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ON THE STRUCTURE OF SOME SUBSETS IN THE SPACE OF INTERVAL FUNCTIONS OF BOUNDED $SC(\Lambda)$-VARIATION

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ABSTRACT. In this paper a generalization of bounded variation of interval functions with respect to local systems is investigated. Some properties and the structure of different subsets of the space of interval functions of bounded $SC(\Lambda)$-variation are studied here.

The concept of real valued functions of one variable of bounded Λ-variation was introduced by D. Waterman [6]. This class of functions was considered by S. Perlman [3] and F. Prus–Wiśniowski [4], too. B. Thomson in his book [5] considered functions of bounded variation with respect to local systems. Here we will extend this concept to real valued interval functions, which have bounded $SC(\Lambda)$-variation with respect to local systems and covers, generated by this local system.

Definition 1. Let $[a,b]$ be an interval. A class $S = \{S(x)\colon x \in [a,b]\}$, where $S(x)$ denotes a family of subsets of $[a,b]$, is called a local system in $[a,b]$ if it satisfies the following conditions:

($S1$) $\{x\} \notin S(x)$;

($S2$) if $V \in S(x)$, then $x \in V$;

($S3$) if $V_1 \in S(x)$ and $V_1 \subset V_2$, then $V_2 \in S(x)$;

($S4$) $(x-\delta, x+\delta) \cap V \in S(x)$ if $\delta > 0$, $V \in S(x)$.

Definition 2. [5] Let S be a local system in $[a,b]$ and C—a family of closed subintervals of $[a,b]$. It will be said that C is an S-cover of $[a,b]$ if the set

$$\{x\} \cup \{y \in [a,b]\colon y < x \wedge [y,x] \in C\} \cup \{y\colon x < y \wedge [x,y] \in C\}$$

belongs to the class $S(x)$ for all x in $[a,b]$.

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Definition 3. [6] A sequence $\Lambda = (\lambda_i)_{i=1}^{\infty}$ of real positive numbers is said to be λ-sequence if it is nondecreasing and the series $\sum_{i=1}^{\infty} \frac{1}{\lambda_i}$ is divergent. Additionally, if Λ is unbounded λ-sequence then it is called a proper λ-sequence.

In the next part of this paper, by an interval function on $[a,b]$ we will understand any real valued function, which is defined only for all closed subintervals of $[a,b]$ and we will consider only nonempty S-covers.

Let S be a local system in $[a,b]$, C be an S-cover of $[a,b]$, $\Lambda = (\lambda_i)_{i=1}^{\infty}$ be a λ-sequence and ψ be a real valued interval function, which is defined on all closed subintervals of $[a,b]$. Then the expression:

$$V_{\Lambda}^{S}(\psi, C) = \sup \sum_{i=1}^{k} \frac{|\psi(I_i)|}{\lambda_i},$$

where supremum is taken for all finite collections of closed non-overlapping subintervals $\{I_1, \ldots, I_k\}$ of C and all positive integers k, is called an $SC(\Lambda)$-variation of the function ψ. Thus, if $V_{\Lambda}^{S}(\psi, C) < \infty$ then the interval function is said to be of bounded $SC(\Lambda)$-variation.
By the symbol $\Psi_{\Lambda}(S,C)$ we will denote the set of all interval functions of bounded $SC(\Lambda)$-variation. Additionally, if $\Lambda = (1)$ (i.e., Λ is constant sequence which terms equal 1) then we obtain a special class of functions, namely a class of functions of bounded SC-variation, and it will be denoted by $\Psi(S,C)$. The symbol $V^{S}(\psi, C)$ will stand for SC-variation of function ψ.

For interval functions ψ, ψ_1, ψ_2 defined in $[a,b]$ and a real number α, we define functions $\psi_1 + \psi_2$ and $\alpha\psi_1$ in usual way, i.e.,

$$(\psi_1 + \psi_2)(I) = \psi_1(I) + \psi_2(I),$$
$$(\alpha\psi)(I) = \alpha \cdot \big(\psi(I)\big)$$

for each closed interval I contained within $[a,b]$.

