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BOUNDEDNESS AND CONVERGENCE THEOREMS IN EFFECT ALGEBRAS

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ABSTRACT. We prove a uniform boundedness theorem and a Vitali-Hahn-Saks theorem for modular measures on D-lattices with values in Dedekind complete ℓ-groups.

Introduction

Two of the most important theorems of Measure theory are Nikodym boundedness theorem and Vitali-Hahn-Saks Theorem, see for example [15, chapter 10]. In the last years, several results concerning these theorems have been obtained for measures defined on non-Boolean structures with values in topological groups. We recall [13] for the projection lattice of a Von Neumann algebra, [4] for measures on MV-algebras, [10] for measures on effect algebras and [1] for modular measures on D-lattices (i.e., lattice ordered effect algebras).

The approach to the study of these theorems is different if the measures take their values in non-topological groups because the lack of the topology does not allow to use classical techniques. In [8] it is obtained Nikodym boundedness theorem and Vitaly-Hahn-Saks theorem for measures on Boolean algebras with values in Dedekind complete ℓ-groups, by introducing the concept of $\mathcal{RD}$-convergence. In this paper, following the idea of [8], we extend these results to measures on effect algebras. Precisely, the paper is organized as follows: In Section 1 we give preliminary results, in Section 2 we prove a Nikodym boundedness type theorem for modular measures on D-lattices and in Section 3 we extend classical results concerning absolute continuity and uniform exhaustivity to the context of effect algebras.

Recall that effect algebras have been introduced by D. J. Foulis and M. K. Bennett in 1994 (see [6]) for modelling unsharp measurement in a quantum mechanical system. They are a generalization of many structures which arises in

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quantum physics (see [5]) and in mathematical economics (see [9] and [12]), in particular of orthomodular lattices in non-commutative measure theory and MV-algebras in fuzzy measure theory. After 1994, there have been a great number of papers concerning effect algebras (see [11] for a bibliography).

1. Preliminaries

An *effect algebra* [11] is a set E, with two distinguished elements 0 and 1, and a partially defined operation $\oplus$ such that for all $a, b, c \in E$:

(E1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.

(E2) If $b \oplus c$ is defined and $a \oplus (b \oplus c)$ is defined, then $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

(E3) There exists a unique $a^\perp \in E$ such that $a \oplus a^\perp$ is defined and $a \oplus a^\perp = 1$.

(E4) If $a \oplus 1$ is defined, then $a = 0$.

It is easily seen that $a \oplus 0$ is always defined and equals a. If $a \oplus b$ is defined, we say that a and b are *orthogonal* and write $a \perp b$.
In an effect algebra E another partially defined operation $\ominus$ can be defined by the following rule: $c \ominus a$ exists and equals b if and only if $a \oplus b$ exists and equals c. In particular, $a \oplus a^\perp = 1$. Moreover, if $a \perp b$, then $a \oplus b = (a^\perp \ominus b)^\perp = (b^\perp \ominus a)^\perp$.

If $a_1, \dots a_n \in E$, we inductively define $a_1 \oplus \dots \oplus a_n = (a_1 \oplus \dots \oplus a_{n-1}) \oplus a_n$ provided that the right hand side exists. The definition is independent of permutations of the elements. We say that a finite subset $\{a_1, \dots a_n\}$ of E is orthogonal if $a_1 \oplus \dots \oplus a_n$ exists.
We say that a sequence (a_n) of E is orthogonal if the set $\{a_1, \dots a_n\}$ is orthogonal, for every $n \in \mathbb{N}$. In this case we put $\bigoplus_{n \geqslant 1} a_n = \bigvee_{n \geqslant 1} (a_1 \oplus \dots \oplus a_n)$, provided that the right hand side exists, and we call it the *sum* of the sequence (a_n).

If the sequence (a_n) of E is orthogonal and $A \subseteq \mathbb{N}$, we put

$$\bigoplus_{n \in A} a_n = \bigvee_{F \in \mathcal{F}} \left(\bigoplus_{k \in F} a_k \right),$$

where $\mathcal{F}$ is the set of finite subsets of A, provided that the right hand side exists.

In an effect algebra E a partial ordering relation $\leqslant$ can be defined as follows: $a \leqslant c$ if and only if, for some $b \in E$, $a \oplus b$ exists and equals c. Hence $c \ominus a$ is defined if and only if $a \leqslant c$. Moreover $a \perp b$ if and only if $a \leqslant b^\perp$.
If $a \vee b$ and $a \wedge b$ exist for all $a, b \in E$, then we say that E is a *lattice ordered effect algebra*, or also a D-*lattice*. Moreover if every orthogonal sequence in E

has a sum, we say that E is *σ-complete*. Let us recall that a D-lattice L is an *MV-algebra* if and only if $(a \vee b) \ominus b = a \ominus (a \wedge b)$ for all $a, b \in L$, while L is an *orthomodular lattice* if and only if $a^{\perp} \wedge a = 0$ for every $a \in L$.

We will make use of the following properties (for the proofs we refer to [11]).

1.1. Proposition. *Let E be an effect algebra. For all $a, b, c \in E$ we have:*

(i) *If $a \leqslant b$, then $b \ominus a \leqslant b$ and $b \ominus (b \ominus a) = a$.*

(ii) *If $a \leqslant b$, then $(b \ominus a)^{\perp} = a^{\perp} \ominus b^{\perp}$.*

(iii) *If $a \leqslant b \leqslant c$, then $c \ominus b \leqslant c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.*

(iv) *If $a \leqslant b \leqslant c$, then $b \ominus a \leqslant c \ominus a$ and $(c \ominus a) \ominus (b \ominus a) = c \ominus b$.*

(v) *$a \leqslant b^{\perp}$ if and only if $b \leqslant a^{\perp}$*

(vi) *If $a \leqslant b^{\perp}$ and $a \oplus b \leqslant c$, then $c \ominus (a \oplus b) = (c \ominus a) \ominus b = (c \ominus b) \ominus a$.*

(vii) *If $a \leqslant b \leqslant c^{\perp}$, then $a \oplus c \leqslant b \oplus c$ and $(b \oplus c) \ominus (a \oplus c) = b \ominus a$.*

