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ON MODULAR SEQUENCE SPACES DETERMINED BY SEQUENCE OF ORLICZ FUNCTIONS

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ABSTRACT. We define a modular sequence space $l(A, \{f_n\})$ determined by the modular $\varrho(x) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{nk} f_k(|\xi_k|)$, where $x = \{\xi_k\}_{n=1}^{\infty}$, $A = [a_{nk}]_{n,k \in N}$ is a regular matrix and $\{f_n\}_{n=1}^{\infty}$ is a sequence of non-degenerate Orlicz functions. This generalizes the modular spaces considered in [J. Ewert, T. Šalát: *Applications of the category method in the theory of modular sequence spaces*, Acta Math. Univ. Comenian. XLVIII–XLIX, (1986), 133–143], [J. Lindenstrauss, L. Tzafriri: *Classical Banach Spaces.* Sequence Spaces I, Springer-Verlag, Berlin, 1977], [J. Musielak: *Orlicz Spaces and Modular Spaces*, Lecture Notes in Math., Vol. 1034, Springer Verlag, Berlin, 1983], [J. Y. T. Woo: *On modular sequence spaces*, Studia Math. **48** (1973), 271–289].

We give some properties of such spaces.

1. Preliminaries

Let X be a real linear space. A functional $\varrho\colon X \to [0,\infty]$ is said to be a convex modular if

(a) $\varrho(x) = 0$ if and only if $x = 0$;

(b) $\varrho(-x) = \varrho(x)$;

(c) $\varrho(\alpha x + \beta y) \leqslant \alpha\varrho(x) + \beta\varrho(y)$ for $x, y \in X$, $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$.

The set

$$X_\varrho = \{x \in X : \exists_{\lambda>0}\ \varrho(\lambda x) < \infty\}$$

is a linear subspace of X and it is called the modular space determined by ϱ. The formula

$$\|x\| = \inf\left\{\lambda > 0 \colon \varrho\left(\frac{x}{\lambda}\right) \leqslant 1\right\}$$

defines a norm on X_ϱ, (cf. [7], [8]).

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Lemma 1.1. ([3]) *Let C be a subset of a modular space X_ϱ. If for each $x \in C$ there exists an $\varepsilon > 0$ such that*

$$\{y \in X_\varrho : \varrho(x-y) < \varepsilon\} \subset C,$$

then C is open in X_ϱ.

A function $f\colon [0,\infty) \to [0,\infty)$ is said to be an Orlicz function if it is continuous, non-decreasing, convex and $\lim_{t\to\infty} f(t) = \infty$. If $f(t) = 0$ for some $t > 0$, then f is said to be a degenerate Orlicz function, (cf. [6, p. 137]).

In what follows we shall use the following notions of porous sets in metric spaces, (cf. [9], [12]). A set M in a metric space (X, ϱ) is called porous at $x \in X$ if there is a positive number α such that for any $\varepsilon > 0$ there is a point y in the open ball $B(x, \varepsilon)$ with the center x and radius ε such that

$$B\big(y, \alpha\varrho(x,y)\big) \cap M = \emptyset.$$

If the above number α can be chosen to 1 as close as we wish, the set M is called strongly porous at x. A set M is called porous (strongly porous) if it is porous (strongly porous) at all points of X. A countable union of strongly porous sets is called σ-strongly porous. Obviously, every strongly porous set is nowhere dense.

Lemma 1.2. ([9]) *Let M be a convex nowhere dense set in a Banach space X. Then M is strongly porous.*

In the next section we will use the following Banach's Theorem.

Theorem 1.1. ([4, p. 93]) *If G is a subset of a linear topological space E such that $G - G \subset G$ (that is, G is a subgroup of E), and if G contains a set A that is of the second category in E and satisfies the condition of Baire, then $G = E$. Hence if G is any proper subgroup of E, then G is either of the first category in E or else fails to satisfy the condition of Baire.*

In the sequel s denotes a linear space of all sequences of real numbers. Further, by l^∞ let us denote the linear space of all bounded sequences of real numbers with a norm $\|x\|_\infty = \sup_{k\in\mathbb{N}} |\xi_k|$ for $x = \{\xi_k\}_{k=1}^\infty$.

2. Modular sequence space determined by a sequence of Orlicz functions

Let $A = [a_{nk}]_{n,k\in\mathbb{N}}$ be a matrix satisfying the conditions:

(A_1) $a_{nk} \geqslant 0$ for each $n, k \in \mathbb{N}$;

(A_2) $\sum_{k=1}^{\infty} a_{nk} < \infty$ for each $n \in \mathbb{N}$;

(A_3) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1$;

(A_4) $\lim_{n\to\infty} a_{nk} = 0$ for each $k \in \mathbb{N}$;

(A_5) $r_k = \sup_{n\in\mathbb{N}} a_{nk} > 0$ for each $k \in \mathbb{N}$.

The matrix A is called a regular matrix, (cf. [5], [10]).