It is easy to prove the following:

Theorem 1. *Let S be a local system in $[a,b]$ and C be an S-cover of $[a,b]$.*

(1) *If $\psi_1 \in \Psi_{\Lambda}(S,C)$ and $\psi_2 \in \Psi_{\Lambda}(S,C)$, then*

$$V_{\Lambda}^{S}(\psi_1 + \psi_2, C) \leqslant V_{\Lambda}^{S}(\psi_1, C) + V_{\Lambda}^{S}(\psi_2, C).$$

(2) *If $\alpha \in \mathbb{R}$ and $\psi \in \Psi_{\Lambda}(S,C)$, then*

$$V_{\Lambda}^{S}(\alpha\psi, C) = |\alpha| \cdot V_{\Lambda}^{S}(\psi, C).$$

In the set $\Psi_\Lambda(S,C)$ we can define the relations

$$\psi_1 \sim_C \psi_2 \Longleftrightarrow V_\Lambda^S(\psi_1 - \psi_2, C) = 0$$

for any interval functions ψ_1, ψ_2 from $\Psi_\Lambda(S,C)$. One can prove that the relation "$\sim_C$" is an equivalence relation in $\Psi_\Lambda(S,C)$. Then the symbol $\widetilde{\Psi}_\Lambda(S,C)$ will stand for the set of all equivalence classes with respect to the relation "$\sim_C$". Of course, the space $\widetilde{\Psi}_\Lambda(S,C)$ is a linear space with the operations of addition and multiplication by scalars defined in the usual sense, i.e.,

$$\begin{gathered}[\psi_1] + [\psi_2] = [\psi_1 + \psi_2]\,,\\ \alpha[\psi] = [\alpha\psi]\,,\end{gathered}$$

where $[\psi_1], [\psi_2], [\psi]$ belong to $\widetilde{\Psi}_\Lambda(S,C)$ and α is a real number.
In $\widetilde{\Psi}_\Lambda(S,C)$ we can define the norm as follows:

$$\big\|[\psi]\big\|_\Lambda^{SC} = V_\Lambda^S(\psi, C) \tag{1}$$

for each $[\psi]$ in $\widetilde{\Psi}_\Lambda(S,C)$.

We will write $\psi \in \widetilde{\Psi}_\Lambda(S,C)$ if the equivalence class $[\psi]$ with respect to the relation "$\sim_C$" belongs to $\widetilde{\Psi}_\Lambda(S,C)$. One can discuss the difference between this relation with the relation "$\cong$" defined by

$$\psi_1 \cong \psi_2 \Longleftrightarrow \psi_1 - \psi_2 = \text{const}\ .$$

It follows that those relations are quite different. In fact, the relation $\psi_1 \sim_C \psi_2$ means that the functions ψ_1 and ψ_2 can be different only for intervals not belonging to the S-cover C.

Theorem 2. *For a local system S and an S-cover C of interval $[a,b]$, the space $\widetilde{\Psi}_\Lambda(S,C)$ is a Banach space with the norm* (1).

P r o o f. We have to prove that the space $\widetilde{\Psi}_\Lambda(S,C)$ is complete. Let $(\psi_i)_{i=1}^\infty$ be an arbitrary Cauchy sequence in $\widetilde{\Psi}_\Lambda(S,C)$. Let us take an arbitrary positive number ε. Then there exists a positive integer n_0 such that

$$V_\Lambda^S(\psi_n - \psi_m, C) < \frac{\varepsilon}{\lambda_1}\,,$$

if $m \geqslant n_0$ and $n \geqslant n_0$. Hence

$$\big|(\psi_n - \psi_m)(I)\big| < \varepsilon$$

if $I \in C$ and $m \geqslant n_0$ and $n \geqslant n_0$.

