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THE NONEXISTENCE IN GENERAL OF A POSITIVE LEFT IDEAL OF THE DESIRED CODIMENSION, EITHER IN THE BOUNDED CASE OR IN THE COMMUTATIVE CASE

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ABSTRACT. A theorem of Thill asserts that if A is a complex algebra, M is a module over A, and $\langle \cdot, \cdot \rangle$ is a Hilbert form on M with a finite rank of negativity k, then there is a submodule of M, of codimension k, which is a positive subspace for $\langle \cdot, \cdot \rangle$, provided that the following two conditions are satisfied: (i) A is commutative; (ii) $\langle \cdot, \cdot \rangle$ is bounded (see the precise statement in the introduction). We show that neither (i) nor (ii) can be omitted.

1. Introduction

Let $(M, \langle \cdot\,, \cdot \rangle)$ be an *inner product space* in the sense of, e.g., Bognár [5], so M is a complex vector space and $\langle \cdot\,, \cdot \rangle$ is an *inner product* on M, i.e., a hermitian sesquilinear form. By the hermitian property, if $x \in M$ then the number $\langle x, x \rangle$ is real; if this number is positive (resp. nonnegative, zero, nonpositive, negative) then we call x *positive* (resp. *nonnegative*, *neutral*, *nonpositive*, *negative*). A linear subspace of M is called *positive definite* (resp. *positive*, *neutral*, *negative*, *negative definite*) if each of its nonzero vectors is positive (resp. nonnegative, neutral, nonpositive, negative). The *rank of negativity* of $\langle \cdot\,, \cdot \rangle$ (or of $(M, \langle \cdot\,, \cdot \rangle)$, or of M if $\langle \cdot\,, \cdot \rangle$ is understood) is the number in $\mathbb{N}_0 \cup \{\infty\}$ which is the supremum of the dimensions of the finite-dimensional negative definite linear subspaces of M; it is denoted by '$\mathrm{r}k\_\langle \cdot\,, \cdot \rangle$' (or '$\mathrm{r}k\_(\langle \cdot\,, \cdot \rangle)$'). (Here $\mathbb{N}_0 = \{0, 1, 2, \dots\}$.) If the rank of negativity of M is finite then it is equal to the least codimension (in M) of any positive linear subspace (see, e.g., [5]). (Another possible reference for indefinite inner product spaces is Iohvidov, Kreĭn, and Langer [6]; we are actually not using any more of the theory than can be found in Sasvári [8, Appendix].)

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Suppose A is a $*$-*algebra*, i.e., a complex algebra equipped with a self-map $a \mapsto a^*$ ('the involution') such that $(\lambda a + \mu b)^* = \bar{\lambda} a^* + \bar{\mu} b^*$, $(ab)^* = b^* a^*$, and $(a^*)^* = a$ for all a and b in A and all λ and μ in the complex field $\mathbb{C}$. Assume that M is a module over A, that is, a module over the ring A which is also a complex vector space (under the same addition) such that the three scalar multiplications involved are connected by the identity $(\lambda a)x = \lambda(ax) = a(\lambda x)$ for λ in $\mathbb{C}$, a in A, and x in M. The form $\langle \cdot, \cdot \rangle$ is called a *Hilbert form* if

$$\langle ax, y \rangle = \langle x, a^* y \rangle$$

for all a in A and all x and y in M. By a *positive submodule* for $\langle \cdot, \cdot \rangle$ is understood a submodule of M which is a positive linear subspace in the sense of the first paragraph. In the following, we assume that $\langle \cdot, \cdot \rangle$ is a Hilbert form with a finite rank of negativity k. A positive submodule, like any positive linear subspace, must then have codimension at least k. If it has codimension exactly k then it will be called 'a positive submodule of the desired codimension'. A theorem of Thill [9] asserts that a positive submodule of the desired codimension exists if the following two conditions are satisfied:

(i) A is commutative;

(ii) $\langle \cdot, \cdot \rangle$ is *bounded*, by which we mean bounded with respect to some vector space seminorm on M with respect to which the operator norm of the linear endomorphism $x \mapsto ax$ of M is finite for each a in A.

(Berg and Sasvári [2][1] show, in the case that $M = A = \mathbb{C}[S]$ for some $*$-semigroup S with identity—for the definition of the semigroup ring $\mathbb{C}[S]$, see the body of the paper—that there exist maximal ideals $L_1, \dots, L_k$ of $\mathbb{C}[S]$ (not necessarily distinct) such that the ideal $\prod_{i=1}^{k} L_i$ is positive; this information is also contained in [9] (in the general case).) The question arises whether one or both of these assumptions can be omitted. It was shown by Cornean and the author [4] that if one assumes neither (i) nor (ii) then there is not in general a positive submodule of the desired codimension, even if $M = A$ (acting on itself on the left by its own multiplication—so a positive submodule may be called 'a positive left ideal'), even if $\langle \cdot, \cdot \rangle$ is *induced* by a linear form φ on A in the sense that

$$\langle x, y \rangle = \varphi(y^* x)$$

for all x and y in A, even if A is the semigroup ring of some $*$-semigroup, even if that $*$-semigroup is the free nonabelian semigroup on finitely many generators (the involution being defined by the condition that the generators are self-adjoint). More precisely, it was shown that for any two positive integers k

[1] In that paper, the words "there is an isomorphism $\varphi\colon V_1 \to V_2$ ..." [2, p. 15] should be modified to a statement about an isomorphism between dense linear subspaces of V_1 and V_2.

and q it may happen that $\langle\cdot,\cdot\rangle$ has rank of negativity k but there is no positive submodule of codimension less than q.

The first purpose of the present paper is to give a simple proof of the fact that even if one assumes (ii) then it may happen that $\langle\cdot,\cdot\rangle$ has finite rank of negativity (in fact, 1) but there is no positive submodule of finite codimension. This may happen even if $M = A$ (as above), even if $\langle\cdot,\cdot\rangle$ is induced by a linear form φ (as above), even if φ is bounded by some submultiplicative seminorm on A, even if A is the semigroup ring of some $*$-semigroup S, even if S is a group with the *inverse involution* $s^* = s^{-1}$, even if S is countable, even if S is a free group. Whether it may happen even if S is finitely generated, is left as a topic for future investigation.

The second purpose is to give a fairly simple proof of the fact that if one assumes (i) but not (ii), it may again happen that $\langle\cdot,\cdot\rangle$ has rank of negativity 1 but no positive submodule of codimension 1, even if $M = A = \mathbb{C}[X,Y]$ (the polynomials in two variables).

As for motivation and context, note that in the case of a commutative A, the finding of a positive (left) ideal has often been the first step towards an integral representation, of which the classical example is the Lévy–Khintchine formula for the logarithm of an infinitely divisible characteristic function. See Berg, Christensen, and Maserick [3] (also Berg [1]) and Sasvári [8].

2. The first example

This section is devoted to the proof of the following theorem and its corollaries. A linear form φ on a $*$-algebra A is called *hermitian* if $\varphi(a^*) = \overline{\varphi(a)}$ for each a in A. If so, then the sesquilinear form $\langle\cdot,\cdot\rangle$ on $A \times A$ induced by φ (in the sense of the Introduction) is hermitian; note that it is automatically a Hilbert form. Saying that φ is *bounded* means that it is bounded with respect to some submultiplicative seminorm on A. If so, then $\langle\cdot,\cdot\rangle$ is bounded in the sense defined in (ii) in the introduction (since if $|\cdot|$ is a submultiplicative seminorm on A then for each a in A the linear endomorphism $x \mapsto ax$ of A is bounded of norm at most $|a|$).