Assume that a regular matrix A and a sequence $\{f_n\}_{n=1}^{\infty}$ of non-degenerate Orlicz functions are given. For $x = \{\xi_k\}_{k=1}^{\infty} \in s$, we put

$$\varrho(x) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{nk} f_k(|\xi_k|)$$

and

$$l(A, \{f_n\}) = \{x \in s : \varrho(\lambda x) < \infty \text{ for some } \lambda > 0\};$$
$$l^o(A, \{f_n\}) = \{x \in s : \varrho(\lambda x) < \infty \text{ for each } \lambda > 0\}.$$

Then ϱ is a convex modular on s and $l(A, \{f_n\})$ is a modular space determined by ϱ. It is normed as usually by the modular norm defined as

$$\|x\| = \inf\left\{\lambda > 0 \colon \varrho\left(\frac{x}{\lambda}\right) \leqslant 1\right\}.$$

For the matrix $A = [a_{nk}]_{n,k\in\mathbb{N}}$, where

$$a_{nk} = \begin{cases} 1; & n = k, \\ 0; & n \neq k, \end{cases}$$

we obtain the space considered in [6, p. 166], [8], [11], [3].

If $f_n = f$ for each natural n, then $l(A, \{f_n\})$ gives the Orlicz sequence space. In particular, if $f_n(x) = x^p$, $p \geqslant 1$, $n \in \mathbb{N}$, we obtain the space l^p. Moreover, in this case the norm given by the modular on $l(A, \{f_n\})$ coincides with the classical norm on l^p. Even in this particular case, all the theorems presented in this paper stand for new results.

Without loss of generality we can suppose that $f_n(1) = 1$ for each $n \in \mathbb{N}$. Assume that $f_n(\alpha_n) = 1$ for $\alpha_n > 0$. Putting $g_n(t) = f_n(\alpha_n t)$ for $t \in [0, \infty)$, we obtain the sequence $\{g_n\}_{n=1}^{\infty}$ of non-degenerate Orlicz functions with $g_n(1) = 1$ for each $n \in \mathbb{N}$. Then it is trivial to show that $l(A, \{f_n\})$ and $l(A, \{g_n\})$ are isometric.

Definition 2.1. ([2]) A sequence $\{f_n\}_{n=1}^{\infty}$ of Orlicz functions is said to satisfy the *strong uniform* Δ_2*-condition*, if there exists $L > 0$ such that the inequality

$$f_j(2t) \leqslant L f_j(t)$$

holds for each $t > 0$ and $j \in \mathbb{N}$.

Theorem 2.1. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-degenerate Orlicz functions with the property $f_n(1) = 1$ for each $n \in \mathbb{N}$. Let A be a regular matrix. Then:*

(1) *$l(A, \{f_n\})$ is a Banach space.*

(2) *If the sequence $\{f_n\}_{n=1}^{\infty}$ satisfies the strong uniform Δ_2-condition, then $l(A, \{f_n\}) = l^o(A, \{f_n\})$.*

Proof. (1) Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $l(A, \{f_n\})$, $x_n = \{\xi_{nj}\}_{j=1}^{\infty}$ and $\varepsilon > 0$. We choose $k(\varepsilon) \in \mathbb{N}$ such that

$$\varrho\left(\frac{x_k - x_m}{\varepsilon}\right) \leqslant 1 \qquad \text{for } m, k \geqslant k(\varepsilon).$$

Hence for all natural numbers n, p and $k, m \geqslant k(\varepsilon)$ we have

$$\sum_{i=1}^{n} \sum_{j=1}^{p} a_{ij} f_j\left(\frac{|\xi_{kj} - \xi_{mj}|}{\varepsilon}\right) \leqslant 1. \tag{2.1}$$

So,

$$\forall_{i,j\in\mathbb{N}} \forall_{k,m\geqslant k(\varepsilon)} \quad a_{ij} f_j\left(\frac{|\xi_{kj} - \xi_{mj}|}{\varepsilon}\right) \leqslant 1.$$

Condition (A_4) implies

$$\forall_{j\in\mathbb{N}} \exists_{i(j)} \forall_{i\geqslant i(j)} \quad a_{ij} < 1.$$

From convexity of functions f_j it follows that

$$\forall_{j\in\mathbb{N}} \forall_{i\geqslant i(j)} \forall_{k,m\geqslant k(\varepsilon)} \quad f_j\left(a_{ij}\frac{|\xi_{kj} - \xi_{mj}|}{\varepsilon}\right) \leqslant 1,$$

so $a_{ij}|\xi_{kj} - \xi_{mj}| \leqslant \varepsilon$ for $j \in \mathbb{N}$, $i \geqslant i(j)$, $k, m \geqslant k(\varepsilon)$. We have shown that $\{\xi_{kj}\}_{k=1}^{\infty}$ is a Cauchy sequence for each $j \in \mathbb{N}$. Let us denote $\xi_j = \lim\limits_{k\to\infty} \xi_{kj}$ for $j \in \mathbb{N}$ and $x = \{\xi_j\}_{j=1}^{\infty}$. Letting $m \to \infty$ in (2.1) we obtain

$$\forall_{n,p\in\mathbb{N}} \forall_{k\geqslant k(\varepsilon)} \quad \sum_{i=1}^{n} \sum_{j=1}^{p} a_{ij} f_j\left(\frac{|\xi_{kj} - \xi_{j}|}{\varepsilon}\right) \leqslant 1,$$

which implies

$$\varrho\left(\frac{x_k - x}{\varepsilon}\right) \leqslant 1 \qquad \text{for } k \geqslant k(\varepsilon).$$

In consequence, $x \in l(A, \{f_n\})$ and $\|x_k - x\| \leqslant \varepsilon$ for $k \geqslant k(\varepsilon)$, which completes the proof of (1).

