I}} \phi(x) &= Ds_h^{\mathcal{I}} f(x) - Ds_h^{\mathcal{I}} g(x) \\ &= Ds_h^{\mathcal{I}} f(x) - g'(x)h = Ds_h^{\mathcal{I}} f(x) - Ds_1^{\mathcal{I}} f(x)h\,. \end{aligned}$$

Thus,

$$Ds_1^{\mathcal{I}} \phi(x) = 0 \quad \text{for all } x \in \mathcal{R}\,.$$

According to the last Corollary 2 we have

$$\phi(x+s) = \phi(s) = \phi(x) \quad \text{for all } x \in \mathcal{R} \text{ and } s \in \mathcal{I}\,.$$

Substituting $x = 0$ we obtain $\phi(0) = f(0) - g(0) = 0$ and

$$\phi(s) = f(s) - g(s) = \phi(0) = 0 \quad \text{for each } s \in \mathcal{I}\,.$$

Now, we will prove that the mapping

$$\mathcal{R} \ni x \to Ds_h^{\mathcal{I}} \phi(x)$$

is constant for each $h \in \mathcal{R}$. Let us fix $h, x, y \in \mathcal{R}$, a sequence (x_n) of elements $x_n \in \mathcal{R}$ with $\lim\limits_{n\to\infty} x_n = x$ and a sequence (r_n) of elements of $\mathcal{I} \setminus \{0\}$ such that $\lim\limits_{n\to\infty} r_n = 0$. For every index n we can choose

$$s_n \in \mathcal{I} \cap U\left(y - x_n, \frac{1}{n}\right) = \left\{u \in \mathcal{I}, \left|u - (y - x_n)\right| < \frac{1}{n}\right\}.$$

Observe that

$$y = \lim_{n\to\infty} (x_n + s_n)$$

and

$$\begin{aligned} Ds_h^{\mathcal{I}} \phi(x) &= \lim_{n\to\infty} \frac{\phi(x_n + r_n h) - \phi(x_n)}{r_n} \\ &= \lim_{n\to\infty} \frac{\phi(x_n + s_n + r_n h) - \phi(x_n + s_n)}{r_n} = Ds_h^{\mathcal{I}} \phi(y)\,. \end{aligned}$$

Now for each $y \in \mathcal{R}$ put

$$\psi_y(x) = \phi(x+y) - \phi(x) \quad \text{for } x \in \mathcal{R}.$$

Then ψ_y is strongly $\mathcal{I}$-differentiable and

$$Ds_h^{\mathcal{I}}\psi_y(x) = Ds_h^{\mathcal{I}}\phi(x+y) - Ds_h^{\mathcal{I}}\phi(x) = 0 \quad \text{for all } x, h \in \mathcal{R}.$$

For any $x \neq 0$ in $\mathcal{R}$ let r denote a positive element of $\mathcal{I}$, $h = x/r$, $u = 0$, and $v = r$. Then $u, v \in \mathcal{R}$ satisfy $v - u = r \in \mathcal{I}$, hence applying Corollary 2 we have

$$\psi_y(0) = \psi_y(uh) = \psi_y(vh) = \psi_y(x).$$

Thus ψ_y is a constant function. There is a function $\kappa\colon \mathcal{R} \to Y$ such that

$$\psi_y(x) = \phi(x+y) - \phi(x) = \kappa(y) \quad \text{for all } x, y \in \mathcal{R}. \tag{1}$$

Substituting $x = 0$ into (1) we obtain that

$$\kappa(y) = \phi(y) - \phi(0) = \phi(y) \quad \text{for each } y \in \mathcal{R}, \text{ i.e., } \kappa = \phi.$$

Equation (1) yields that ϕ is additive and the proof is finished. $\square$

Since every additive function $f\colon \mathcal{R} \to \mathcal{R}$, which is bounded from above (or below) on a measurable set of positive Lebesgue measure or on a set of the second category with the Baire property, is continuously differentiable (it is of the form $f(x) = ax$ ([2])), each strongly $\mathcal{I}$-differentiable function bounded from above or bounded from below on a measurable set of positive measure or on a set of the second category with the Baire property is continuously differentiable.

The measurable functions (the functions with the Baire property) are bounded on some measurable subsets of positive Lebesgue measure (on some subsets of the second category with the Baire property). Hence the measurable (with the Baire property) strongly $\mathcal{I}$-differentiable functions from $\mathcal{R}$ to $\mathcal{R}$ are continuously differentiable.

This is not true in the case where $\mathcal{K} = \mathcal{C}$ ($\mathcal{C}$ denotes the field of all complex numbers). For example, if $f(x+yi) = x - yi$ for $x, y \in \mathcal{R}$ and $i = \sqrt{-1}$, then $f\colon \mathcal{C} \to \mathcal{C}$ is additive and continuous, but it is not differentiable in an ordinary sense. An additive function f is strongly $\mathcal{I}$-differentiable and

$$Ds_h^{\mathcal{I}} f(z) = f(h) \quad \text{for each } z, h \in \mathcal{C}.$$

The authors are thankful to the referee for substantial improvements of Lemma 1, Corollary 2, and Remark 15.

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Received September 27, 2005

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