Theorem 1. *There exist a (necessarily non-commutative) $*$-algebra A with identity (whose underlying algebra is the algebra of bounded operators on a Hilbert space) and a bounded hermitian linear form φ on A such that the Hilbert form $\langle\cdot,\cdot\rangle$ on A induced by φ has exactly 1 negative square but there is no positive submodule (=positive left ideal) of finite codimension.*

Proof. Choose a separable infinite-dimensional Hilbert space H with an orthonormal basis $(e_n)_{n=0}^{\infty}$. Let $[\cdot|\cdot]$ be the (positive definite) inner product of

H, and $\|\cdot\|$ the corresponding norm. Let P be the closed linear subspace of H spanned by the set $\{e_n : n > 0\}$, so $H = \mathbb{C}e_0 \oplus P$ (orthogonal direct sum). Define a complex-valued function $(\cdot|\cdot)$ on $H \times H$ by continuity, sesquilinearity, and the condition that

$$\big(e_m \,|\, e_n\big) = \begin{cases} -1 & \text{if } m = n = 0\,, \\ 1 & \text{if } m = n > 0\,, \\ 0 & \text{if } m \neq n\,. \end{cases}$$

Note that $(\cdot|\cdot)$ is hermitian, hence an (indefinite) inner product on H (in fact, with rank of negativity exactly 1, though this will not be needed). Let A be the algebra of bounded linear operators on H, equipped with the involution $*$ defined by the condition that if $a \in A$ then a^* is the adjoint of a with respect to $(\cdot|\cdot)$, characterized by the condition that

$$\big(ax \,|\, y\big) = \big(x \,|\, a^* y\big)$$

for all x and y in H (see, e.g., [5]). Define a linear form φ on A by the condition that

$$\varphi(a) = \big(ae_0 \,|\, e_0\big)$$

for each a in A. Then $\varphi(a^*) = \big(a^* e_0 \,|\, e_0\big) = \big(e_0 \,|\, ae_0\big) = \overline{\big(ae_0 \,|\, e_0\big)} = \overline{\varphi(a)}$ for each a in A, i.e., φ is hermitian. The Hilbert form induced by φ is

$$A \times A \ni (a, b) \mapsto \langle a, b\rangle = \varphi(b^* a) = \big(b^* a e_0 \,|\, e_0\big) = \big(ae_0 \,|\, be_0\big)\,.$$

To see that it has at least one negative square, it suffices to note that if 1 is the identity of A then $\varphi(1^*1) = \big(e_0 \,|\, e_0\big) = -1 < 0$. To see that it has at most one, let L be the set of those a in A such that $ae_0 \in P$. Then L is the kernel of the linear form on A mapping each a in A to the unique complex number λ such that $a - \lambda e_0 \in P$. Therefore, L has codimension 1 in A. If $a \in L$ then $\varphi(a^* a) = \big(ae_0 \,|\, ae_0\big) = [ae_0, ae_0] = \|ae_0\|^2 \geqslant 0$ (since $ae_0 \in P$). Thus L is positive. This means that $\langle \cdot\,, \cdot\rangle$ has exactly 1 negative square. It is bounded since $\big|\varphi(a)\big| = \big|\big(ae_0 \,|\, e_0\big)\big| \leqslant \|ae_0\|\,\|e_0\|$ (see, e.g., [5]), so that $|\varphi(a)|$ is bounded by the operator norm of a with respect to the given norm on H. To see that there is no positive left ideal of finite codimension, suppose K is a positive left ideal; we have to show $\dim(A/K) = \infty$. If $a \in K$ then $0 \leqslant \varphi(a^* a) = \big(ae_0 \,|\, ae_0\big)$. If $b \in A$ then ba is also in K (since K is a left ideal), hence by the preceding: $0 \leqslant \big(bae_0 \,|\, bae_0\big)$. In other words, the vector ae_0 is carried by *every* bounded linear endomorphism of H to a nonnegative vector. Since the space contains a negative vector (such as e_0), it easily follows that $ae_0 = 0$. In other words, K is contained in the space J of those a in A such that $ae_0 = 0$. It therefore suffices to show $\dim(A/J) = \infty$. For this, it suffices to exhibit a linear subspace I of A such that $I \cap J = \{0\}$ and $\dim I = \infty$. For each n in $\mathbb{N} = \{1, 2, 3, \dots\}$ choose

f_n in A such that $f_n e_0 = e_n$. (To be specific, let f_n annihilate P.) Letting I be the linear subspace of A spanned by the set $\{ f_n : n > 0 \}$, we leave the verification of the required properties as an easy exercise. □

We note for later use that although the involution $*$ is an unusual one, the algebra A of the preceding proof, with the operator norm and the involution $*$, is a Banach $*$-algebra.

If $\langle \cdot, \cdot \rangle$ is an inner product on a module M over a $*$-algebra, let $\mathrm{psm}\langle \cdot, \cdot \rangle$ or $\mathrm{psm}(\langle \cdot, \cdot \rangle)$ be the least codimension (in M) of any positive submodule (with respect to $\langle \cdot, \cdot \rangle$). Then we know that $\mathrm{psm}\langle \cdot, \cdot \rangle \geqslant \mathrm{r}k\_\langle \cdot, \cdot \rangle$.

Proposition 1. *Let L and M be complex vector spaces, let h be a linear mapping of L into M, let $\langle \cdot, \cdot \rangle$ be an inner product on M, and let $\langle \cdot, \cdot \rangle_{\circ h}$ be the inner product $(x, y) \mapsto \big\langle h(x), h(y) \big\rangle$ on L. Then*

(i) *$rk\_\langle \cdot, \cdot \rangle_{\circ h} \leqslant rk\_\langle \cdot, \cdot \rangle$, with equality if $M = h(L)$.*

Assume now that L and M are modules over the same $*$-algebra A and that h is A-linear (so h is now a homomorphism of A-modules). Then

(ii) *$\mathrm{psm}\langle \cdot, \cdot \rangle_{\circ h} \leqslant \mathrm{psm}\langle \cdot, \cdot \rangle$, with equality if $M = h(L)$.*

Proof. (i): Let D be a negative definite linear subspace of L (with respect to $\langle \cdot, \cdot \rangle_{\circ h}$); we have to show $\dim D \leqslant \mathrm{r}k\_\langle \cdot, \cdot \rangle$. By the definition of the right-hand side, it suffices to show that the linear subspace $h(D)$ of M is negative definite and $\dim h(D) = \dim D$. Given a nonzero y in $h(D)$, we can choose a (necessarily nonzero) x in D such that $y = h(x)$, and then

$$0 > \langle x, x \rangle_{\circ h} = \big\langle h(x), h(x) \big\rangle = \langle y, y \rangle . \qquad (*)$$

This being so for all such y, the space $h(D)$ is negative definite. To see that $\dim h(D) = \dim D$, it suffices to show that h is one-to-one on D. But if x is a nonzero vector in D then by $(*)$, $h(x)$ cannot be zero. This proves the inequality in (i).

We also have to show the converse in case $M = h(L)$. So let E be a negative definite linear subspace of M; we have to show $\dim E \leqslant \mathrm{r}k\_\langle \cdot, \cdot \rangle_{\circ h}$. Since $M = h(L)$, we can choose a linear subspace D of L such that $E = h(D)$. By linear algebra, we can do this in such a way that h is one-to-one on D. If now x is a nonzero vector in D then (since h is one-to-one on D) we have $h(x) \neq 0$ in E, hence (since E is negative definite) $0 > \big\langle h(x), h(x) \big\rangle = \langle x, x \rangle_{\circ h}$. This being so for all such x, D is a negative definite space, so $\dim E = \dim D \leqslant \mathrm{r}k\_\langle \cdot, \cdot \rangle_{\circ h}$ where the equality is true because h is one-to-one on D.

(ii): Let J be a positive submodule of L (with respect to $\langle \cdot, \cdot \rangle_{\circ h}$); we have to show $\dim \frac{L}{J} \leqslant \mathrm{psm}\langle \cdot, \cdot \rangle$. Since h is now A-linear, the set $h(J)$ is a submodule of M. Given y in $h(J)$, we can choose x in J such that $y = h(x)$, and then (since J is positive) $0 \leqslant \langle x, x \rangle_{\circ h} = \langle y, y \rangle$. This being so for all such y, the submodule $h(J)$ is positive. Therefore, $\dim \frac{M}{h(J)} \leqslant \operatorname{psm}\langle\cdot\,,\cdot\rangle$. It remains to note that $\dim \frac{L}{J} \leqslant \dim \frac{M}{h(J)}$, which follows from linear algebra. (For example, choose a linear complement Y of $h(J)$ in M, so $\dim Y = \dim \frac{M}{h(J)}$. Choose a linear subspace X of L such that h is one-to-one on X and maps X onto Y. Then $\dim X = \dim Y$. It now suffices to verify that $X \cap J = \{0\}$. But if $x \in X \cap J$ then the element $y = h(x)$ is in $Y \cap h(J) = \{0\}$, i.e., $y = 0$, whence by injectivity, $x = 0$, as desired.) This proves the inequality in (ii).