(viii) *If $a \leqslant b \leqslant c$, then $a \oplus (c \ominus b) = c \ominus (b \ominus a)$ and $(c \ominus b) \oplus (b \ominus a) = c \ominus a$*

(ix) *If $a \leqslant b^{\perp} \leqslant c^{\perp}$, then $a \oplus (b \ominus c) = (a \oplus b) \ominus c$.*

1.2. Proposition. *Let L be a D-lattice. For all $a, b, c \in L$ we have:*

(i) *If $a \leqslant c$ and $b \leqslant c$, then $c \ominus (a \vee b) = (c \ominus a) \wedge (c \ominus b)$ and $c \ominus (a \wedge b) = (c \ominus a) \vee (c \ominus b)$.*

(ii) *If $a \leqslant b^{\perp}$, then $(a \vee b) \leqslant (a \wedge b)^{\perp}$ and $a \oplus b = (a \vee b) \oplus (a \wedge b)$.*

(iii) *If $c \leqslant a$ and $c \leqslant b$, then $(a \wedge b) \ominus c = (a \ominus c) \wedge (b \ominus c)$ and $(a \vee b) \ominus c = (a \ominus c) \vee (b \ominus c)$.*

(iv) *If $a \leqslant c^{\perp}$ and $b \leqslant c^{\perp}$, then $(a \vee b) \oplus c = (a \oplus c) \vee (b \oplus c)$ and $(a \wedge b) \oplus c = (a \oplus c) \wedge (b \oplus c)$.*

For convenience, we recall the following results.

1.3. Proposition. [1, Prop. 2.5 and 2.7] *Let (a_n) be an orthogonal sequence in an effect algebra E and A, $B \subseteq \mathbb{N}$ such that $a = \bigoplus_{n \in A} a_n$ and $b = \bigoplus_{n \in B} a_n$ exist. Then:*

(i) *If A and B are disjoint, then $a \perp b$ and $a \oplus b = \bigoplus_{n \in A \cup B} a_n$.*

(ii) *If $B \subseteq A$, then $\bigoplus_{n \in A \setminus B} a_n$ exists and $a \ominus b = \bigoplus_{n \in A \setminus B} a_n$.*

For a given sequence (x_n) in a given lattice A and for an element x of A, we will write $x_n \downarrow_n x$ (resp. $x_n \uparrow_n x$) if the sequence (x_n) is decreasing (resp. increasing) and $\bigwedge_n x_n = x$ (resp. $\bigvee_n x_n = x$).

1.4. Proposition. [3, Prop. 3.2] *Let (a_n) be a sequence in an effect algebra E and $a, b \in E$. Then:*

(i) *If $a_n \uparrow a$ and $b \geqslant a_n$ for each $n \in \mathbb{N}$, then $b \ominus a_n \downarrow b \ominus a$.*

(ii) *If $a_n \downarrow a$ and $b \leqslant a_n$ for each $n \in \mathbb{N}$, then $a_n \ominus b \downarrow a \ominus b$.*

1.5 Proposition. [2, Lemma 2.1] *Let $a_0, a_1, \dots, a_n$ be in a D-lattice L with $a_0 \leqslant a_1 \leqslant \dots \leqslant a_n$ and let $b_i = a_i \ominus a_{i-1}$ for every $i \in \{1, \dots, n\}$. Then $\{b_1, \dots, b_n\}$ is orthogonal and $b_1 \oplus \dots \oplus b_n = a_n \ominus a_0$.*

An *ℓ-group* is an Abelian group G with an ordering $\leqslant$ such that

$$(\forall x, y, z \in G) x \leqslant y \implies x + z \leqslant y + z\,.$$

An ℓ-group G is said to be *Dedekind σ-complete* if every nonempty countable subset of G, bounded from above, has a supremum.

In an ℓ-group G we define, for every a in G, the *positive part*, the *negative part* and the *modulus* of a, respectively, as

$$a^+ = a \vee 0, \;\; a^- = (-a) \vee 0 \;\text{ and }\; |a| = a^+ + a^-.$$

For basic properties on ℓ-groups, see [7].

If (x_n) is a sequence of positive elements of an ℓ-group G, we set $\sum\limits_{n=1}^{\infty} x_n = \bigvee\limits_n \left(\sum\limits_{k=1}^{n} x_k \right)$ if the right hand side exists.

1.6. Definition. According to [8], we shall call a *regulator* (or *$\mathcal{D}$-sequence*) in an ℓ-group G, a bounded double sequence $(x_{i,j})$ of elements of G such that, for every $i \in \mathbb{N}$, $x_{i,j} \downarrow_j 0$.

We will denote by Φ the set of all mappings $\phi \colon \mathbb{N} \longrightarrow \mathbb{N}$

1.7. Definition. An ℓ-group is said to be *weakly σ-distributive* (see [8] and [16]) if for every bounded double sequence $(y_{i,j})$ such that $y_{i,j} \geqslant y_{i,j+1}$ for all i, j, we have

$$\bigvee_{i=1}^{\infty} \left(\bigwedge_{j=1}^{\infty} y_{i,j} \right) = \bigwedge_{\phi \in \Phi} \left(\bigvee_{i=1}^{\infty} y_{i,\phi(i)} \right).$$

It is easy to check that an ℓ-group is weakly σ-distributive if and only if for every regulator $(x_{i,j})$, we have

$$\bigwedge_{\phi \in \Phi} \left(\bigvee_{i=1}^{\infty} x_{i,\phi(i)} \right) = 0\,.$$

We will need the following result of [14].

1.8. Lemma. *Let $(x^k_{i,j})$ be a countable family of regulators in an ℓ-group G. Then, for every $u \in G$, there exists a regulator $(x_{i,j})$ such that, $\forall \phi \in \Phi$,*

$$u \wedge \sum_{k=1}^{\infty} \Big(\bigvee_i x^k_{i,\phi(i+k)} \Big) \leqslant \bigvee_i x_{i,\phi(i)} \,.$$

(where $u \wedge \sum\limits_{k=1}^{\infty} \Big(\bigvee\limits_i x^k_{i,\phi(i+k)} \Big) = u$ if $\sum\limits_{k=1}^{\infty} \Big(\bigvee\limits_i x^k_{i,\phi(i+k)} \Big)$ is not defined).