(2) Let $x = \{\xi_j\}_{j=1}^{\infty} \in l(A, \{f_n\})$ and $\lambda > 0$. There exists $\lambda_1 > 0$ such that $\varrho(\lambda_1 x) < \infty$. We can choose $k \in \mathbb{N}$ such that $\lambda \leqslant 2^k \lambda_1$. Then $\varrho(\lambda x) \leqslant \varrho(2^k \lambda_1 x) \leqslant L^k \varrho(\lambda_1 x) < \infty$, so $\varrho(\lambda x) < \infty$ for each $\lambda > 0$. Thus $x \in l^o(A, \{f_n\})$. □

PROPOSITION 2.1. *For a regular matrix $A = [a_{nk}]_{n,k\in\mathbb{N}}$ and a sequence $\{f_n\}_{n=1}^{\infty}$ of non-degenerate Orlicz functions the following conditions are equivalent:*

(a) $l^\infty \subset l(A, \{f_n\})$;

(b) *there exists $t > 0$ such that* $\sum\limits_{n=1}^{\infty} \sum\limits_{k=1}^{\infty} a_{nk} f_k(t) < \infty$.

Proof. If $l^\infty \subset l(A, \{f_n\})$, then $x_1 = \{1, 1, 1, \dots\} \in l(A, \{f_n\})$. Thus for some $t > 0$ we have $\infty > \varrho(tx) = \sum\limits_{n=1}^{\infty} \sum\limits_{k=1}^{\infty} a_{nk} f_k(t)$. Conversely, let $t > 0$ be such that $\sum\limits_{n=1}^{\infty} \sum\limits_{k=1}^{\infty} a_{nk} f_k(t) < \infty$ and let $x = \{\xi_j\}_{j=1}^{\infty} \in l^\infty$. We can choose $p \in \mathbb{N}$ such that $\frac{1}{2^p} \sup\limits_{j\in\mathbb{N}} |\xi_j| < t$. So

$$\varrho\left(\frac{1}{2^p} x\right) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{nk} f_k\left(\frac{1}{2^p} |\xi_j|\right) \leqslant \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{nk} f_k(t) < \infty$$

and $x \in l(A, \{f_n\})$. □

THEOREM 2.2. *For each sequence $\{f_n\}_{n=1}^{\infty}$ of non-degenerate Orlicz functions there exists a regular matrix A such that $l^\infty \neq l(A, \{f_n\})$.*

Proof. In [1, Th. 2.1] we constructed the matrix $A = [a_{nk}]_{n,k\in\mathbb{N}}$ in the following way: at first by induction we choose an increasing sequence $\{p_j\}_{j=1}^{\infty}$ of natural numbers such that

$$p_1 = 1;$$
$$p_{j+1} \geqslant f_{2j}^{-1}(2^{p_j}) > j;$$
$$f_{2(j+1)}^{-1}(2^{p_{j+1}}) > f_{2j}^{-1}(2^{p_j}),$$

then we denote

$$
\begin{aligned}
a_{nk} &= 0 && \text{for } k = 1, 2, \ldots, 2n-2\,;\\
a_{n2n-1} &= \sum_{i=1}^{n} \frac{1}{2^i}\,; && \\
a_{n2n} &= 2^{-p_n - n}\,; && \\
a_{nk} &= 0 && \text{for } k = 2n+1, 2n+2, \ldots
\end{aligned}
$$

Thus A is a regular matrix. For the sequence

$$x_o = \{\xi_j\}_{j=1}^{\infty}\,, \qquad \text{where } \xi_{2j-1} = 0\,,\ \xi_{2j} = f_{2j}^{-1}(2^{p_j}) \text{ for } j \in \mathbb{N},$$

we have

$$\sum_{k=1}^{\infty} a_{nk} f_k(|\xi_k|) = \frac{1}{2^{p_n+n}} 2^{p_n} = \frac{1}{2^n}\,.$$

Hence $\varrho(x_o) = 1$, so $x_o \in l(A, \{f_n\})$. But $x_o \notin l^\infty$. $\square$

Proposition 2.2. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-degenerate Orlicz functions with the property $f_n(1) = 1$ for each $n \in \mathbb{N}$. Let $A = [a_{nk}]_{n,k\in\mathbb{N}}$ be a regular matrix. If $\inf\limits_k \sup\limits_n a_{nk} > 0$, then $l(A, \{f_n\}) \subset l^\infty$ and*

$$\|x\|_\infty \leqslant \max\left\{1, \left(\inf_k \sup_n a_{nk}\right)^{-1}\right\} \cdot \|x\|$$

for $x \in l(A, \{f_n\})$.