We also have to prove the converse in case $M = h(L)$. Let K be a positive submodule of M (with respect to $\langle\cdot\,,\cdot\rangle$); we have to show $\dim \frac{M}{K} \leqslant \operatorname{psm}\langle\cdot\,,\cdot\rangle_{\circ h}$. The set $h^{-1}(K)$ is a submodule of L, and its codimension in L is exactly the codimension of K in M (we here use the hypothesis that $M = h(L)$). Thus it suffices to verify that $h^{-1}(K)$ is positive (with respect to $\langle\cdot\,,\cdot\rangle_{\circ h}$); this is trivial. □

A $*$-*semigroup* is a semigroup S equipped with a self-map $s \mapsto s^*$ ('the involution') such that $(st)^* = t^*s^*$ and $(s^*)^* = s$ for all s and t in S. Suppose S is a $*$-semigroup. The *semigroup ring* $\mathbb{C}[S]$ of S is the space $\mathbb{C}^{(S)}$ of finitely supported complex-valued functions on S, made into a $*$-algebra by setting $ab(s) = \sum_{u,v\in S:\ uv=s} a(u)b(v)$ and $a^*(s) = \overline{a(s^*)}$ for all a and b in $\mathbb{C}^{(S)}$ and s in S. For each s in S one defines an element δ_s of $\mathbb{C}[S]$ by the condition that $\delta_s(t) = \delta(s,t)$ (the Kronecker delta) for each t in S. Then $\{\,\delta_s : s \in S\,\}$ is a linear basis of $\mathbb{C}[S]$ which is also a $*$-subsemigroup of the multiplicative $*$-semigroup of $\mathbb{C}[S]$, isomorphic to S itself (since $\delta_{st} = \delta_s\delta_t$ and $\delta_{s^*} = \delta_s^*$ for all s and t in S). There is a one-to-one correspondence between complex-valued functions f on S and linear forms φ on $\mathbb{C}[S]$, given by $\varphi(a) = \sum_{s\in S} a(s)f(s)$ for a in $\mathbb{C}[S]$ (or equivalently, $f(s) = \varphi(\delta_s)$ for s in S). With every such φ is of course associated an inner product $\langle\cdot\,,\cdot\rangle$ on $\mathbb{C}[S]$ (induced by φ as in the Introduction). One says that f (supposed to be hermitian) has *rank of negativity* k if $\langle\cdot\,,\cdot\rangle$ has rank of negativity k, and a *positive left ideal* for f is a positive left ideal of $\mathbb{C}[S]$ (with respect to $\langle\cdot\,,\cdot\rangle$). Saying that f has rank of negativity k simply means that the maximum number of negative eigenvalues (counted with multiplicity) of any matrix of the form $\big(f(s_j^*s_i)\big)_{i,j=1}^n$ with n in $\mathbb{N}$ and the s_i in S is k; describing a positive left ideal in similar terms is less easy.

Proposition 2. *If A is a $*$-algebra, S a $*$-subsemigroup (of the multiplicative $*$-semigroup of A) which spans A, and φ a hermitian linear form on A then*

(i) *the rank of negativity of φ as a function on S is equal to the rank of negativity of φ as a linear form on A (i.e., the rank of negativity of the corresponding inner product on A);*

(ii) *the least codimension of any positive left ideal for φ as a function on S is equal to the least codimension of any positive left ideal for φ as a linear form on A.*

Proof. Define a $*$-algebra homomorphism h of $\mathbb{C}[S]$ onto A by the condition that $h(x) = \sum_{s\in S} x(s)s$ for each x in $\mathbb{C}[S]$, and apply Proposition 1, noting that $\mathbb{C}[S]$ and A are both $\mathbb{C}[S]$-modules, the former by its internal multiplication, the latter by the scalar multiplication $\mathbb{C}[S]\times A \ni (x,a) \mapsto h(x)a$. Note also that (since $A = h(\mathbb{C}[S])$) $\mathbb{C}[S]$-submodules of A are the same as A-submodules of A, i.e., left ideals. $\square$

An *absolute value* on a $*$-semigroup S is a nonnegative real-valued function α such that $\alpha(s^*) = \alpha(s)$ and $\alpha(st) \leqslant \alpha(s)\alpha(t)$ for all s and t in S. A complex-valued function f on S is *bounded with respect to* α (where α is an absolute value) if there is some $C \geqslant 0$ such that $|f| \leqslant C\alpha$ (pointwise), and *exponentially bounded* if it is bounded with respect to some absolute value on S. Now f is exponentially bounded if and only if the corresponding linear form φ on $\mathbb{C}[S]$ is bounded (recall that this means bounded with respect to some submultiplicative seminorm on $\mathbb{C}[S]$). Indeed, if φ is bounded with respect to some submultiplicative seminorm $|\cdot|$ on $\mathbb{C}[S]$, let $\|\cdot\|$ be the 'auxiliary $*$-norm' defined by $\|a\| = \max\{|a|, |a^*|\}$ for a in $\mathbb{C}[S]$ (so φ is trivially bounded with respect to $\|\cdot\|$) and define $\alpha(s) = \|\delta_s\|$ for $s \in S$; then f is bounded with respect to α. Conversely, if f is bounded with respect to some in absolute value α then φ is easily seen to be bounded with respect to the norm $\|\cdot\|$ on $\mathbb{C}[S]$ defined by $\|a\| = \sum_{s\in S} |a(s)|\alpha(s)$ for a in $\mathbb{C}[S]$.

Corollary 1. *There exist a group G with the involution $s^* = s^{-1}$ and an (automatically exponentially bounded) hermitian function on G which has rank of negativity 1 but no positive left ideal of finite codimension.*

Note. That a hermitian function with finite rank of negativity on a group with the involution $x^* = x^{-1}$ is automatically exponentially bounded was shown in [9].

Proof. With A and everything else as in the theorem, let G be the unitary group of A, that is, the set of those u in A such that $u^*u = uu^* = 1$ where 1 is the identity of A and where adjoints are defined as in the proof of the theorem. By Ford's square root lemma, G spans A. (Indeed, since A is a Banach $*$-algebra then Ford's square root Lemma [10, p. 27] says that if x is a hermitian element of A with $\|x\| < 1$ then $1-x$ has a hermitian square root; a well-known application of this (see Pedersen [7]) says that if x is a hermitian element of A with $\|x\| < 1$ then x is a linear combination of the two unitaries $u = x + i\sqrt{1-x^2}$ and u^*.) Apply Proposition 2. $\square$

The next proposition will also be useful in later work. If M is a module over a $*$-algebra A and if A' and M' are linear subspaces of A and M, respectively, let $A'M'$ be the linear hull of the set $\{ax\colon a \in A',\ x \in M'\}$. This applies, in particular, if $M = A$. For every complex vector space E, let $\mathcal{C}(E)$ be the set of all finite-dimensional linear subspaces of E.

Proposition 3. *For a module M over a $*$-algebra A, a Hilbert form $\langle\cdot,\cdot\rangle$ on M, and a nonnegative integer l, the following two conditions are equivalent:*

(i) $\operatorname{psm}\langle\cdot,\cdot\rangle \leqslant l$;

(ii) *for every finite-dimensional linear subspace A' of A and every finite-dimensional linear subspace M' of M, there is a linear subspace X of M', of codimension at most l (in M'), such that the space $X + A'X$ is positive (with respect to $\langle\cdot,\cdot\rangle$).*

Note. It would be nice if in (ii) the ugly space $X + A'X$ could be replaced by $A'X$. It can be done if A has an identity but not in general. For a counterexample, let A be the (commutative) $*$-algebra (without identity) having a presentation with the single self-adjoint generator a and the single relation $a^3 = 0$. Let $M = A$ (as a module over itself in the usual way) and define a hermitian linear form φ on A by $\varphi(a^2) = -1$ and $\varphi(a)$ arbitrary (real). Let $\langle\cdot,\cdot\rangle$ be the Hilbert form induced by φ and take $l = 0$. Then (i) is not satisfied (since it is not true that A itself is a positive subspace, because $\varphi(a^*a) = \varphi(a^2) = -1 < 0$) but (ii) would be if we had only $A'X$ instead of $X + A'X$ (since we could then take $X' = M'$, which has codimension $0 = l$ in M'; then $A'X \subset AA$ and the latter is a positive subspace since $AAAA = \{0\}$).