1.9. Definition. For a given sequence (r_n) and an element r of an ℓ-group G, we say that (r_n) *$\mathcal{D}$-converges to r* and we write

$$\mathcal{D}\text{-}\lim_n r_n = r$$

if there exists a regulator $(x_{i,j})$ such that $\forall \phi \in \Phi$, $\exists n_0 \in \mathbb{N}$ such that

$$\forall n \geqslant n_0, |r_n - r| \leqslant \bigvee_i x_{i,\phi(i)} \,.$$

In this case we say also that (r_n) *$\mathcal{D}$-converges to r with respect to $(x_{i,j})$* (or relatively to $(x_{i,j})$).

Throughout the paper, the symbol G will always denote a *Dedekind-σ-complete weakly σ-distributive ℓ-group*, E an *effect algebra* and L a D-*lattice.*

A function $\mu\colon E \longrightarrow G$ is said to be a *measure* if, for any $a, b \in E$ with $a \perp b$, $\mu(a \oplus b) = \mu(a) + \mu(b)$. If E is a D-lattice, the measure μ is said to be *modular* if, for any $a, b \in E$, $\mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b)$.

1.10. Proposition. [2, Prop. 2.4] Let $\mu\colon E \longrightarrow [0, \infty[$ be a measure. Then the following conditions are equivalent:

(i) For every orthogonal sequence (a_n) in L such that $a = \bigoplus\limits_n a_n$ exists, $\mu(a) = \sum\limits_{n=1}^{\infty} \mu(a_n)$.

(ii) For every sequence (a_n) and every a in E, if $a_n \uparrow a$, then $\mu(a) = \lim\limits_n \mu(a_n)$.

(iii) For every sequence (a_n) and every a in E, if $a_n \downarrow a$, then $\mu(a) = \lim\limits_n \mu(a_n)$.

(iv) For every sequence (a_n) in E, if $a_n \downarrow 0$, then $\lim\limits_n \mu(a_n) = 0$.

2. Uniform boundedness

In this section we prove a Nikodym boundedness type theorem for modular measures on D-lattices (see Theorem 2.19).

If $\mu: E \longrightarrow G$ is a measure, we define

$$\forall a \in E \quad \bar{\mu}(a) = \sup\{|\mu(b)|: b \leqslant a\}.$$

According to [8], *we give the following definitions.*

2.1. Definition. We say that a function $\mu: E \longrightarrow G$ is *bounded* if there exists $u \in G$ such that $|\mu(a)| \leqslant u$ for every $a \in E$.

2.2. Definition. We say that a measure $\mu: E \longrightarrow G$ is *σ-additive* if there exists a regulator $(x_{i,j})$ in G such that, for every sequence (c_n) in E such that $c_n \downarrow 0$, then $\bar{\mu}(c_n)$ $\mathcal{D}$-converges to 0 with respect to $(x_{i,j})$. In this case we say also that μ is *σ-additive with respect to* $(x_{i,j})$.

2.3. Definition. We say that function $\mu: E \longrightarrow G$ is *exhaustive* if there exists a regulator $(x_{i,j})$ in G such that, for every orthogonal sequence (a_n) in E, then $\mu(a_n)$ $\mathcal{D}$-converges to 0 with respect to $(x_{i,j})$. In this case we say also that μ is *exhaustive with respect to* $(x_{i,j})$.

It is easy to see that μ is exhaustive with respect to $(x_{i,j})$ if and only if for every orthogonal sequence (a_n) in E, $\bar{\mu}(a_n)$ $\mathcal{D}$-converges to 0 with respect to $(x_{i,j})$.
Moreover, if G is the set of all real numbers, the definitions of σ-additivity and exhaustivity coincide with the classical ones.

2.4. Definition. We say that a sequence (μ_n) of G-valued measures on E is *uniformly bounded* if the set $\{\mu_n(a): a \in E, n \in \mathbb{N}\}$ is bounded.

2.5. Definition. Let (μ_n) be a sequence of exhaustive G-valued measures on E. We say that (μ_n) is *uniformly exhaustive* if there exists a regulator $(x_{i,j})$ such that, for any orthogonal sequence (a_k) in E,

$$\mathcal{D}\text{-}\lim_k \mu_n(a_k) = 0$$

relatively to $(x_{i,j})$ uniformly with respect to n. In this case we say also that (μ_n) is *uniformly exhaustive with respect to* $(x_{i,j})$.

2.6. Definition. We say that be a sequence (μ_n) of G-valued measures on E is *$\mathcal{RD}$-convergent* to a function $\mu: G \longrightarrow E$ if there exists a regulator $(x_{i,j})$ such that, for every $a \in E$, $(\mu_n(a))$ $\mathcal{D}$-converges to $\mu(a)$. In this case we say also that (μ_n) *$\mathcal{RD}$-converges to μ with respect to* $(x_{i,j})$ and we write

$$\mu = \mathcal{RD}\text{-}\lim_n \mu_n .$$

It is easy to prove the following.