P r o o f. Let us denote $M = \inf\limits_k \sup\limits_n a_{nk}$. Let $x = \{\xi_j\}_{j=1}^{\infty} \in l(A, \{f_n\})$ and $\varepsilon > 0$. From the following inequality $\varrho\left(\frac{x}{\|x\|+\varepsilon}\right) \leqslant 1$ we have $\sum\limits_{n=1}^{\infty} \sum\limits_{k=1}^{\infty} a_{nk} f_k\left(\frac{|\xi_k|}{\|x\|+\varepsilon}\right) \leqslant 1$, so $a_{nk} f_k\left(\frac{|\xi_k|}{\|x\|+\varepsilon}\right) \leqslant 1$ for each $n \in \mathbb{N}$, $k \in \mathbb{N}$. Hence

$$M f_k\left(\frac{|\xi_k|}{\|x\|+\varepsilon}\right) \leqslant \left(\sup_n a_{nk}\right) f_k\left(\frac{|\xi_k|}{\|x\|+\varepsilon}\right) \leqslant 1$$

for each $k \in \mathbb{N}$. So

$$f_k\left(\frac{|\xi_k|}{\|x\|+\varepsilon}\right) \leqslant \frac{1}{M} \qquad \text{for each } k \in \mathbb{N}. \tag{2.2}$$

If $M \geqslant 1$, then $f_k\left(\frac{|\xi_k|}{\|x\|+\varepsilon}\right) \leqslant 1 = f_k(1)$ for $k \in \mathbb{N}$. Applying the conditions of f_k we obtain $|\xi_k| \leqslant \|x\| + \varepsilon$, and $|\xi_k| \leqslant \|x\|$ for each $k \in \mathbb{N}$. This means that

$$\|x\|_\infty \leqslant \|x\|\,.$$

If $M < 1$, then from (2.2) we have $M \cdot f_k\left(\frac{|\xi_k|}{\|x\|+\varepsilon}\right) \leqslant 1$ for each $k \in \mathbb{N}$. Hence $f_k\left(\frac{M\cdot|\xi_k|}{\|x\|+\varepsilon}\right) \leqslant 1 = f_k(1)$. Thus $M \cdot |\xi_k| \leqslant \|x\|$ for each $k \in \mathbb{N}$, so

$$M \cdot \|x\|_\infty \leqslant \|x\|.$$

In consequence we get that $x \in l^\infty$ and $\|x\|_\infty \leqslant \max\{1, \frac{1}{M}\}\|x\|$. □

Proposition 2.3. *Let $\{f_n\}_{n=1}^\infty$ be a sequence of non-degenerate Orlicz functions. If a regular matrix $A = [a_{nk}]_{n,k\in\mathbb{N}}$ is such that $\sum\limits_{n=1}^\infty a_{nk} < \infty$ for each $k \in \mathbb{N}$, then each sequence with only a finite number of non-zero terms belongs to $l(A, \{f_n\})$.*

Proof. Let $x = \{\xi_j\}_{j=1}^\infty$ be a sequence such that $\xi_{k_i} \neq 0$ for $i = 1, 2, \ldots, m$, where $m \in \mathbb{N}$, and $\xi_k = 0$ for $k \notin \{k_1, k_2, \ldots, k_m\}$. Then

$$\varrho(x) = \sum_{n=1}^\infty \Big(a_{nk_1} f_{k_1}(|\xi_{k_1}|) + \cdots + a_{nk_m} f_{k_m}(|\xi_{k_m}|)\Big) < \infty,$$

i.e., $x \in l(A, \{f_n\})$. □

In what follows we shall use the following notations: $e_k = \{\alpha_{kj}\}$, where $\alpha_{kj} = 1$ for $k = j$ and $\alpha_{kj} = 0$ for $k \neq j$, for $k \in \mathbb{N}$.

Theorem 2.3. *Let $\{f_n\}_{n=1}^\infty$ be a sequence of non-degenerate Orlicz functions, which satisfies the strong uniform Δ_2-condition. If the regular matrix $A = [a_{nk}]_{n,k\in\mathbb{N}}$ is such that $\sum\limits_{n=1}^\infty a_{nk} < \infty$ for each $k \in \mathbb{N}$, then for each $x = \{\xi_j\}_{j=1}^\infty \in l(A, \{f_n\})$ we obtain $x = \sum\limits_{j=1}^\infty \xi_j e_j$.*

Proof. Let $\lambda > 0$. For the fixed $x = \{\xi_j\}_{j=1}^\infty \in l(A, \{f_n\})$ and $\varepsilon > 0$ we choose $n_o \in \mathbb{N}$ with

$$\sum_{i=n_o+1}^\infty \sum_{k=1}^\infty a_{ik} f_k(\lambda|\xi_k|) < \frac{\varepsilon}{2}.$$