Since the sequence $\big(\psi_i(I)\big)_{i=1}^{\infty}$ satisfies Cauchy condition in complete space $\mathbb{R}$, then it is convergent to some real number. For example, let us define

$$\phi(I) := \begin{cases} \lim\limits_{i\to\infty} \psi_i(I) & \text{if } I \in C\,, \\ 0 & \text{if } I \notin C\,. \end{cases}$$

We will show that $\phi \in \widetilde{\Psi}_\Lambda(S, C)$.
Let $\{I_1, \ldots, I_{j_0}\}$ be a finite collections of non-overlapping intervals from C, where $j_0 \in \mathbb{N}$. Then there exists $k_0 \in \mathbb{N}$ such that

$$\big|\psi_n(I_j) - \phi(I_j)\big| < \frac{\lambda_1 \varepsilon}{j_0}\,,$$

for every $n \geqslant k_0$ and $j \in \{1, \ldots, j_0\}$. Thus,

$$\sum_{i=1}^{j_0} \frac{\big|\phi(I_i)\big|}{\lambda_i} \leqslant \sum_{i=1}^{j_0} \frac{\big|\phi(I_i) - \psi_n(I_i)\big|}{\lambda_i} + \sum_{i=1}^{j_0} \frac{\big|\psi_n(I_i)\big|}{\lambda_i} < \varepsilon + V_\Lambda^S(\psi_n, C) \tag{2}$$

if $n \geqslant k_0$.
Now we will see that

$$\big|V_\Lambda^S(\psi_n, C) - V_\Lambda^S(\psi_m, C)\big| < V_\Lambda^S(\psi_n - \psi_m, C) \tag{3}$$

if $m \geqslant m_0$ and $n \geqslant m_0$, where $m_0 = \max\{n_0, k_0\}$. Indeed, if we take a finite family of intervals $\{J_1, \ldots, J_l\}$ from C, which do not overlap, then

$$\sum_{i=1}^{l} \frac{\big||\psi_n(J_i)| - |\psi_m(J_i)|\big|}{\lambda_i} \leqslant \sum_{i=1}^{l} \frac{\big|\psi_n(J_i) - \psi_m(J_i)\big|}{\lambda_i} \leqslant V_\Lambda(\psi_n - \psi_m, C)$$

whence,

$$\sum_{i=1}^{l} \frac{\big|\psi_n(J_i)\big|}{\lambda_i} \leqslant V_\Lambda^S(\psi_n - \psi_m, C) + \sum_{i=1}^{l} \frac{\big|\psi_m(J_i)\big|}{\lambda_i} \leqslant V_\Lambda^S(\psi_n - \psi_m, C) + V_\Lambda(\psi_m, C)$$

and, consequently,

$$V_\Lambda^S(\psi_n, C) - V_\Lambda^S(\psi_m, C) \leqslant V_\Lambda^S(\psi_n - \psi_m, C)\,, \tag{4}$$

if $n \geqslant m_0$ and $m \geqslant m_0$. In the same way we can prove that

$$V_\Lambda^S(\psi_n, C) - V_\Lambda^S(\psi_m, C) \geqslant -V_\Lambda^S(\psi_n - \psi_m, C)\,, \tag{5}$$

for each $m \geqslant m_0$ and $n \geqslant m_0$. Inequalities (4) and (5) imply inequality (3) if $m \geqslant n_0$ and $n \geqslant n_0$. Then $\big|V_\Lambda^S(\psi_n, C) - V_\Lambda^S(\psi_m, C)\big| < \frac{\varepsilon}{\lambda_1}$, where m, n are positive integers not less then m_0. Hence, the sequence $\big(V_\Lambda^S(\psi_i, C)\big)_{i=1}^{\infty}$ satisfies Cauchy condition, which means that the limit $\lim\limits_{i\to\infty} V_\Lambda^S(\psi_i, C)$ exists and is finite.