2.7. Proposition. *If (μ_n) is a sequence of G-valued measures on E, then (μ_n) $\mathcal{RD}$-converges to a function μ with respect to a regulator $(x_{i,j})$ if and only if $\forall a \in E$ and $\forall \phi \in \Phi$, $\exists n_0 \in \mathbb{N}$ such that*

$$\forall n \geqslant n_0, \forall p \in \mathbb{N}, \quad \left|\mu_n(a) - \mu_{n+p}(a)\right| \leqslant \bigvee_i x_{i,\phi(i)} \,.$$

2.8. Lemma. *Let (a_n) be an orthogonal sequence in E such that $\bigoplus_{n\in\mathbb{N}} a_n$ exists and (A_k) a decreasing sequence of subsets of $\mathbb{N}$ such that $\bigcap_k A_k = \emptyset$. Set, for each $k \in \mathbb{N}$, $b_k = \bigoplus_{n\in A_k} a_n$. Then $b_k \downarrow 0$.*

Proof. It is clear that the sequence (b_k) is decreasing. We prove that $\bigwedge_n b_n = 0$. Set $a = \bigoplus_{n\in\mathbb{N}} a_n$ and, for every $j \in \mathbb{N}$, $B_j = \{n \in \mathbb{N} \colon n > j\}$. From Proposition 1.3 we have $\bigoplus_{n\in B_j} a_n = a \ominus \Big(\bigoplus_{n=1}^{j} a_n\Big)$. Since $\bigoplus_{n=1}^{j} a_n \uparrow a$, from Proposition 1.4 we get that $\inf_j \bigoplus_{n\in B_j} a_n = 0$. We can suppose that $A_k \neq \emptyset$ for each $k \in \mathbb{N}$ and set $m_k = \min A_k$. Since (A_k) is decreasing, (m_k) is increasing and so $m = \lim_k m_k$ exists. Since $\bigcap_k A_k = \emptyset$, we have that $m = +\infty$. Therefore, for every $j \in \mathbb{N}$, we can find an integer k_j such that, for every $k \geqslant k_j$, $A_k \subseteq B_k$. Then $\inf_n b_n \leqslant \inf_j \bigoplus_{n\in A_{k_j}} a_n \leqslant \inf_j \bigoplus_{n\in B_j} a_n = 0$. □

2.9. Proposition. *Suppose that E is σ-complete and let $\mu\colon E \longrightarrow G$ be a σ-additive measure. Then μ is exhaustive.*

Proof. Let μ be σ-additive with respect to the regulator $(x_{i,j})$ and (a_n) be an orthogonal sequence in E. Denote by $\mathcal{A}$ the power set of $\mathbb{N}$ and define the function $\tilde{\mu}\colon \mathcal{A} \longrightarrow G$ as $\tilde{\mu}(A) = \mu\Big(\bigoplus_{n\in A} a_n\Big)$ for all $A \in \mathcal{A}$. By Lemma 2.8, $\tilde{\mu}$ is a σ-additive measure. From [8, Prop. 2.13], $\tilde{\mu}$ is exhaustive. Hence, for every $\phi \in \Phi$, an integer n_0 can be found such that, for every $n > n_0$, $\left|\mu(a_n)\right| = \left|\tilde{\mu}(\{n\})\right| < \bigvee_i x_{i,\phi(i)}$. Since (a_n) is arbitrary, we obtain that μ is exhaustive. □

2.10. Proposition. *If the measure $\mu\colon E \longrightarrow G$ is exhaustive with respect to the regulator $(x_{i,j})$ and (a_k) is a monotone sequence in E, then the limit $\mathcal{D}\text{-}\lim_k \mu(a_k)$ with respect to $(x_{i,j})$ exists.*

Proof. Without loss of generality, we can suppose that (a_k) is decreasing. Suppose, by contradiction, that the limit $\mathcal{D}\text{-}\lim_k \mu(a_k)$ with respect to $(x_{i,j})$

does not exist. Then there exist $\phi \in \Phi$ and sequences of positive integers (n_k) and (p_k) with $n_{k+1} > n_k + p_k$ such that

$$\left|\mu(a_{n_k}) - \mu(a_{n_k+p_k})\right| \nleqslant \bigvee_i x_{i,\phi(i)} .$$

Setting now $b_k = a_{n_k} \ominus a_{n_k+p_k}$, by Proposition 1.5 it is easily seen that sequence (b_k) is orthogonal, and

$$\forall k, \quad \left|\mu(b_k)\right| = \left|\mu(a_{n_k}) - \mu(a_{n_k+p_k})\right| \nleqslant \bigvee_i x_{i,\phi(i)} ,$$

which contradicts exhaustivity of μ with respect to $(x_{i,j})$. □

2.11. Proposition. *Let (μ_n) be a sequence of measures on E and suppose that it is uniformly exhaustive with respect to the regulator $(x_{i,j})$. Let (a_k) be a monotone sequence in E. Then the limit $\mathcal{RD}$-$\lim\limits_k \mu_n(a_k)$ with respect to $(x_{i,j})$ exists uniformly in n.*

Proof. The proof is the same as [8, Prop. 3.4]. Without loss of generality, we can suppose that (a_k) is decreasing. The existence of the limit follows from Proposition 2.10. To prove the uniformity in n, by contradiction suppose that there exist $\phi \in \Phi$ and increasing sequences of positive integers (n_k), (p_k) and (h_k) such that $h_k + p_k < h_{k+1}$ and

$$\left|\mu_{n_k}(a_{h_k}) - \mu_{n_k}(a_{h_k+p_k})\right| \nleqslant \bigvee_i x_{i,\phi(i)} .$$

Proceeding as in the proof of Proposition 2.10, this contradicts uniform exhaustivity of the sequence (μ_n) with respect to $(x_{i,j})$. □

2.12. Definition. We say that a sequence (μ_n) of G-valued measures on E is *uniformly σ-additive* if there exists a regulator $(x_{i,j})$ in G such that, for every sequence (c_k) in E such that $c_k \downarrow 0$ and for every $\phi \in \Phi$, there exists $k_0 \in \mathbb{N}$ such that, for each $n \in \mathbb{N}$,

$$\bar{\mu}_n(c_{k_0}) \leqslant \bigvee_i x_{i,\phi(i)} .$$

In this case we say also that (μ_n) is *uniformly σ-additive with respect to* $(x_{i,j})$.

Proposition 2.11 has the following consequence.

2.13. Corollary. *Suppose that E is σ-complete. Let (μ_n) be a uniformly exhaustive sequence of G-valued σ-additive measures on E. Then (μ_n) is uniformly σ-additive. Moreover, if (μ_n) $\mathcal{RD}$-converges to a function μ_0, then μ_0 is a σ-additive measure, too.*

The proof of the following lemma follows faithfully that of [8, Lemma 2.15].