For $n = 1, 2, \ldots, n_o$ the series

$$\sum_{k=1}^\infty a_{1k} f_k(\lambda|\xi_k|), \sum_{k=1}^\infty a_{2k} f_k(\lambda|\xi_k|), \ldots, \sum_{k=1}^\infty a_{n_ok} f_k(\lambda|\xi_k|)$$

are convergent. Thus there exists $k_o \in \mathbb{N}$ such that

$$\sum_{j=k_o+1}^\infty a_{mj} f_j(\lambda|\xi_j|) < \frac{\varepsilon}{2n_o}$$

for $m = 1, 2, \dots, n_o$. Let us take $k \geqslant k_o$. Now putting

$$y_k = \sum_{j=1}^{k} \xi_j e_j = \{\xi_1, \xi_2, \dots, \xi_k, 0, 0, \dots\}$$

we have $x - y_k = \{0, \dots, 0, \xi_{k+1}, \xi_{k+2}, \dots\}$. Let us denote $x - y_k = \{\eta_j\}_{j=1}^{\infty}$. Then

$$\begin{aligned}\varrho\big(\lambda(x-y_k)\big) &= \sum_{i=1}^{n_o}\sum_{j=1}^{\infty} a_{ij} f_j\big(\lambda|\eta_j|\big) + \sum_{i=n_o+1}^{\infty}\sum_{j=1}^{\infty} a_{ij} f_j\big(\lambda|\eta_j|\big) < \\ &< \sum_{i=1}^{n_o}\sum_{j=k+1}^{\infty} a_{ij} f_j\big(\lambda|\eta_j|\big) + \frac{\varepsilon}{2} < \sum_{i=1}^{n_o} \frac{\varepsilon}{2n_o} + \frac{\varepsilon}{2} = \varepsilon\,.\end{aligned}$$

In consequence we obtain the condition

$$\forall_{\lambda>0}\,\forall_{\varepsilon>0}\,\exists_{k_o}\,\forall_{k\geqslant k_o}\quad \varrho\big(\lambda(x-y_k)\big) < \varepsilon\,,$$

i.e., $\|x - y_k\| \to 0$, $k \to \infty$. So $x = \lim\limits_{k\to\infty} y_k = \lim\limits_{k\to\infty} \sum_{j=1}^{k} \xi_j e_j = \sum_{j=1}^{\infty} \xi_j e_j$. □

Remark. With assumptions from the last theorem the set of all sequences with only a finite number of non-zero terms constitutes a base of the space $l\big(A, \{f_n\}\big)$.

Definition 2.2. ([6, p. 167]) A sequence $\{f_n\}_{n=1}^{\infty}$ of Orlicz functions is said to satisfy the *uniform* Δ_2*-condition* if there exist $K > 0$ and $k_o \in \mathbb{N}$ such that

$$f_k(2u) \leqslant K \cdot f_k(u)$$

for each $k \geqslant k_o$ and $u \in \big[0, \frac{1}{2}\big]$.

Theorem 2.4. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-degenerate Orlicz functions with the property $f_n(1) = 1$ for each $n \in \mathbb{N}$. Let $A = [a_{nk}]_{n,k\in\mathbb{N}}$ be a regular matrix such that $\sum\limits_{n=1}^{\infty} a_{nk} < \infty$ for each $k \in \mathbb{N}$. If the sequence $\{f_n\}_{n=1}^{\infty}$ satisfies the uniform Δ_2-condition, then for each $t \neq 0$ we have $x = \{t, t, \dots\} \notin l\big(A, \{f_n\}\big)$.*

P r o o f. From (A_3)

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_{nk} f_k(t) \geqslant \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_{nk} f_k(1) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_{nk} = \infty$$

for every $t > 1$. From the uniform Δ_2-condition it follows the existence $K > 0$ and $k_o \in \mathbb{N}$ such that

$$f_k(2u) \leqslant K \cdot f_k(u)$$

for each $k \geqslant k_o$ and $u \in \left[0, \frac{1}{2}\right]$. Thus $f_k(1) \leqslant K \cdot f_k(\frac{1}{2})$ for every $k \geqslant k_o$, so

$$\sum_{k=1}^{\infty} a_{nk} f_k(1) \leqslant \sum_{k=1}^{k_o} a_{nk} + K \cdot \sum_{k=k_o+1}^{\infty} a_{nk} f_k\left(\frac{1}{2}\right).$$

From assumption we have $\sum\limits_{n=1}^{\infty} \sum\limits_{k=1}^{k_o} a_{nk} < \infty$. Then

$$\infty = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{nk} - \sum_{n=1}^{\infty} \sum_{k=1}^{k_o} a_{nk} \leqslant K \cdot \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{nk} f_k\left(\frac{1}{2}\right),$$

what means $\sum\limits_{n=1}^{\infty} \sum\limits_{k=1}^{\infty} a_{nk} f_k(\frac{1}{2}) = \infty$. Now, let us suppose that

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{nk} f_k\left(\frac{1}{2^p}\right) = \infty$$

for every $p \in \mathbb{N}$. Then from the uniform Δ_2-condition $f_k(\frac{1}{2^p}) \leqslant K \cdot f_k(\frac{1}{2^{p+1}})$ for $k \geqslant k_o$. Thus using the same arguments we obtain

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{nk} f_k\left(\frac{1}{2^{p+1}}\right) = \infty.$$

In consequence

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{nk} f_k(t) = \infty$$

for each $t \neq 0$, i.e., $\varrho(x) = \infty$ for $x = \{t, t, \dots\}$. It means that $x = \{t, t, \dots\} \notin l(A, \{f_n\})$ for each $t \neq 0$. $\square$

Remark. With assumptions from the last theorem and Proposition 2.1 we obtain the fact $l^\infty \not\subset l(A, \{f_n\})$.