P r o o f. (i)$\Rightarrow$(ii): Choose a positive submodule L of codimension at most l in M. Given A' and M' as in (ii), let $X = L \cap M'$, which is a linear subspace of M', of codimension at most l. Now $X + A'X \subset X + AX \subset L + AL \subset L + L \subset L$. Since the ultimate right-hand side is a positive subspace, so is the ultimate left-hand side.

(ii)$\Rightarrow$(i): We may assume that A has an identity. To see this, use the facts that: A can be embedded in a $*$-algebra $\widetilde{A}$ with identity (as an ideal of codimension 1); M can be made uniquely a module over $\widetilde{A}$ such that $1x = x$ for all x in M; submodules of M as a module over $\widetilde{A}$ are the same as submodules of M as a module over A; positivity of a subspace has nothing to do with any module structure; if $A' \in \mathcal{C}(A)$ then the space $\widetilde{A}' := \mathbb{C}1 + A'$ is in $\mathcal{C}(\widetilde{A})$ and (for X as in (ii)) $X + A'X = \widetilde{A}'X = X + \widetilde{A}'X$; and finally, every finite-dimensional linear subspace of $\widetilde{A}$ is contained in $\widetilde{A}'$ for some A' in $\mathcal{C}(A)$.

Henceforth, assume that A has an identity, so assuming $1 \in A'$, $X + A'X$ reduces to $A'X$. Let $\mathcal{C}_1(A)$ be the set of those A' in $\mathcal{C}(A)$ which contain 1. Consider both $C_1(A)$ and $\mathcal{C}(M)$ with the inclusion orderings, and $C_1(A) \times \mathcal{C}(M)$

with the product ordering $\leqslant$, so $(A', M') \leqslant (A'', M'')$ if and only if $A' \subset A''$ and $M' \subset M''$. For each (A', M') in $C_1(A) \times \mathcal{C}(M)$ and every nonnegative integer k, let $\mathcal{P}_k(A', M')$ be the set of all linear subspaces X of M', of codimension exactly k, such that $A'X$ is a positive subspace. By hypothesis, the set $\mathcal{P}_{\leqslant l}(A', M') := \bigcup_{k=0}^{l} \mathcal{P}_k(A', M')$ is nonempty, so choose $X(A', M')$ in this set. We shall use the easily established fact that $\mathcal{P}_{\leqslant l}(A', M')$ is compact in the topology defined by the condition that for each k the set $\mathcal{P}_k(A', M')$ is considered with the topology inherited from the appropriate Grassmann manifold (which is compact) while $\mathcal{P}_{\leqslant l}(A', M')$ is the topologically disjoint union of the spaces $\mathcal{P}_k(A', M')$ $(0 \leqslant k \leqslant l)$. Select a universal subnet (A_i, M_i) of the identical net on the upwards filtering partially ordered set $\big(C_1(A) \times \mathcal{C}(M), \leqslant\big)$. For each (A', M') in $C_1(A) \times \mathcal{C}(M)$ the net $\big(X(A_i, M_i) \cap M'\big)$, indexed by all i large enough so that $M_i \supset M'$, is a universal net in the compact set $\mathcal{P}_{\leqslant l}(A', M')$ and so converges to some $Y(A', M')$ in that set. If $(A', M') \leqslant (A'', M'')$ then $Y(A', M') = Y(A'', M'') \cap M'$ by continuity. In particular, if $A', A'' \in C_1(A)$, $A' \subset A''$, and $M^* \in \mathcal{C}(M)$ then $Y(A', M^*) = Y(A'', M^*)$, whence we see that there is a family $\big(Y(M^*)\big)_{M^* \in \mathcal{C}(M)}$ such that $Y(A', M^*) = Y(M^*)$, for all (A', M^*) in $C_1(A) \times \mathcal{C}(M)$. By the preceding, $Y(M') = Y(M'') \cap M'$ whenever $M' \subset M''$, so if we define $Y = \bigcup_{M' \in \mathcal{C}(M)} Y(M')$ then Y is a linear subspace of M. It has codimension at most l in M (since its intersection with an arbitrary M' has codimension at most l in M'). Now Y need not be a submodule of M, but the space $L = AY$ certainly is, and moreover contains Y (since A has an identity) and so has condimension (in M) at most l. It now suffices to show that L is positive, i.e., if $x \in L$ then $\langle x, x\rangle \geqslant 0$. Since $x \in L = AY$ then there exist n in $\mathbb{N}$, $a_1, \dots, a_n$ in A, and $y_1, \dots, y_n$ in Y such that $x = a_1 y_1 + \cdots + a_n y_n$. Let A' and M' be the (finite-dimensional) linear subspaces of A and M (respectively) spanned by $\{a_1, \dots, a_n\}$ and $\{y_1, \dots, y_n\}$ (respectively). Then $x \in A'(Y \cap M')$. Since $Y = \bigcup_{M'' \in \mathcal{C}(M)} Y(M'') = \bigcup_{M'' \in \mathcal{C}(M):\, M'' \supset M'} Y(M'')$, we have $Y \cap M' = \bigcup_{M'' \in \mathcal{C}(M):\, M'' \supset M'} Y(M'') \cap M' = Y(M')$, so $x \in A'Y(M')$. But $Y(M') = Y(A', M') \in \mathcal{P}_{\leqslant l}(A', M')$, so by the definition of the latter, $A'Y(M')$ is a positive subspace, whence it follows that $\langle x, x\rangle \geqslant 0$, completing the proof. □

Proposition 4. *Let k be a positive integer, and l a positive integer or ∞.* (i) *Every example $(A, M, \langle \cdot, \cdot\rangle)$ of a $*$-algebra A, an A-module M, and a Hilbert form $\langle \cdot, \cdot\rangle$ on M, such that $rk\_\langle \cdot, \cdot\rangle = k$ but $\operatorname{psm}\langle \cdot, \cdot\rangle \geqslant l$, gives rise to a similar example $(A^*, M^*, \langle \cdot, \cdot\rangle^*)$ where A^* and M^* are* countable-dimensional*; in fact, a countable-dimensional $*$-subalgebra of A and a countable-dimensional linear subspace of M such that $A^*M^* \subset M^*$, and $\langle \cdot, \cdot\rangle^* = \langle \cdot, \cdot\rangle_{|M^* \times M^*}$.*

(ii) *If* $M = A$ *then we can take* $M^* = A^*$. (iii) *If furthermore* $\langle\cdot,\cdot\rangle$ *is induced by a linear form* φ *on* A *then we can take* $\langle\cdot,\cdot\rangle^*$ *to be induced by a linear form* φ^* *on* A^*, *in fact* $\varphi^* = \varphi\big|_{A^*}$. (iv) *If furthermore* $A = \mathbb{C}[S]$ *for some* $*$*-semigroup* S *then we can take* A^* *to be* $\mathbb{C}[S^*]$ *for some* $*$*-semigroup* S^*, *in fact a* $*$*-subsemigroup of* S. (v) *If* S *is a group then we can take* S^* *to be a group.*