2.14. Lemma. *Let $\mu\colon E \longrightarrow G$ be a measure which is exhaustive with respect to the regulator $(x_{i,j})$ and u be a majorant for $(x_{i,j})$. Then for any element $b \in E$ such that $|\mu(b)| \nleqslant u$, there exists $a \in E$ such that $a \leqslant b$, $|\mu(a)| \nleqslant u$ and the set $\{|\mu(c)|\colon c \leqslant a\}$ is bounded.*

Proof. Let $b \in E$ be with $|\mu(b)| \nleqslant u$. We can suppose that the set $\{|\mu(c)|\colon c \leqslant b\}$ is unbounded, so that there exists $b_1 \leqslant b$ such that $|\mu(b_1)| \nleqslant u + |\mu(b)|$. Then we have $|\mu(b \ominus b_1)| = |\mu(b) - \mu(b_1)| \nleqslant u$. Again we can suppose that $\{|\mu(c)|\colon c \leqslant b_1\}$ is unbounded and then there exists $b_2 \leqslant b_1$ such that $|\mu(b_2)| \nleqslant u + |\mu(b_1)|$ and hence $|\mu(b_1 \ominus b_2)| \nleqslant u$. We can proceed to construct a decreasing sequence (b_n) in E such that $|\mu(b_n \ominus b_{n+1})| \nleqslant u$ and $\{|\mu(c)|\colon c \leqslant b_n\}$ is unbounded. Since the sequence $(b_n \ominus b_{n+1})$ is orthogonal, this contradicts the exhaustivity of μ with respect to $(x_{i,j})$. Then the set $\{|\mu(c)|\colon c \leqslant b_n\}$ is bounded for some n. □

2.15. Lemma. *Let $\mu\colon L \longrightarrow G$ be a modular measure and let a, a_1 and $a_2 \in L$ and r, s, $t \in G$.*

(i) *If both the sets $\{|\mu(c)|\colon c \leqslant a\}$ and $\{|\mu(c)|\colon c \geqslant a\}$ are bounded, with upper bounds r and s respectively, then μ is bounded, with an upper bound for the set $\{|\mu(c)|\colon c \in L\}$ equal to $r + s + |\mu(a)|$.*

(ii) *If the set $\{|\mu(c)|\colon c \leqslant a^\perp\}$ is bounded with upper bound r, then the set $\{|\mu(c)|\colon c \geqslant a\}$ is bounded by $r + |\mu(a)|$.*

(iii) *If $a_1 \perp a_2$ and the sets $\{|\mu(c)|\colon c \leqslant a_1\}$, $\{|\mu(c)|\colon c \leqslant a_2\}$ and $\{|\mu(c)|\colon c \leqslant (a_1 \oplus a_2)^\perp\}$ are bounded with upperbounds r, s and t, respectively, then μ is bounded with an upper bound for the set $\{|\mu(c)|\colon c \in L\}$ equal to $r + s + t + 2|\mu(a_1)| + 4|\mu(a_2)|$.*

Proof. (1) It follows from the equality $\mu(c) + \mu(a) = \mu(c \vee a) + \mu(c \wedge a)$.
(ii) For any $c \geqslant a$, we have $\mu(c) = \mu(c \ominus a) + \mu(a)$. Since $(c \ominus a) \perp a$, we have that $|\mu(c)|$ is bounded by $r + |\mu(a)|$.
(iii) By (ii), if $a_1 \oplus a_2 \leqslant c$, then $|\mu(c)|$ is bounded by $t + |\mu(a_1 \oplus a_2)|$. Moreover, if $a_2 \leqslant d \leqslant a_1 \oplus a_2$, then $d = (d \ominus a_2) \oplus a_2$ and $d \ominus a_2 \leqslant a_1$. Then $\mu(d) = \mu(d \ominus a_2) + \mu(a_2)$, and so $|\mu(d)|$ is bounded by $r + |\mu(a_2)|$. Hence, if $a_2 \leqslant c$, from $\mu(c) + \mu(a_1 \oplus a_2) = \mu\big((a_1 \oplus a_2) \wedge c\big) + \mu\big((a_1 \oplus a_2) \vee c\big)$ we have that $|\mu(c)|$ is bounded by $r + t + |\mu(a_2)| + 2|\mu(a_1 \oplus a_2)|$, so that, from (i), $\{|\mu(c)|\colon c \in L\}$ is bounded by $r + s + t + 2|\mu(a_1)| + 4|\mu(a_2)|$. □

2.16. Corollary. *Let (μ_n) be a sequence of G-valued measures on L and let $a, b \in L$.*

(i) *If the sets $\{\bar{\mu}_n(a)\colon n \in \mathbb{N}\}$ and $\{\bar{\mu}_n(a^\perp)\colon n \in \mathbb{N}\}$ are both bounded, then (μ_n) is uniformly bounded.*

(ii) *If the sets* $\{\bar{\mu}_n(a) \colon n \in \mathbb{N}\}$, $\{\bar{\mu}_n(b) \colon n \in \mathbb{N}\}$ *and* $\{\bar{\mu}_n((a \oplus b)^{\perp}) \colon n \in \mathbb{N}\}$ *are bounded, then* (μ_n) *is uniformly bounded.*

Proof. (i) follows from the definition of $\bar{\mu}_n$ and from Lemma 2.15 (i)(ii), and (ii) follows from Lemma 2.15(iii). □

2.17. Theorem. *Let* $\mu \colon L \longrightarrow G$ *be an exhaustive modular measure. Then* μ *is bounded.*

Proof. By contradiction we assume that μ is unbounded. Suppose that μ is exhaustive with respect to the regulator $(x_{i,j})$ and let u be a majorant for $(x_{i,j})$. Let $b_1 \in L$ such that $|\mu(b_1)| \not\leqslant u$. By Lemma 2.14, there exists $a_1 \leqslant b_1$ such that $|\mu(a_1)| \not\leqslant u$ and the set $\{|\mu(c)| \colon c \leqslant a_1\}$ is bounded. Then by Lemma 2.15(i)–(ii) the set $\{|\mu(c)| \colon c \leqslant a_1^{\perp}\}$ is unbounded. Again by Lemma 2.14, there exists $a_2 \leqslant a_1^{\perp}$ such that $|\mu(a_2)| \not\leqslant u$ and the set $\{|\mu(c)| \colon c \leqslant a_2\}$ is bounded.
Since μ is unbounded, by Lemma 2.15(iii) there exists an element $b_3 \perp (a_1 \oplus a_2)$ such that $|\mu(b_3)| \not\leqslant u$. Again by By Lemma 2.14, there exists $a_3 \leqslant b_3$ such that $|\mu(a_3)| \not\leqslant u$ and the set $\{|\mu(c)| \colon c \leqslant a_3\}$ is bounded. It is now clear that proceeding in this way we can construct an orthogonal sequence (a_n) in L such that $|\mu(a_n)| \not\leqslant u$, for all n, which contradicts the exhaustivity of μ with respect to $(x_{i,j})$. □