Corollary 2.1. *If the sequence $\{f_n\}_{n=1}^{\infty}$ satisfies the uniform Δ_2-condition and $f_n(1) = 1$ for each $n \in \mathbb{N}$, the matrix $A = [a_{nk}]_{n,k\in\mathbb{N}}$ is regular with the property $\sum\limits_{n=1}^{\infty} a_{nk} < \infty$ for each $k \in \mathbb{N}$, then we have $\inf\limits_{j\in\mathbb{N}} |\xi_j| = 0$ for each $x = \{\xi_j\}_{j=1}^{\infty} \in l(A, \{f_n\})$.*

Proof. Let $x = \{\xi_j\}_{j=1}^{\infty} \in l(A, \{f_n\})$. Suppose, contrary to our claim, that $\inf\limits_{j\in\mathbb{N}} |\xi_j| > 0$. Then there exists $p \in \mathbb{N}$ such that

$$\frac{1}{2^p} < \inf_{j\in\mathbb{N}} |\xi_j| \leqslant |\xi_j|.$$

Thus

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_{nk} f_k(|\xi_k|) = \infty\,,$$

which is impossible. □

THEOREM 2.5. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-degenerate Orlicz functions with the property $f_n(1) = 1$ for each $n \in \mathbb{N}$. If $A = [a_{nk}]_{n,k\in\mathbb{N}}$ is a regular matrix such that $l^\infty \cap l(A,\{f_n\}) \neq l^\infty$, then the space $l^\infty \cap l(A,\{f_n\})$ is an F_σ, σ-strongly porous set in the space l^∞.*

P r o o f. By assumption and Proposition 2.1 we have

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} f_j(|t|) = \infty$$

for each $t > 0$. We will consider the set

$$\mathfrak{D}_{nkpm} = \left\{ x = \{\xi_j\}_{j=1}^{\infty} \in l^\infty \,:\, \sum_{i=1}^{n}\sum_{j=1}^{k} a_{ij} f_j\left(\frac{1}{p}|\xi_j|\right) > m \right\}$$

for every $n,k,p,m \in \mathbb{N}$. Let us denote

$$\mathfrak{D} = \bigcap_{p=1}^{\infty}\bigcap_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty} \mathfrak{D}_{nkpm}\,.$$

For $x_1 = \{1,1,1,\dots\}$ we obtain $\varrho(\frac{1}{p}x_1) = \infty$ for every $p \in \mathbb{N}$. So $x_1 \in \mathfrak{D}$ and $\mathfrak{D} \neq \emptyset$. Now, fix the natural numbers n,k,p,m. Let us consider $x_o = \{\xi_j^o\}_{j=1}^{\infty} \in \mathfrak{D}_{nkpm}$. We can choose such $\varepsilon > 0$ that

$$\sum_{i=1}^{n}\sum_{j=1}^{k} a_{ij} f_j\left(\frac{1}{p}|\xi_j^o|\right) - \varepsilon > m\,.$$

Since the functions $f_1, f_2, \dots, f_k$ are continuous there is $\delta > 0$ such that

$$\begin{aligned} &\text{if } |\xi_j - \xi_j^o| < \delta \text{ for } j = 1,2,\dots,k\,,\\ &\text{then } \sum_{i=1}^{n}\sum_{j=1}^{k} a_{ij}\left| f_j\left(\frac{1}{p}|\xi_j|\right) - f_j\left(\frac{1}{p}|\xi_j^o|\right)\right| < \varepsilon\,. \end{aligned} \tag{2.3}$$

Choose $x = \{\xi_j\}_{j=1}^{\infty} \in l^\infty$ such that $\|x - x_o\|_\infty < \delta$. Then according to (2.3) we get

$$\sum_{i=1}^{n}\sum_{j=1}^{k} a_{ij}\left| f_j\left(\frac{1}{p}|\xi_j|\right) - f_j\left(\frac{1}{p}|\xi_j^o|\right)\right| < \varepsilon$$

and

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{k} a_{ij} f_j\Big(\frac{1}{p}|\xi_j|\Big) &\\ &\geqslant \sum_{i=1}^{n}\sum_{j=1}^{k} a_{ij} f_j\Big(\frac{1}{p}|\xi_j^o|\Big) - \sum_{i=1}^{n}\sum_{j=1}^{k} a_{ij}\left|f_j\Big(\frac{1}{p}|\xi_j^o|\Big) - f_j\Big(\frac{1}{p}\,|\xi_j|\Big)\right| \\ &> \sum_{i=1}^{n}\sum_{j=1}^{k} a_{ij} f_j\Big(\frac{1}{p}|\xi_j^o|\Big) - \varepsilon > m\,.\end{aligned}$$