By (2) we get

$$\sum_{i=1}^{k} \frac{|\phi(I_i)|}{\lambda_i} \leqslant \varepsilon + \lim_{n\to\infty} V_\Lambda^S(\psi_n, C)$$

and finally $V_\Lambda^S(\phi, C) < \infty$. Thus, $\phi \in \widetilde{\Psi}_\Lambda(S, C)$.

Moreover, it is easy to prove that $\lim\limits_{n\to\infty} \|\psi_n - \phi\|_\Lambda^{SC} = 0$. We have proved that the space $\widetilde{\Psi}_\Lambda(S, C)$ is complete and the proof is finished. □

THEOREM 3. *If S is a local system in $[a, b]$, C is an S-cover and $\Lambda = (\lambda_i)_{i=1}^{\infty}$ is a λ-sequence then*

$$\widetilde{\Psi}(S, C) \subset \widetilde{\Psi}_\Lambda(S, C).$$

Proof. Let us see $V_\Lambda^S(\psi, C) \leqslant \frac{1}{\lambda_1} \cdot V^S(\psi, C)$, where $\psi \in \widetilde{\Psi}_\Lambda(S, C)$. By this we infer that $\widetilde{\Psi}(S, C) \subset \widetilde{\Psi}_\Lambda(S, C)$. □

In the next part of the article we will make use of the next theorem:

THEOREM 4. [3] *Let $(a_i)_{i=1}^{\infty}$ be a sequence of positive numbers converging to zero. Then there exists a nonincreasing sequence $(b_i)_{i=1}^{\infty}$ converging to zero such that $\sum\limits_{i=1}^{\infty} b_i = \infty$ and $\sum\limits_{i=1}^{\infty} a_i b_i < \infty$.*

THEOREM 5. *Let S be a local system in $[a, b]$ and C be an S-cover of $[a, b]$. Then*

$$\widetilde{\Psi}(S, C) = \widetilde{\Psi}_\Lambda(S, C) \tag{6}$$

iff the sequence $\Lambda = (\lambda_i)_{i=1}^{\infty}$ is bounded.

Proof. Let us assume that equality (6) holds and suppose that the sequence $\Lambda = (\lambda_i)_{i=1}^{\infty}$ is unbounded. Since the sequence $\left(\frac{1}{\lambda_i}\right)_{i=1}^{\infty}$ satisfies assumptions of Theorem 4, there exists a nonincreasing sequence $(b_i)_{i=1}^{\infty}$ of positive numbers such that

$$\sum_{i=1}^{\infty} b_i = \infty \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{b_i}{\lambda_i} < \infty.$$

Let i_0 be a positive integer such that $a + \frac{1}{i_0} < b$. Define

$$c_i := a + \frac{1}{i}$$

if $i > i_0$. Since each S-cover is Vitali cover, we can find for each $i > i_0$ some closed intervals $I_i \in C$ such that $c_i \in I_i$ and

$$\sup\{x \colon x \in I_{i+1}\} \leqslant \inf\{x \colon x \in I_i\} \tag{7}$$

for every $i > i_0$. By this remark we can find a countable family $(I_i)_{i=i_0}^{\infty}$ of non-overlapping intervals of C which satisfy condition (7). Let us put:

$$\psi(I) := \begin{cases} b_i\,, & \text{if } I = I_i\,, \\ 0\,, & \text{if } I \neq I_i\,, \end{cases} \quad i \in \{i_0, i_0+1, \dots\}\,.$$

Then $V^S(\psi, C) \geqslant \sum_{i=i_0}^{\infty} b_i$, so $V^S(\psi, C) = \infty$. On the other hand, $V_\Lambda^S(\psi, C) \leqslant \sum_{i=i_0}^{\infty} \frac{b_i}{\lambda_i}$, and by this, $V_\Lambda^S(\psi, C) < \infty$. Then we conclude that the set $\widetilde{\Psi}_\Lambda(S, C) \setminus \widetilde{\Psi}(S, C)$ is nonempty and it contradicts to assumption.