2.18. Lemma. *Let* (μ_n) *be a sequence of* G*-valued measures on* E *which is uniformly exhaustive with respect to the regulator* $(x_{i,j})$. *Let* u *be a majorant for* $(x_{i,j})$ *and* $b \in E$ *be such that* $\bigvee_n |\mu_n(b)| \not\leqslant u$. *Then there exists* $a \in E$ *such that* $a \leqslant b$, $\bigvee_n |\mu_n(a)| \not\leqslant u$ *and the set* $\{\bar{\mu}_n(a) \colon n \in \mathbb{N}\}$ *is bounded.*

Proof. Set, for every $a \in L$,

$$v(a) = \bigvee_{n \in \mathbb{N}} |\mu_n(a)| \,.$$

By contradiction, assume that the assertion is false. Then for all $a \in L$ such that $a \leqslant b$ and $v(a) \not\leqslant u$, the set $\{\bar{\mu}_n(a) \colon n \in \mathbb{N}\}$ is unbounded, so that there exists $n \in \mathbb{N}$ with $\bar{\mu}_n(a) \not\leqslant u + v(b)$, and hence there exists an element c_1 in E such that $c_1 \leqslant b$ and $v(c_1) \not\leqslant u + v(b)$. Analogously we can find an element c_2 in E such that $c_2 \leqslant c_1$ and $v(c_2) \not\leqslant u + v(c_1)$, and so on. So we have a decreasing sequence (c_k) in E and a sequence (n_k) of integers such that

$$|\mu_{n_k}(c_{k+1})| \not\leqslant u + |\mu_{n_k}(c_k)| \,,$$

and so

$$\forall k \in \mathbb{N}, \ |\mu_{n_k}(c_k \ominus c_{k+1})| \not\leqslant u \,,$$

which contradicts uniform exhaustivity with respect to $(x_{i,j})$. □

2.19. Theorem. (**Nikodym boundedness Theorem**) *Let (μ_n) be a uniformly exhaustive sequence of G-valued modular measures on L. Then (μ_n) is uniformly bounded if and only if, for any $a \in L$, the set $\{\mu_n(a)\colon n \in \mathbb{N}\}$ is bounded in G.*

Proof. Let $(x_{i,j})$ be a regulator with respect to which (μ_n) is uniformly exhaustive and let u be a majorant for $(x_{i,j})$. As in the proof of Lemma 2.18, set, for every $a \in L$,

$$v(a) = \bigvee_{n\in\mathbb{N}} |\mu_n(a)| .$$

By contradiction, assume that the set $\{|\mu_n(a)|\colon n \in \mathbb{N}\}$ is bounded in G for each $a \in L$ and (μ_n) is not uniformly bounded. So there exists a_1 in L such that $v(a_1) \nleqslant u$. By Lemma 2.18, we can suppose that the set $\{\bar{\mu}_n(a_1)\colon n \in \mathbb{N}\}$ is bounded. By Corollary 2.16(i), there exists an element $a_2 \leqslant a_1^\perp$ such that $v(a_2) \nleqslant u$ and again, by Lemma 2.18, the element a_2 can be chosen so that the set $\{\bar{\mu}_n(a_2)\colon n \in \mathbb{N}\}$ is bounded. By Corollary 2.16(ii), there exists an element a_3 in L such that $a_3 \leqslant (a_1 \oplus a_2)^\perp$ and $v(a_3) \nleqslant u$. Proceeding in this fashion, we construct an orthogonal sequence (a_k) in L such that $v(a_k) \nleqslant u$ for all k, which contradicts uniform exhaustivity with respect to $(x_{i,j})$.
The other implication is obvious. □

3. Absolute continuity and uniform exhaustivity

In this section we extend classical results concerning uniform exhaustivity and absolute continuity to the context of effect algebras.

3.1. Definition. If $\mu\colon E \longrightarrow G$ and $\nu\colon E \longrightarrow [0,+\infty[$ are measures, we say that μ is *absolutely continuous with respect to* ν and write

$$\mu \ll \nu$$

if there exists a regulator $(x_{i,j})$ in G such that, for every sequence (a_k) in E with $\lim_k \nu(a_k) = 0$ and every $\phi \in \Phi$, there exists an integer k_0 such that

$$\forall k \geqslant k_0, \quad |\mu(a_k)| \leqslant \bigvee_i x_{i,\phi(i)} .$$

In this case we say also that $\mu \ll \nu$ *with respect to* $(x_{i,j})$ (or relatively to $(x_{i,j})$).

3.2. Definition. Let (μ_n) be a sequence of G-valued measures defined on E and $\nu\colon E \longrightarrow [0,+\infty[$ be a measure. We say that (μ_n) is *uniformly absolutely continuous with respect to* ν and write $(\mu_n) \ll \nu$ if there exists a regulator

$(x_{i,j})$ in G such that, for every sequence (a_k) in E with $\lim_k \nu(a_k) = 0$ and every $\phi \in \Phi$, there exists an integer k_0 such that

$$\forall k \geqslant k_0, \forall n \in \mathbb{N} \quad |\mu_n(a_k)| \leqslant \bigvee_i x_{i,\phi(i)}\,.$$

In this case we say also that $(\mu_n) \ll \nu$ *with respect to* $(x_{i,j})$ (or relatively to $(x_{i,j})$).