P r o o f . (i): For each pair (A', M') in $\mathcal{C}(A) \times \mathcal{C}(M)$ let $\operatorname{psm}(A', M', \langle\cdot,\cdot\rangle)$ be the least integer l' for which there is a linear subspace X of M', of codimension at most l' (in M'), such that the space $X + A'X$ is positive (with respect to $\langle\cdot,\cdot\rangle$). Then we know from Proposition 3 that $\operatorname{psm}\langle\cdot,\cdot\rangle = \sup\big\{ \operatorname{psm}\big(A', M', \langle\cdot,\cdot\rangle\big) : (A', M') \in \mathcal{C}(A) \times \mathcal{C}(M)\big\}$. We can therefore take a sequence (A_n, M_n) in $\mathcal{C}(A)\times\mathcal{C}(M)$ such that $\operatorname{psm}\langle\cdot,\cdot\rangle = \sup_n \operatorname{psm}(A_n, M_n, \langle\cdot,\cdot\rangle)$. Let A^* be the $*$-subalgebra of A generated by $\bigcup_n A_n$ (which is countable-dimensional since it is generated by a countable set) and let M^* be the A^*-submodule of M generated by $\bigcup_n M_n$ (which is also countable-dimensional). Let $\langle\cdot,\cdot\rangle^*$ be the restriction of $\langle\cdot,\cdot\rangle$ to $M^* \times M^*$. By Proposition 3 again, $\operatorname{psm}\langle\cdot,\cdot\rangle^* = \sup\big\{\operatorname{psm}\big(A', M', \langle\cdot,\cdot\rangle^*\big) : (A', M') \in \mathcal{C}(A^*) \times \mathcal{C}(M^*)\big\}$. Since $\operatorname{psm}\big(A', M', \langle\cdot,\cdot\rangle^*\big)$, for each (A', M') in $\mathcal{C}(A^*) \times \mathcal{C}(M^*)$, is the same as $\operatorname{psm}\big(A', M', \langle\cdot,\cdot\rangle\big)$, we infer $\operatorname{psm}\langle\cdot,\cdot\rangle^* = \sup\big\{\operatorname{psm}\big(A', M', \langle\cdot,\cdot\rangle\big) : (A', M') \in \mathcal{C}(A^*) \times \mathcal{C}(M^*)\big\} \geqslant \sup_n \operatorname{psm}(A_n, M_n, \langle\cdot,\cdot\rangle) = \operatorname{psm}\langle\cdot,\cdot\rangle \geqslant l$, as desired. It is clear that $\operatorname{r}k\_\langle\cdot,\cdot\rangle^* \leqslant \operatorname{r}k\_\langle\cdot,\cdot\rangle$, and we can get equality simply by replacing $\mathcal{C}(M)$ in the definitions by the set $\mathcal{C}_k(M)$ of those M' in $\mathcal{C}(M)$ such that $\operatorname{r}k\_\langle\cdot,\cdot\rangle\big|_{M'\times M'} = k$ (this is satisfied by all sufficiently large finite-dimensional linear subspaces M' of M).

(ii): Let A_0 and M_0 be the pair (A^*, M^*) from (i), and consider the $*$-subalgebra A^* of A generated by $A_0 \cup M_0$.

(iii): If we take $\langle\cdot,\cdot\rangle^*$ equal to $\langle\cdot,\cdot\rangle\big|_{A^*\times A^*}$ then it is automatically induced by $\varphi^* := \varphi\big|_{A^*}$.

(iv): Being a countable-dimensional $*$-subalgebra of A, A^* is spanned by a countable set E. Each member e of E belongs to the linear hull of $\big\{\delta_s : s \in S_e\big\}$ for some finite subset S_e of S. Let S^* be the $*$-subsemigroup of S generated by the countable set $\bigcup\limits_{e\in E} S_e$.

(v): Let S_0 be the S^* of (iv), and let S^* be the $*$-subsemigroup of S generated by the set $S_0 \cup \{\, s^{-1} : s \in S_0 \,\}$. □

Corollary 2. *There exist a countable group* G *with the involution* $x^* = x^{-1}$ *and an exponentially bounded hermitian function on* G *which has rank of negativity* 1 *but no positive left ideal of finite codimension.*

P r o o f . The group G of the present corollary can be taken to be a suitable countable subgroup of the group G of Corollary 1 (we are using Proposition 4). $\square$

Corollary 3. *The free group on a countable infinity of generators, with the involution $x^* = x^{-1}$, admits an exponentially bounded hermitian function with rank of negativity 1 but no positive left ideal of finite codimension.*

P r o o f . With G as in Corollary 2 and F the free group on a countable infinity of generators, choose a homomorphism h of F onto G, and apply Proposition 1 with $L = \mathbb{C}[F]$, $M = \mathbb{C}[G]$, and the h of Proposition 1 the 'linearization' of the present homomorphism h, i.e., $h(x)(g) = \sum_{f \in h^{-1}(g)} x(f)$ for x in $\mathbb{C}[F]$ and g in G. $\square$

3. The second example

An inner product $\langle \cdot, \cdot \rangle$ on a complex vector space M is said to be *non-degenerate* if 0 is the only vector x in M which is *orthogonal* to every vector y in M in the sense that $\langle x, y \rangle = 0$ (or equivalently, $\langle y, x \rangle = 0$). (If M happens to be finite-dimensional, say with a basis $(e_k)_{k=1}^n$, an equivalent condition is that the matrix $\big(\langle e_j, e_k \rangle\big)_{j,k=1}^n$ should be non-singular.) A hermitian linear form φ on a $*$-algebra A is said to be *non-degenerate* if the Hilbert form it induces so is. A hermitian function f on a $*$-semigroup S is said to be *non-degenerate* if the corresponding linear form on $\mathbb{C}[S]$ so is.

For the example which ends this section (a non-degenerate real-valued function on $(\mathbb{N}_0^2, +, n^* = n)$ with rank of negativity 1 but no positive ideal of codimension 1), we shall need a lemma and an extension result for non-degenerate functions on certain semigroups. We begin with the lemma, which is rather difficult.

The *rank of negativity* of a hermitian matrix is its number of negative eigenvalues, counted with multiplicity. The trace of a square matrix D is denoted by 'tr D'.

Lemma. *There is a symmetric real matrix $B = (b_{jk})_{j,k=1}^4$ such that the following four conditions are satisfied:*

(a) *all the determinants* $\det\big[(b_{jk})_{j,k=1}^m\big]$, $m = 1, 2, 3, 4$, *are nonzero.*

(b) *all diagonal elements of B are negative.*

(c) *the rank of negativity of B is* 1.

(d) *there is no triple (z_1, z_2, z_3) of complex numbers such that the following two conditions are satisfied:*

(d1) $z_1 z_2 z_3 = 1$.

(d2) *the matrix* $E = (b_{jk} - b_{4k} z_j - b_{4j} \bar{z}_k + b_{44} z_j \bar{z}_k)_{j,k=1}^3$ *has nonnegative trace.*

P r o o f . We first define a symmetric real matrix $B^{(0)} = \big(b_{jk}^{(0)}\big)_{j,k=1}^4$ satisfying all the conditions except (a). Indeed, choose a 4-tuple $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)$ of nonzero reals such that

$$\beta_1 \beta_2 \beta_3 \neq \beta_4^3 . \tag{$*$}$$

Now define $B^{(0)}$ by the condition that $b_{jk}^{(0)} = -\beta_j \beta_k$ for all j and k in $\{1,2,3,4\}$. Since each β_k is nonzero, $b_{kk}^{(0)} = -\beta_k^2 < 0$, i.e., (b) is satisfied. We leave it as an exercise to verify that if $x = (x_1, x_2, x_3, x_4)$ is a 4-tuple of reals which is orthogonal to β then $B^{(0)} x = 0$. Since there is a 3-dimensional space of such x then 0 is an eigenvalue of $B^{(0)}$ of multiplicity at least 3. It cannot have multiplicity 4 since $B^{(0)} \neq 0$. Thus $B^{(0)}$ has exactly one nonzero eigenvalue (counted with multiplicity). This value cannot be positive since $B^{(0)}$ is not positive semidefinite (for example, $b_{11}^{(0)} < 0$). Thus it must be negative, proving (c). To verify (d), suppose there is a triple (z_1, z_2, z_3) of complex numbers satisfying (d1) and (d2), where in (d2) we read

$$E^{(0)} = \big(b_{jk}^{(0)} - b_{4k}^{(0)} z_j - b_{4j}^{(0)} \bar{z}_k + b_{44}^{(0)} z_j \bar{z}_k\big)_{j,k=1}^3 .$$

With $D^{(0)} := \big(b_{jk}^{(0)}\big)_{j,k=1}^3$ we have $\operatorname{tr} D^{(0)} = -\sum_{k=1}^3 \beta_k^2 < 0$. Also recall that $b_{44}^{(0)} < 0$. Now

$$0 \leqslant \operatorname{tr} E^{(0)} = \operatorname{tr} D^{(0)} - 2 \operatorname{Re} \sum_{k=1}^3 b_{4k}^{(0)} \bar{z}_k \; + \; b_{44}^{(0)} \sum_{k=1}^3 |z_k|^2 . \tag{$**$}$$

By the Cauchy–Schwarz inequality,

$$\left(\operatorname{Re} \sum_{k=1}^3 b_{4k}^{(0)} \bar{z}_k \right)^2 \leqslant \sum_{k=1}^3 {b_{4k}^{(0)}}^2 \; \sum_{k=1}^3 |z_k|^2 ,$$

with equality iff (z_1, z_2, z_3) is proportional to $\big(b_{41}^{(0)}, b_{42}^{(0)}, b_{43}^{(0)}\big)$.