Hence $x \in \mathfrak{D}_{nkpm}$. Thus we have proved that the set $\mathfrak{D}_{nkpm}$ is open in l^∞. In consequence the set $\mathfrak{D}$ is G_δ, so $l^\infty \setminus \mathfrak{D} = l^\infty \cap l(A,\{f_n\})$ is an F_σ-set in l^∞. From the assumption $l^\infty \cap l(A,\{f_n\})$ is a non-trivial subspace of Banach space l^∞. So according to the Banach's Theorem we have that $l^\infty \cap l(A,\{f_n\})$ is of the first category in l^∞. Thus the set $\mathfrak{D}$ is dense, which gives that

$$\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty}\mathfrak{D}_{nkpm}$$

is dense for each $p, m \in \mathbb{N}$. The sets

$$l^\infty \setminus \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty}\mathfrak{D}_{nkpm}$$

are closed, nowhere dense, convex. Moreover, applying Lemma 1.2, these sets are strongly porous. Because

$$l^\infty \cap l(A,\{f_n\}) = l^\infty \setminus \mathfrak{D} = \bigcup_{p=1}^{\infty}\bigcup_{m=1}^{\infty}\Big(l^\infty \setminus \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty}\mathfrak{D}_{nkpm}\Big),$$

we have shown that the set $l^\infty \cap l(A,\{f_n\})$ is σ-strongly porous in l^∞. $\square$

The subspace $l^\infty \cap l(A,\{f_n\})$ need not be dense in l^∞, because we have:

PROPOSITION 2.4. *Let $\{f_n\}_{n=1}^\infty$ be a sequence of non-degenerate Orlicz functions satisfying the uniform Δ_2-condition and the property $f_n(1) = 1$ for each $n \in \mathbb{N}$. Let $A = [a_{nk}]_{n,k\in\mathbb{N}}$ be a regular matrix such that $\sum\limits_{n=1}^{\infty} a_{nk} < \infty$ for each $k \in \mathbb{N}$. Then the subspace $l^\infty \cap l(A,\{f_n\})$ is not dense in l^∞.*

Proof. Let $x_1 = \{1,1,1,\dots\}$, $0 < \varepsilon < \frac{1}{2}$. For $x = \{\xi_j\}_{j=1}^\infty \in l^\infty$ with the property $\|x - x_1\|_\infty < \varepsilon$ we have $|\xi_j| > 1 - \varepsilon$ for every $j \in \mathbb{N}$, so $\inf\limits_{j\in\mathbb{N}} |\xi_j| > 0$. Thus from Corollary 2.1

$$x \notin l(A,\{f_n\})\,,$$

which means that

$$\{x \in l^\infty : \|x - x_1\|_\infty < \varepsilon\} \cap l(A, \{f_n\}) = \emptyset .$$

□

Now, let us consider two sequences $\{f_n\}_{n=1}^\infty$, $\{g_n\}_{n=1}^\infty$ of non-degenerate Orlicz functions. Denote by $l(A, \{f_n\})$ and $l(A, \{g_n\})$ the modular spaces determined by these sequences, respectively.

Theorem 2.6. *Let $\{f_n\}_{n=1}^\infty$, $\{g_n\}_{n=1}^\infty$ be the sequences of non-degenerate Orlicz functions satisfying the strong uniform Δ_2-condition. If the regular matrix $A = [a_{nk}]_{n,k\in\mathbb{N}}$ is such that*

(a) $\sum\limits_{n=1}^\infty a_{nk} < \infty$ *for each* $k \in \mathbb{N}$,

(b) $l(A, \{f_n\}) \cap l(A, \{g_n\}) \neq l(A, \{f_n\})$,

then $l(A, \{f_n\}) \cap l(A, \{g_n\})$ is a dense, F_σ and σ-strongly porous set in $l(A, \{f_n\})$.

Proof. By ϱ_F denote the modular introduced by the sequence $F = \{f_n\}_{n=1}^\infty$ of Orlicz functions. According to Proposition 2.3 each sequence with only a finite number of non-zero terms belongs to $l(A, \{f_n\}) \cap l(A, \{g_n\})$. The set of all such sequences is dense in $l(A, \{f_n\})$, which follows from Theorem 2.3. Therefore $l(A, \{f_n\}) \cap l(A, \{g_n\})$ is dense in $l(A, \{f_n\})$.

We shall prove that $l(A, \{f_n\}) \cap l(A, \{g_n\})$ is an F_σ and σ-strongly porous set in $l(A, \{f_n\})$. For $p, m, n, k \in \mathbb{N}$ we put

$$\mathfrak{D}_{mnpk} = \left\{ x = \{\xi_j\}_{j=1}^\infty \in l(A, \{f_n\}) \colon \sum_{i=1}^n \sum_{j=1}^k a_{ij} g_j\left(\frac{1}{p}|\xi_j|\right) > m \right\}.$$

Since $l(A, \{f_n\}) \cap l(A, \{g_n\}) \neq l(A, \{f_n\})$ there exists such $x = \{\xi_j\}_{j=1}^\infty \in l(A, \{f_n\})$ that $x \notin l(A, \{g_n\})$. Hence $\sum\limits_{n=1}^\infty \sum\limits_{k=1}^\infty a_{nk} g_k\left(\frac{1}{p}|\xi_k|\right) = \infty$ for each $p \in \mathbb{N}$ and

$$\mathfrak{D} = \bigcap_{p=1}^\infty \bigcap_{m=1}^\infty \bigcup_{n=1}^\infty \bigcup_{k=1}^\infty \mathfrak{D}_{mnpk} \neq \emptyset .$$