From the definition it is clear that, if ν is exhaustive (or σ-additive) and $\mu \ll \nu$, then μ is exhaustive (or σ-additive, respectively) too.
The absolute continuity can be characterized in the following way.

3.3. Proposition. *Let $\mu\colon E \longrightarrow G$ and $\nu\colon E \longrightarrow [0,+\infty[$ be measures and let $(x_{i,j})$ be a regulator in G. Then $\mu \ll \nu$ with respect to $(x_{i,j})$ if and only if for every $\phi \in \Phi$, there exists $\delta > 0$ satisfying the following condition:*

$$(\forall a \in E)\ \nu(a) \leqslant \delta \implies |\mu(a)| \leqslant \bigvee_i x_{i,\phi(i)}\,. \tag{1}$$

P r o o f . The sufficiency of the condition (1) is trivial. Conversely, if $\mu \ll \nu$ with respect to $(x_{i,j})$, suppose by contradiction that (1) is not satisfied. Then there exists $\phi \in \Phi$ such that, for every integer k, we can find $a_k \in E$ with $\nu(a_k) \leqslant \frac{1}{k}$ and $|\mu(a_k)| \nleqslant \bigvee_i x_{i,\phi(i)}$. This contradicts the assumption that $\mu \ll \nu$ with respect to $(x_{i,j})$. $\square$

3.4. Lemma. *Let $\nu\colon L \longrightarrow [0,+\infty[$ be a σ-additive modular measure and (a_n) a sequence in L such that $a = \bigvee_k a_k$ exists. Then $\nu(a) \leqslant \sum_{n=1}^{\infty} \nu(a_n)$.*

P r o o f . For each $n \in \mathbb{N}$, set $b_n = \bigvee_{i \leqslant n} a_i$. Since $\nu(a_1 \vee a_2) = \nu(a_1) + \nu(a_2) - \nu(a_1 \wedge a_2) \leqslant \nu(a_1) + \nu(a_2)$, it is very easy to show by induction that

$$\nu(b_n) \leqslant \sum_{i=1}^{n} \nu(a_i)\,. \tag{2}$$

Now, since $b_n \uparrow a$ and ν is σ-additive, by Proposition 1.10 we have $\lim_n \nu(b_n) = \nu(a)$. By (2) we obtain $\nu(a) \leqslant \lim_n \sum_{i=1}^{n} \nu(a_i) = \sum_{n=1}^{\infty} \nu(a_n)$. $\square$

3.5. Proposition. *Suppose that L is σ-complete and let $\mu\colon L \longrightarrow G$ and $\nu\colon L \longrightarrow [0,+\infty[$ be σ-additive modular measures. Then $\mu \ll \nu$ if and only if*

$$(\forall a \in L)\ \nu(a) = 0 \implies \mu(a) = 0\,. \tag{3}$$

Proof. Suppose that $\mu \ll \nu$ with respect to the regulator $(x_{i,j})$. Then, if $\nu(a) = 0$, we obtain $|\mu(a)| \leqslant \bigvee_i x_{i,\phi(i)}$ for every $\phi \in \Phi$. Since G is weakly distributive, we obtain $|\mu(a)| \leqslant \inf_{\phi\in\Phi} \bigvee_i x_{i,\phi(i)} = 0$ and therefore $\mu(a) = 0$. Conversely, suppose that (3) holds and let μ be σ-additive with respect to the regulator $(x_{i,j})$. Assume by contradiction that μ is not absolutely continuous with respect to ν. Then we can find a sequence (a_k) in L with $\lim_k \nu(a_k) = 0$ and a $\phi \in \Phi$ such that, for infinitely many integers k, $\bar{\mu}(a_k) \nleqslant \bigvee_i^k x_{i,\phi(i)}$. By taking a subsequence, we may assume that $\bar{\mu}(a_k) \nleqslant \bigvee_i x_{i,\phi(i)}$ for every $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} \nu(a_k) < \infty$. Now set $b_k = \bigvee_{j\geqslant k} a_k$ and $b = \bigwedge_k b_k$. From Lemma 3.4, we have that $\nu(b_k) \leqslant \sum_{j=k}^{\infty} \nu(a_j)$ and therefore $\lim_k \nu(b_k) = 0$. On the other hand we have that $b_k \downarrow b$. Hence, by Proposition 1.10, $\lim_k \nu(b_k) = \nu(b)$ and then $\nu(b) = 0$, so that $\mu(b) = 0$.
Moreover, since $b_k \downarrow b$, by Proposition 1.4 we have $b_k \ominus b \downarrow 0$. Hence, since μ is σ-additive, we can find $\bar{k} \in \mathbb{N}$ such that

$$\forall k \geqslant \bar{k}, \quad \bar{\mu}(b_k) = \bar{\mu}(b_k \ominus b) \leqslant \bigvee_i x_{i,\phi(i)}.$$

Since $a_k \leqslant b_k$, we have a contradiction. □

3.6. Theorem. *Suppose that L is σ-complete. Let $\nu\colon L \longrightarrow [0,+\infty[$ be a σ-additive modular measure and (μ_n) a uniformly exhaustive sequence of σ-additive G-valued modular measures on L, with $\mu_n \ll \nu, \forall n \in \mathbb{N}$. Then (μ_n) is uniformly absolutely continuous with respect to ν.*

Proof. The proof is similar to that of [8. Th. 4.4]. We may assume that (μ_n) is positive. Suppose that (μ_n) is uniformly exhaustive with respect to the regulator $(x_{i,j})$ and suppose, by contradiction, that (μ_n) is not uniformly absolutely continuous with respect to ν. Then we can find a sequence (a_k) in L, a $\phi \in \Phi$ and an increasing sequence (n_k) of integers such that, for each $k \in \mathbb{N}$, $\nu(a_k) < 2^{-k}$ and

$$\mu_{n_k}(a_k) \nleqslant \bigvee_i x_{i,\phi(i)}. \tag{4}$$

Set $b_k = \bigvee_{j\geqslant k} a_j$. By Lemma 3.4 we have that $\nu(b_k) \leqslant \sum_{j=k}^{\infty} \nu(a_j) < \sum_{j=k}^{\infty} \frac{1}{2^j} = 2^{1-k}$, from which $\lim_k \nu(b_k) = 0$ and therefore $\mathcal{D}\text{-}\lim_k \mu_n(b_k) = 0$, since $\mu_n \ll \nu$ for all $n \in \mathbb{N}$. By 2.11, we have that $\mathcal{D}\text{-}\lim_k \mu_n(b_k) = 0$ uniformly with respect to n.