By the identity in $(**)$, it follows that in order to verify $\operatorname{tr} E^{(0)} \leqslant 0$ we only have to show $\sum_{k=1}^3 {b_{4k}^{(0)}}^2 \leqslant b_{44}^{(0)} \operatorname{tr} D^{(0)}$. But using the definition of $B^{(0)}$, this is trivial. Thus $\operatorname{tr} E^{(0)} \leqslant 0$. Since by $(**)$ we also have the opposite inequality, $\operatorname{tr} E^{(0)} = 0$. Therefore, equality must hold in the above application of the Cauchy-Schwarz inequality, i.e., there is some λ in $\mathbb{C}$ such that

$$(z_1, z_2, z_3) = \big(\lambda b_{41}^{(0)}, \lambda b_{42}^{(0)}, \lambda b_{43}^{(0)}\big) . \tag{$***$}$$

Using this, $(**)$, the definition of $B^{(0)}$, and the fact that tr $E^{(0)} = 0$, we make a computation which we omit, ending up with

$$0 = \operatorname{tr} E^{(0)} = -\left|1 + \beta_4^2 \lambda\right|^2 \sum_{k=1}^{3} \beta_k^2 .$$

Since the sum is positive, it follows that

$$\lambda = -\beta_4^{-2} ,$$

whence by (d1) and $(***)$:

$$1 = z_1 z_2 z_3 = \lambda^3 b_{41}^{(0)} b_{42}^{(0)} b_{43}^{(0)} = (-\beta_4^{-2})^3 (-\beta_4 \beta_1)(-\beta_4 \beta_2)(-\beta_4 \beta_3) = \beta_4^{-3} \beta_1 \beta_2 \beta_3 ,$$

contradicting $(*)$. This completes the verification that $B^{(0)}$ satisfies all the requirements except (a).

To get (a), for each n in $\mathbb{N}$ consider the symmetric real matrix $B^{(n)} = \left(b_{jk}^{(n)}\right)_{j,k=1}^{4}$ defined by

$$B^{(n)} = B^{(0)} + \frac{1}{n} I_4 ,$$

where I_4 is the 4×4 identity matrix, or equivalently,

$$b_{jk}^{(n)} = b_{jk}^{(0)} + \frac{1}{n} \delta(j,k) ,$$

where δ is the Kronecker delta. Since the determinant of $\left(b_{jk}^{(n)}\right)_{j,k=1}^{m}$ is a degree m polynomial in $1/n$, there are only finitely many n for which for some m (in $\{1,2,3,4\}$) that determinant vanishes. In other words, $B^{(n)}$ satisfies (a) for all sufficiently large n. Since all diagonal elements of $B^{(0)}$ are negative, so are all diagonal elements of $B^{(n)}$ for all sufficiently large n. For such n, $B^{(n)}$ cannot be positive semidefinite, so it has rank of negativity at least 1. This rank cannot be greater than 1 since $B^{(0)}$ has rank of negativity 1 and the rank of negativity of a matrix cannot be increased by adding to it a positive semidefinite matrix (such as $(1/n) I_4$).

Summing up, $B^{(n)}$ satisfies (a)–(c) for all sufficiently large n. Among such n, choose n such that (d) also holds, and define $B = B^{(n)}$. To see that this is possible (thus completing the proof), suppose not; we shall derive a contradiction. For each n let K_n be the set of all triples (z_1, z_2, z_3) of complex numbers such that $z_1 z_2 z_3 = 1$ and such that the matrix

$$E^{(n)} = \left(b_{jk}^{(n)} - b_{4k}^{(n)} z_j - b_{4j}^{(n)} \bar{z}_k + b_{44}^{(n)} z_j \bar{z}_k\right)_{j,k=1}^{3}$$

has nonnegative trace. By hypothesis, $K_n \neq \emptyset$ for all sufficiently large n. It is also compact for all sufficiently large n—more precisely, for n as in the first sentence of this paragraph. To see this, note that K_n is obviously closed, so we only need show that it is bounded. Suppose not; we shall derive a contradiction. Choose a sequence $\big((z_1^{(m)}, z_2^{(m)}, z_3^{(m)})\big)_{m=1}^{\infty}$ in K_n which diverges. By hypothesis,

$$0 \leqslant \operatorname{tr} D^{(n)} - 2\operatorname{Re}\sum_{k=1}^{3} b_{4k}^{(n)}\,\bar{z}_k^{(m)} + b_{44}^{(n)}\sum_{k=1}^{3}\big|z_k^{(m)}\big|^2, \quad \text{where } D^{(n)} := \big(b_{jk}^{(n)}\big)_{j,k=1}^{3}$$

for each m. Dividing by $\sum_{k=1}^{3}|z_k^{(m)}|^2$ and letting m tend to ∞, we obtain $0 \leqslant b_{44}^{(n)}$, a contradiction. This proves that K_n is compact [for those n as in the first sentence of this paragraph]. A short computation, using only definitions, shows

$$\operatorname{tr}\ E^{(n)} = \operatorname{tr}\ E^{(0)} + \frac{1}{n}\left(3 + \sum_{k=1}^{3}|z_k|^2\right), \tag{$\dagger$}$$

hence for $n' < n''$:

$$\operatorname{tr}\ E^{(n')} - \operatorname{tr}\ E^{(n'')} = \left[\frac{1}{n'} - \frac{1}{n''}\right]\left(3 + \sum_{k=1}^{3}|z_k|^2\right) > 0\,,$$

whence it immediately follows that

$$K_{n''} \subset K_{n'}\,.$$

In other words, the sets K_n, for sufficiently large n, form a nonincreasing sequence of nonempty compact sets, whence it follows that the intersection

$$\bigcap_n K_n$$

is again nonempty. Choose (z_1, z_2, z_3) in this set. For each n we have, since (z_1, z_2, z_3) is in K_n, and by $(\dagger)$:

$$\operatorname{tr}\ E^{(0)} = \operatorname{tr}\ E^{(n)} - \frac{1}{n}\left(3 + \sum_{k=1}^{3}|z_k|^2\right) \geqslant -\frac{1}{n}\left(3 + \sum_{k=1}^{3}|z_k|^2\right).$$

Letting n tend to ∞, we obtain $\operatorname{tr}\ E^{(0)} \geqslant 0$, contradicting what we showed about the matrix $B^{(0)}$. This completes the proof. □

Consider the semigroup $S = \big(\mathbb{N}_0^{(\mathbb{N})}, +, s^* = s\big)$, i.e., the direct sum of a countable infinity of copies of $(\mathbb{N}_0, +, n^* = n)$. It is contained as an additive subsemigroup of the real vector space $E := \mathbb{R}^{(\mathbb{N})}$. If U is a subset of S, let $\operatorname{Co}(U)$ be the intersection of S with the convex hull of U (the latter computed in E). If U is finite, so is $\operatorname{Co}(U)$. We call $\operatorname{Co}(U)$ the *relative convex hull* of U, or expanded a bit: the *convex hull of U relative to S*. Also, U is said to be *relatively convex* if $U = \mathrm{Co}(U)$. Note that the mapping Co (of the set of all subsets of S into itself) is a closure operation, i.e., $U \subset \mathrm{Co}(U)$, $\mathrm{Co}\big(\mathrm{Co}(U)\big) = \mathrm{Co}(U)$, and if $U \subset V$ then $\mathrm{Co}(U) \subset \mathrm{Co}(V)$.

For every set U, let $\mathbb{C}^{(U)}$ be the space of finitely supported complex-valued functions on U.

Suppose U is a relatively convex subset of S. We then define

$$\sqrt{U} = \{s \in S \colon 2s \in U\}.$$

If s and t are in $\sqrt{U}$ then $2s$ and $2t$ are in U, so their convex combination

$$s + t = \frac{1}{2}(2s + 2t)$$

is both in S and in the convex hull of U, hence (by definition) in $\mathrm{Co}(U)$, which is equal to U (since U is relatively convex). Thus

$$\sqrt{U} + \sqrt{U} \subset U.$$

With a real-valued function f on U we can therefore associate an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^{(\sqrt{U})}$ by the condition that

$$\langle a, b \rangle = \sum_{s,t \in \sqrt{U}} a(s)\,\overline{b(t)}\,f(s+t) \qquad \big(a, b \in \mathbb{C}^{(\sqrt{U})}\big).$$

We say that f is *non-degenerate* (resp. has *rank of negativity* k) if $\langle \cdot, \cdot \rangle$ is non-degenerate (resp. has rank of negativity k). We leave it as an exercise to verify that if $U = S$ (hence $\sqrt{U} = S$) then these definitions agree with the original ones.