Now, let us assume that the sequence $\Lambda = (\lambda_i)_{i=1}^{\infty}$ is bounded. Let ψ be a function of bounded $SC(\Lambda)$-variation. By assumption there exists a positive number M such that $\lambda_i \leqslant M$ for all $i \in \mathbb{N}$. Then $V^S(\psi, C) \leqslant M V_\Lambda^S(\psi, C)$ and from this we get that $\psi \in \widetilde{\Psi}(S, C)$. Thus $\widetilde{\Psi}_\Lambda(S, C) \subset \widetilde{\Psi}(S, C)$. By Theorem 3 the inclusion $\widetilde{\Psi}(S, C) \subset \widetilde{\Psi}_\Lambda(S, C)$ holds and, consequently, we get the required equality. □

Definition 4. [2] Let (Y, d) be a metric space. A set M is said to be porous at $x \in Y$ if there exists a positive number α such that for each $\varepsilon > 0$ there exists a point y in an open ball $K^0(x, \varepsilon)$ with the center x and radius ε such that

$$K^0\big(x, \alpha \cdot d(x, y)\big) \cap M = \emptyset\,.$$

If α from the above condition can be chosen arbitrarily close to 1, then the set M is said to be strongly porous at x.

A set which is porous (strongly porous) at every point x from Y is called porous (strongly porous).

Definition 5. [2] A set which is a countable union of porous (strongly porous) sets is said to be σ-porous (σ-strongly porous).

Theorem 6. [1] *Let (X, τ) be a linear topological space. If G is a subgroup of X and G contains a set A, which is of second category of Baire in (X, τ), then $G = E$. Hence, if G is any proper subgroup of X, then it is of the first category of Baire or fails to satisfy Baire condition in (X, τ).*

Theorem 7. [2] *Any nowhere dense and convex set in a Banach space is strongly porous.*

Theorem 8. *Let S be a local system in $[a, b]$, C be an S-cover of $[a, b]$ and $(\psi_i)_{i=1}^{\infty}$ be a converging sequence of interval functions. If*

$$\psi(I) = \lim_{i \to \infty} \psi_i(I)$$

for all closed subintervals $I \in C$, then

$$V_{\Lambda}^{S}(\psi, C) \leqslant \liminf_{i\to\infty} V_{\Lambda}^{S}(\psi_i, C). \tag{8}$$

Proof. Let us denote $L = \liminf\limits_{i\to\infty} V_{\Lambda}^{S}(\psi_i, C)$. If $L = \infty$ then inequality (8) is obvious, so we consider the case when $L < \infty$. Choose any $\varepsilon > 0$. Then

$$\sup_i \inf_k V_{\Lambda}^{S}(\psi_{i+k}, C) < L + \varepsilon .$$

By this condition we can find a subsequence $(\psi_{n_i})_{i=1}^{\infty}$ of the sequence $(\psi_i)_{i=1}^{\infty}$ such that

$$V_{\Lambda}^{S}(\psi_{n_i}, C) < L + \varepsilon$$

holds for every positive integer i. Let $\{I_1, \dots, I_k\}$ be a finite family of non-overlapping intervals of C. Then:

$$\sum_{k=1}^{\infty} \frac{|\psi_{n_i}(I_k)|}{\lambda_i} < L + \varepsilon ,$$

$$\lim_{i\to\infty} \sum_{k=1}^{\infty} \frac{|\psi_{n_i}(I_k)|}{\lambda_i} \leqslant L + \varepsilon$$

and at last

$$\sum_{k=1}^{\infty} \frac{|\psi(I_k)|}{\lambda_i} \leqslant L + \varepsilon .$$

Since the family $\{I_1, \dots, I_k\}$ was chosen in an arbitrary way, $V_{\Lambda}^{S}(\psi, C) \leqslant L + \varepsilon$ for each positive ε.