Since $a_k \leqslant b_k$ and μ_n are positive, we can find an integer k_0 such that

$$\forall k \geqslant k_0, \bigvee_n \mu_n(a_k) \leqslant \bigvee_i x_{i,\phi(i)},$$

a contradiction with (4) □

3.7. Lemma. *Let (μ_n) be a uniformly bounded sequence of G-valued measures on E and $\nu: E \longrightarrow [0,+\infty[$ a measure such that, for each $n \in \mathbb{N}$, $\mu_n \ll \nu$. Then there exists a regulator $(x_{i,j})$ in G such that, for every $n \in \mathbb{N}$ and $\phi \in \Phi$, there exists $\delta > 0$ such that*

$$(\forall a \in E)\ \nu(a) \leqslant \delta \implies |\mu_n(a)| \leqslant \bigvee_i x_{i,\phi(i)}.$$

Proof. For every $n \in \mathbb{N}$ and $\phi \in \Phi$, define $\tilde{\phi}_n(i) = \phi(i+n)$ for each $i \in \mathbb{N}$. Since each $\mu_n \ll \nu$, we can find, for any $n \in \mathbb{N}$, a regulator $(x^n_{i,j})$ in G such that, for every $\phi \in \Phi$, there exists $\delta > 0$ such that

$$(\forall a \in E)\ \nu(a) \leqslant \delta \implies |\mu_n(a)| \leqslant \bigvee_i x^n_{i,\phi(i)}.$$

Moreover, since (μ_n) is uniformly bounded, we can take $u \in G$ such that $|\mu_n(a)| \leqslant u$ for every $n \in \mathbb{N}$ and $a \in E$. Then, for every $n \in \mathbb{N}$ and $a \in E$, we have

$$|\mu_n(a)| \leqslant u \wedge \left(\bigvee_i x^n_{i,\tilde{\phi}_n(i)}\right) \leqslant u \wedge \sum_{n=1}^{\infty}\left(\bigvee_i x^n_{i,\tilde{\phi}_n(i)}\right).$$

On the other hand, by Lemma 1.8 we can find a regulator $(x_{i,j})$ in G such that, for every $\phi \in \Phi$,

$$u \wedge \sum_{n=1}^{\infty}\left(\bigvee_i x^n_{i,\tilde{\phi}_n(i)}\right) \leqslant \bigvee_i x_{i,\phi(i)}.$$

Hence the assertion follows. □

3.8. Theorem. *Suppose that E is σ-complete. Let (μ_n) be a uniformly bounded sequence of G-valued measures on E, which is $\mathcal{RD}$-convergent to a function μ_0 and $\nu: E \longrightarrow [0,+\infty[$ a measure. If $\mu_n \ll \nu$ for each $n \in \mathbb{N}$, then (μ_n) is uniformly exhaustive. Moreover, if E is a D-lattice and μ_n and ν are σ-additive and modular, then (μ_n) is uniformly absolutely continuous with respect to ν.*

Proof. Let $(x_{i,j})$ be a regulator according to Lemma 3.7 and $(y_{i,j})$ a regulator with respect to which (μ_n) is $\mathcal{RD}$-convergent. Denote by $\mathcal{A}$ the power set of $\mathbb{N}$. Let (a_n) be an orthogonal sequence in E and define the functions $\tilde{\mu}_n: \mathcal{A} \longrightarrow G$ $(n \in \mathbb{N})$, $\tilde{\mu}: \mathcal{A} \longrightarrow G$ and $\tilde{\nu}: \mathcal{A} \longrightarrow [0,+\infty[$ as $\tilde{\mu}_n(A) =$

$\mu_n\Big(\bigoplus_{k\in A} a_k\Big)$, $\tilde{\mu}(A) = \mu\Big(\bigoplus_{k\in A} a_k\Big)$ and $\tilde{\nu}(A) = \nu\Big(\bigoplus_{k\in A} a_k\Big)$, for each $A \in \mathcal{A}$. By Proposition 1.3, it is clear that $\tilde{\mu}_n$, $\tilde{\mu}$ and $\tilde{\nu}$ are measures on $\mathcal{A}$, $(\tilde{\mu}_n)$ is uniformly bounded and $\mathcal{RD}$-convergent to $\tilde{\mu}_0$ with respect to $(y_{i,j})$ and, for each $n \in \mathbb{N}$, $\tilde{\mu}_n \ll \tilde{\nu}$ with respect to $(x_{i,j})$. Then, applying [8, Cor. 5.7], we obtain that $(\tilde{\mu}_n)$ is uniformly exhaustive with respect to the regulator $(6z_{i,j})$, where $z_{i,j} = x_{i,j} \vee y_{i,j}$. Hence, for every $\phi \in \Phi$, we can find $k_0 \in \mathbb{N}$ such that, for every $k \geqslant k_0$,

$$\bigvee_n \mu_n(a_k) = \bigvee_n \tilde{\mu}_n(\{k\}) \leqslant 6 \bigvee_i z_{i,\phi(i)} .$$

Since (a_k) is arbitrary and $(z_{i,j})$ is independent of (a_k), we obtain that (μ_n) is uniformly exhaustive. The last assertion follows from Theorem 3.6. ☐

3.9. Corollary. *Suppose that E is σ-complete. Let (μ_n) be a uniformly bounded $\mathcal{RD}$-convergent sequence of G-valued σ-additive measures on E.*

Then (μ_n) is uniformly σ-additive and the $\mathcal{RD}$-limit is a σ-additive measure.

Proof. From Theorem 3.8 we obtain that (μ_n) is uniformly exhaustive. Hence the assertion follows from Corollary 2.13. ☐

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