If X is a set, let $|X|$ be its cardinality.

Proposition 5. *If U is a finite relatively convex subset of $\mathbb{N}_0^{(\mathbb{N})}$ then every non-degenerate real-valued function on U extends to a non-degenerate real-valued function on all of $\mathbb{N}_0^{(\mathbb{N})}$ with the same rank of negativity.*

Proof. By an *enumeration* of a set we understand an injective finite or infinite sequence with that set as its range. Choose an enumeration $(s_1, \ldots, s_m)$ of U (where $m = |U|$), then extend it to an enumeration $(s_n)_{n=1}^{\infty}$ of all of S. For each n in $\mathbb{N}_0$, let $S_n = \{s_1, \ldots, s_n\}$ (where $\{s_1, \ldots, s_0\} = \emptyset$; it may be debated whether this is a fact or a convention). Then $(S_n)_{n=0}^{\infty}$ is an increasing sequence of finite subsets of S with union S. Moreover, $S_0 = \emptyset$ and $S_m = U$.

For each n define $T_n = \mathrm{Co}(S_n)$. Since the relatively convex hull of a finite set is finite, each T_n is finite. Since Co is a closure operation, $S_n \subset T_n = \mathrm{Co}(T_n)$ for each n and $T_0 \subset T_1 \subset T_2 \subset \cdots$. Thus $(T_n)_{n=0}^{\infty}$ is a nondecreasing sequence of finite relatively convex subsets of S with union S. Moreover, $T_0 = \emptyset$ and $T_m = U$ (since U is relatively convex). Extend the sequence (T_n) to a similar sequence $(U_n)_{n=0}^{\infty}$ which is *maximal* in the sense that if $n \in \mathbb{N}_0$ and if V is a relatively convex set such that $U_n \subset V \subset U_{n+1}$ then V is either U_n or U_{n+1}; we also require U_n to be a *proper* subset of U_{n+1} for each n (this can be achieved by deleting repetitions). (By saying that the sequence (U_n) extends (T_n) is meant that there is an sequence $0 = n_0 \leqslant n_1 \leqslant n_2 \leqslant \cdots$ such that $T_k = U_{n_k}$ for each k.)

Let us show

$$\text{for each } n,\ |U_{n+1} \setminus U_n| = 1.$$

Indeed, since U_n is a proper subset of U_{n+1}, we have $|U_{n+1} \setminus U_n| \geqslant 1$. Suppose the left-hand side is greater than 1; we shall derive a contradiction. Choose two distinct elements p and q of $U_{n+1} \setminus U_n$. If we define $V = \mathrm{Co}(U_n \cup \{p\})$ then: V is relatively convex (since Co is a closure operation); it is contained in U_{n+1} (since $U_n \cup \{p\}$ so is, and since Co is a closure operation, and since U_{n+1} is relatively convex); and it properly contains U_n (it contains U_n since Co is a closure operation, and it does so properly since it contains p, which is not in U_n). Therefore, by maximality, $V = U_{n+1}$. That is, $\mathrm{Co}(U_n \cup \{p\}) = U_{n+1}$. Similarly, $\mathrm{Co}(U_n \cup \{q\}) = U_{n+1}$. In particular, $\mathrm{Co}\big(U_n \cup \{p\}\big) = \mathrm{Co}\big(U_n \cup \{q\}\big)$. Since q is in the right-hand side (recall that Co is a closure operation!), it is in the left-hand side, which is contained in the convex hull of $U_n \cup \{p\}$. Thus there exist a nonnegative real-valued function ϕ on U_n and a nonnegative real α such that

$$\alpha + \sum_{u \in U_n} \phi(u) = 1 \qquad \text{and} \qquad q = \alpha p + \sum_{u \in U_n} \phi(u) u\,. \tag{$*$}$$

It is impossible that $\alpha = 0$, which would mean $q \in \mathrm{Co}(U_n) = U_n$. It is also impossible that $\alpha = 1$, which would mean $q = p$. Thus $0 < \alpha < 1$. Similarly, there exist a nonnegative real-valued function ψ on U_n and a real β such that $0 < \beta < 1$,

$$\beta + \sum_{u \in U_n} \psi(u) = 1, \qquad \text{and} \qquad p = \beta q + \sum_{u \in U_n} \psi(u) u\,.$$

Inserting $(*)$, we obtain

$$p = (1 - \alpha\beta)^{-1} \sum_{u \in U_n} \big(\beta\phi(u) + \psi(u)\big) u\,.$$

Since the coefficients satisfy

$$(1 - \alpha\beta)^{-1} \sum_{u \in U_n} \big(\beta\phi(u) + \psi(u)\big) = (1 - \alpha\beta)^{-1} \big[\beta(1 - \alpha) + (1 - \beta)\big] = 1\,,$$

this shows $p \in \mathrm{Co}(U_n) = U_n$, a contradiction which proves (a).

Note that $U = U_m$. The proof will be complete when we shall have shown

(b) *for each n, every non-degenerate real-valued function on U_n extends to a non-degenerate real-valued function on U_{n+1} with the same rank of negativity.*

Indeed, we can then apply induction.

Thus we only have to show (b). Because of (a), this means that we only have to show, changing notation, that if U and V are finite relatively convex subsets of S such that $U \subset V$ and $|V \setminus U| = 1$ then every non-degenerate real-valued function f on U extends to a non-degenerate real-valued function on V with the same rank of negativity.

Let w be the unique element of $V \setminus U$. Since $U \subset V$, we have $\sqrt{U} \subset \sqrt{V}$. Thus we can choose an enumeration $(s_1, \ldots, s_n)$ of $\sqrt{V}$ such that $\sqrt{U} = \{s_1, \ldots, s_m\}$ for some m less than or equal to n. If $m = n$ then there is nothing to prove (define $f(w)$ to be an arbitrary real). Thus we may (and do!) assume $m < n$. On the other hand, if j and k are any two members of $\{m+1, \ldots, n\}$ then $2s_j = w$ and $2s_k = w$, in particular $2s_j = 2s_k$, whence $s_j = s_k$ and therefore $j = k$. Thus $n = m + 1$.

If κ is the rank of negativity of the function f then the matrix

$$M = \big(f(s_j + s_k)\big)_{j,k=1}^{m}$$

has rank of negativity κ. Since it is non-singular (by the non-degeneracy of f), its rank of positivity is $m - \kappa$. In other words, if we define an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^m$ by the condition that

$$\langle a, b \rangle = \sum_{j,k=1}^{m} a(j)\,\overline{b(k)} f(s_j + s_k) \qquad (a, b \in \mathbb{C}^m) \tag{$**$}$$

then $\langle \cdot, \cdot \rangle$ has rank of positivity $m - \kappa$. Choose a linear subspace P of $\mathbb{C}^m$, of dimension $m - \kappa$, which is a positive subspace for $\langle \cdot, \cdot \rangle$. Choose an orthonormal basis $(e_k)_{k=1}^{m-\kappa}$ of P, so

$$\langle e_j, e_k \rangle = \delta(j,k) \qquad (j, k = 1, \ldots, m - \kappa).$$

Embed $\mathbb{C}^m$ in $\mathbb{C}^{m+1}$ via $\mathbb{C}^m \ni \big(a(1), \ldots, a(m)\big) \mapsto \big(a(1), \ldots, a(m), 0\big) \in \mathbb{C}^{m+1}$. Extend the family $(e_k)_{k=1}^{m-\kappa}$ (now regarded as a family of elements of $\mathbb{C}^{m+1}$) to a family $(e_k)_{k=1}^{m-\kappa+1}$ of elements of $\mathbb{C}^{m+1}$ by setting $e_{m-\kappa+1} = (0, \ldots, 0, 1)$.