Now, we can state that $V_{\Lambda}^{S}(\psi, C) \le L$, which completes the proof. □

For a subset A of $\widetilde{\Psi}_{\Lambda}(S, C)$ the symbol $\mathrm{cl}_{\Lambda}(A)$ will denote the closure of set A in $\widetilde{\Psi}_{\Lambda}(S, C)$.

THEOREM 9. *Let S be a local system, C—an S-cover of $[a, b]$. If*

$$\widetilde{\Psi}_{\Gamma}(S, C) \subsetneq \widetilde{\Psi}_{\Lambda}(S, C),$$

then $\widetilde{\Psi}_{\Gamma}(S, C)$ is σ-strongly porous in the space $\widetilde{\Psi}_{\Lambda}(S, C)$.

Proof. Let us put

$$A_n := \{\psi \in \widetilde{\Psi}_{\Gamma}(S, C) \colon V_{\Gamma}^{S}(\psi, C) \leqslant n\}, \qquad n \in \mathbb{N}.$$

(*i*) First, we prove that A_n is a closed subset of $\widetilde{\Psi}(S, C)$. If we take $\psi \in \mathrm{cl}_{\Lambda}(A_n)$, then there exists a sequence of interval function $(\psi_i)_{i=1}^{\infty}$ such that $\psi_i \in A_n$ for all positive integers i and $\psi(I) = \lim_{i\to\infty} \psi_i(I)$ for each $I \in C$. By Theorem 8 we have: $V_\Gamma^S(\psi, C) \leqslant n$, whence $\psi \in A_n$.

(ii) Now we will show that A_n is a nowhere dense set.

From assumptions the set $\widetilde{\Psi}_\Gamma(S,C)$ forms a proper subgroup of $\widetilde{\Psi}_\Lambda(S,C)$ and

$$\widetilde{\Psi}_\Gamma(S,C) = \bigcup_{n=1}^{\infty} A_n\,. \tag{9}$$

By this and (i) we infer that $\widetilde{\Psi}_\Gamma(S,C)$ is an F_σ set. By Theorem 6 the set $\widetilde{\Psi}_\Gamma(S,C)$ is of the first category of Baire in a complete space $\widetilde{\Psi}_\Lambda(S,C)$ and consequently it is a boundary set. From that it follows that A_n is nowhere dense as a boundary and a closed set.

(iii) It is not difficult to see that each of the sets A_n is convex in $\widetilde{\Psi}_\Lambda(S,C)$.

From the previous theorem and Theorem 7 we infer that each of the sets A_n is strongly porous. By this and equality (9) we infer that $\widetilde{\Psi}_\Gamma(S,C)$ is σ-strongly porous. □

Immediately from Theorem 5 and Theorem 9 we get

COROLLARY 1. *For every proper λ-sequence Λ the set of all SC-bounded interval functions is σ-strongly porous in the space of all $SC(\Lambda)$-bounded interval functions.*

We will see that if $\Lambda = (\lambda_i)_{i=1}^\infty$ and $\Gamma = (\gamma_i)_{i=1}^\infty$ are two λ-sequences then the sequence Σ, where $\Sigma = \left(\frac{\lambda_i\gamma_i}{\lambda_i+\gamma_i}\right)_{i=1}^\infty$, is a λ-sequence, too. It is easy to prove that

$$\widetilde{\Psi}_\Sigma(S,C) = \widetilde{\Psi}_\Lambda(S,C) \cap \widetilde{\Psi}_\Gamma(S,C)$$

for each local system S and S-cover C of $[a,b]$.

COROLLARY 2. *Let S be a local system and C be an S-cover of $[a,b]$. If Λ and Γ are λ-sequences such that*

$$\widetilde{\Psi}_\Lambda(S,C) \cap \widetilde{\Psi}_\Gamma(S,C) \subsetneq \widetilde{\Psi}_\Lambda(S,C)$$

then $\widetilde{\Psi}_\Lambda(S,C) \cap \widetilde{\Psi}_\Gamma(S,C)$ is σ-strongly porous set in the space $\widetilde{\Psi}_\Lambda(S,C)$.

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