Let us consider $x_o = \{\xi_j^o\}_{j=1}^\infty \in \mathfrak{D}$. Then for each $p, m \in \mathbb{N}$ there exist $n, k \in \mathbb{N}$ with the property

$$\sum_{i=1}^n \sum_{j=1}^k a_{ij} g_j\left(\frac{1}{p}|\xi_j^o|\right) > m .$$

We can choose $\varepsilon > 0$ such that

$$\sum_{i=1}^{n} \sum_{j=1}^{k} a_{ij} g_j\left(\frac{1}{p}|\xi_j^o|\right) - \varepsilon > m\,.$$

Since the functions $g_1, g_2, \dots, g_k$ are continuous there is $\delta_1 > 0$ such that

$$\begin{aligned} &\text{if } |\xi_j - \xi_j^o| < \delta_1 \text{ for } j = 1, 2, \dots, k\,, \\ &\text{then } \sum_{i=1}^{n} \sum_{j=1}^{k} a_{ij} \left| g_j\left(\frac{1}{p}\,|\xi_j^o|\right) - g_j\left(\frac{1}{p}\,|\xi_j|\right)\right| < \varepsilon\,. \end{aligned} \tag{2.4}$$

From (A_5) we know that $r_j = \sup\limits_i a_{ij} > 0$ for $j \in \mathbb{N}$. So we put $\delta = \min\limits_{1 \leqslant j \leqslant k} r_j f_j\left(\frac{\delta_1}{2}\right)$. Then $\delta > 0$. Let $x = \{\xi_j\}_{j=1}^{\infty}$ be such that $\varrho_F(x - x_o) < \delta$, i.e.,

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} f_j\big(|\xi_j - \xi_j^o|\big) < \delta\,.$$

Then for each natural i, j we have $a_{ij} f_j\big(|\xi_j - \xi_j^o|\big) < \delta$. Thus $\sup\limits_i a_{ij} f_j\big(|\xi_j - \xi_j^o|\big) \leqslant \delta$ for each $j \in \mathbb{N}$. So $r_j f_j\big(|\xi_j - \xi_j^o|\big) \leqslant r_j f_j\left(\frac{\delta_1}{2}\right)$ for $j = 1, 2, \dots, k$, $f_j\big(|\xi_j - \xi_j^o|\big) \leqslant f_j\left(\frac{\delta_1}{2}\right)$ for $j = 1, 2, \dots, k$, and we get $|\xi_j - \xi_j^o| \leqslant \frac{\delta_1}{2} < \delta_1$, $(j = 1, 2, \dots, k)$. Then according to (2.4) we have

$$\sum_{i=1}^{n} \sum_{j=1}^{k} a_{ij} \left| g_j\left(\frac{1}{p}|\xi_j^o|\right) - g_j\left(\frac{1}{p}\,|\xi_j|\right)\right| < \varepsilon$$

and

$$\begin{aligned} &\sum_{i=1}^{n} \sum_{j=1}^{k} a_{ij} g_j\left(\frac{1}{p}|\xi_j|\right) \\ &\qquad \geqslant \sum_{i=1}^{n} \sum_{j=1}^{k} a_{ij} g_j\left(\frac{1}{p}|\xi_j^o|\right) - \sum_{i=1}^{n} \sum_{j=1}^{k} a_{ij} \left| g_j\left(\frac{1}{p}\,|\xi_j^o|\right) - g_j\left(\frac{1}{p}\,|\xi_j|\right)\right| \\ &\qquad \geqslant \sum_{i=1}^{n} \sum_{j=1}^{k} a_{ij} g_j\left(\frac{1}{p}|\xi_j^o|\right) - \varepsilon > m\,. \end{aligned}$$

Hence $x \in \mathfrak{D}_{mnpk}$. In consequence we proved that the set $\mathfrak{D}_{mnpk}$ is open in $l(A, \{f_n\})$. Let us remark that

$$l(A, \{f_n\}) \setminus l(A, \{g_n\}) = \bigcap_{p=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \mathfrak{D}_{mnpk}\,,$$

so $l(A,\{f_n\}) \cap\ l(A,\{g_n\})$ is an F_σ set in $l(A,\{f_n\})$. From the assumption $l(A,\{f_n\}) \cap l(A,\{g_n\})$ is a non-trivial subspace of the Banach space $l(A,\{f_n\})$. According to the Banach's Theorem we have that $l(A,\{f_n\}) \cap l(A,\{g_n\})$ is of the first Baire category in $l(A,\{f_n\})$. Thus the set $l(A,\{f_n\}) \setminus l(A,\{g_n\})$ is dense and

$$\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty}\mathfrak{D}_{mnpk}$$

is dense for each $p, m \in \mathbb{N}$. The sets

$$l(A,\{f_n\}) \setminus \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty}\mathfrak{D}_{mnpk}$$

are closed, nowhere dense, convex. Using Lemma 1.2 these sets are strongly porous. Because

$$l(A,\{f_n\}) \cap l(A,\{g_n\}) = \bigcup_{p=1}^{\infty}\bigcup_{m=1}^{\infty}\left(l(A,\{f_n\}) \setminus \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty}\mathfrak{D}_{mnpk}\right),$$

then we have shown that the set $l(A,\{f_n\}) \cap l(A,\{g_n\})$ is σ-strongly porous in $l(A,\{f_n\})$. ☐

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