For each real t, define a function f_t on V by the conditions that $f_t|_U = f$ and $f_t(w) = t$. Let $\langle \cdot, \cdot \rangle_t$ be the associated inner product on $\mathbb{C}^{m+1}$, defined by analogy with $(**)$, only reading $m+1$ instead of m. We claim that there is a real A such that

$$\det\left(\left(\langle e_j, e_k\rangle_t\right)_{j,k=1}^{m-\kappa+1}\right) = t - A \qquad (t \in \mathbb{R}). \tag{$***$}$$

To see that this is so, we write out in full the matrix involved:

$$\left(\langle e_j, e_k\rangle_t\right)_{j,k=1}^{m-\kappa+1}$$

$$= \begin{pmatrix} 1 & 0 & \dots & 0 & \langle e_1, e_{m-\kappa+1}\rangle_t \\ 0 & 1 & \dots & 0 & \langle e_2, e_{m-\kappa+1}\rangle_t \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \langle e_{m-\kappa}, e_{m-\kappa+1}\rangle_t \\ \langle e_{m-\kappa+1}, e_1\rangle_t & \langle e_{m-\kappa+1}, e_2\rangle_t & \dots & \langle e_{m-\kappa+1}, e_{m-\kappa}\rangle_t & t \end{pmatrix}.$$

(We used the following facts: (1) if $j \leqslant m-\kappa$ and $k \leqslant m-\kappa$ then e_j and e_k are in $\mathbb{C}^m (\subset \mathbb{C}^{m+1})$, so $\langle e_j, e_k\rangle_t = \langle e_j, e_k\rangle = \delta(j,k)$ [since $f_t|_U = f$]; (2) $\langle e_{m-\kappa+1}, e_{m-\kappa+1}\rangle_t = f_t(w) = t$.) By expansion after the last row [or column], $(***)$ follows if we show that if $k \leqslant m-\kappa$ then $\langle e_k, e_{m-\kappa+1}\rangle_t$ does not depend on t. It is readily seen that $\langle e_k, e_{m-\kappa+1}\rangle_t$ is a linear combination of numbers of the form $f_t(s_j + s_{m+1})$ with $j \leqslant m$, so we only have to show that the latter does not depend on t, which is to say that if $j \leqslant m$ then $s_j + s_{m+1} \neq w$. Suppose $s_j + s_{m+1} = w$; we shall derive a contradiction. Indeed, since $w \notin U = \mathrm{Co}(U)$, the point w is an extreme point (vertex) of the convex polytope which is the convex hull of $U \cup \{w\} = w$. Now the points $2s_j$ and $2s_{m+1}$ are in that polytope and their convex combination $(1/2)[2s_j + 2s_{m+1}]$ is equal to w, which is a vertex, so in $2s_j = w = 2s_{m+1}$, whence $s_j = s_{m+1}$ and therefore $j = m+1 > m$, a contradiction which establishes $(***)$.

The remainder of the proof is now easy. Indeed, if we take t such that $t > A$ then the matrix in $(***)$ has positive determinant. Since its principal $(m-\kappa)\times(m-\kappa)$ submatrix in the upper left corner is positive definite (cf. the above expression) then by the principal subdeterminant criterion for positive definiteness it follows that the matrix in question, i.e., $\left(\langle e_j, e_k\rangle_t\right)_{j,k=1}^{m-\kappa+1}$, is positive definite. This proves that the inner product $\langle\cdot,\cdot\rangle_t$ has rank of positivity at least $m-\kappa+1$. Therefore, its rank of negativity is at most $(m+1)-(m-\kappa+1) = \kappa$. On the other hand, since $\langle\cdot,\cdot\rangle$ has rank of negativity κ, the rank of negativity of the extension $\langle\cdot,\cdot\rangle_t$ must be exactly κ. Hence its rank of positivity is at most $m-\kappa+1$ and therefore, by the preceding, is exactly $m-\kappa+1$. Since the sum of the ranks of positivity and negativity is $m+1$, the inner product $\langle\cdot,\cdot\rangle_t$ is non-degenerate. This proves that f_t is the desired extension of f, completing the proof. □

THEOREM 2. *There is a (necessarily non-exponentially bounded) non-degenerate real-valued function on $(\mathbb{N}_0^2, +, s^* = s)$ which has rank of negativity 1 but no positive ideal of codimension 1.*

Proof. Write $S = (\mathbb{N}_0^2, +, s^* = s)$ for brevity. Define $s_1 = (0,0)$, $s_2 = (2,1)$, $s_3 = (1,2)$ and $s_4 = (1,1)$, let $S_4 = \{s_1, s_2, s_3, s_4\}$, and define $U = S_4 + S_4$. It is readily verified (for example, by a drawing on paper) that U is relatively convex. Choose a matrix $B = (b_{jk})_{j,k=1}^4$ as in the lemma. Define a real-valued function f on U by the condition that $f(s_j + s_k) = b_{jk}$ for $j, k = 1, 2, 3, 4$. (This is well defined since the right-hand side is a symmetric function of j and k and since the mapping $(j,k) \mapsto s_j + s_k$ is one-to-one on the set $\{(j,k) \colon j \leqslant k\}$.) By the properties of B from the lemma, f is non-degenerate and has rank of negativity 1. By Proposition 5, f extends to a real-valued function on all of S, denoted by the same symbol, which is non-degenerate and has rank of negativity 1. All that remains to be shown is that there is no positive ideal of codimension 1. Suppose there is such an ideal, say L. Identifying $\mathbb{C}[S]$ with $\mathbb{C}[X,Y]$ (by identifying $\delta_{(m,n)}$ with $X^m Y^n$ whenever $(m,n) \in S$—it is a matter of taste whether this is actually the definition of $\mathbb{C}[X,Y]$), let φ be the linear form on $\mathbb{C}[X,Y]$ associated with f, so $\varphi(X^m Y^n) = f(m,n)$ whenever $(m,n) \in S$. Now L, being an ideal of codimension 1, is a maximal ideal and therefore (since $\mathbb{C}[X,Y]$ has an identity) is the kernel of an algebra homomorphism of $\mathbb{C}[X,Y]$ onto $\mathbb{C}$. Such a homomorphism has the form $\mathbb{C}[X,Y] \ni p \mapsto p(u,v)$ for some u and v in $\mathbb{C}$. Summing up, there is some (u,v) in $\mathbb{C}^2$ such that

$$L = \{p \in \mathbb{C}[X,Y] \colon p(u,v) = 0\}.$$

Given any $c = (c_1, c_2, c_3)$ in $\mathbb{C}^3$, we next seek c_4 in $\mathbb{C}$ such that the polynomial

$$p_c = \sum_{k=1}^{4} c_k \delta_{s_k}$$

is in L (recall that δ_{s_k} is 1, X^2Y, XY^2, or XY according as k is 1, 2, 3, or 4). We first note that $uv \neq 0$ (if not, then $q := XY \in L$, hence $0 \leqslant \varphi(q^* q) = \varphi(X^2Y^2) = f(2s_4) = b_{44}$, contradicting the fact that $b_{44} < 0$, cf. the lemma). It will be seen that the unique solution is

$$c_4 = -\frac{c_1 + c_2 u^2 v + c_3 u v^2}{uv} = -\sum_{k=1}^{3} c_k z_k ,$$

where $z_1 = (uv)^{-1}$, $z_2 = u$, and $z_3 = v$, so

$$z_1 z_2 z_3 = 1$$

and

$$p_c = \sum_{k=1}^{3} c_k \delta_{s_k} - \left(\sum_{k=1}^{3} c_k z_k \right) \delta_{s_4} = \sum_{k=1}^{3} c_k (\delta_{s_k} - z_k \delta_{s_4}),$$

hence

$$p_c^* p_c = \sum_{j,k=1}^{3} c_j \bar{c}_k (\delta_{s_j} - z_j \delta_{s_4})(\delta_{s_k} - \bar{z}_k \delta_{s_4})$$
$$= \sum_{j,k=1}^{3} c_j \bar{c}_k (\delta_{s_j+s_k} - z_j \delta_{s_4+s_k} - \bar{z}_k \delta_{s_4+s_j} + z_j \bar{z}_k \delta_{2s_4}) .$$

Since $p_c \in L$, we have

$$0 \leqslant \varphi(p_c^* p_c) = \sum_{j,k=1}^{3} c_j \bar{c}_k (b_{jk} - b_{4k} z_j - b_{4j} \bar{z}_k + b_{44} z_j \bar{z}_k) = \langle Ec, c \rangle ,$$

where E is the matrix from the lemma. This being so for all such c, the matrix E is positive semidefinite and in particular has nonnegative trace, contradicting the lemma and completing the proof of the theorem. □

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