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ANOTHER ALGEBRAIC EQUIVALENT OF THE CONTINUUM HYPOTHESIS

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ABSTRACT. A renowned theorem due to Erdős and Kakutani states that the Continuum Hypothesis holds if and only if the set of all non-zero reals is a union of countably many Hamel bases. Adapting Erdős and Kakutani's argument, here is shown that the Continuum Hypothesis holds if and only if the set of all transcendental reals is a union of countably many transcendence bases.

1. Introduction

Hamel bases for $\mathbb{R}$, i.e., maximal subsets of $\mathbb{R}$ linearly independent over $\mathbb{Q}$, have been extensively investigated since the beginning of the modern real analysis. It is well-known that the existence of a Hamel basis for $\mathbb{R}$, guaranteed by the Axiom of Choice, implies the existence not only of subsets of $\mathbb{R}$ being both Lebesgue non-measurable and without the Baire property, but also of discontinuous (necessarily nonmeasurable) additive real functions. (Kuczma's monograph [4] and the sixth chapter of Morgan's book [6] contain a lot of classical material on Hamel bases and additive real functions.)

An old result by Sierpiński ([8]) states that no Hamel basis for $\mathbb{R}$ can be an analytic set; on the other hand, Hamel bases that are Lebesgue zero (hence Lebesgue measurable) or Marczewski zero (hence Marczewski measurable) or of first category (hence with the Baire property) do exist ([5], [8]).

It is also known that any union of fewer than continuum many translates of a Hamel basis for $\mathbb{R}$ has Lebesgue inner measure zero and contains no second category subset with the Baire property (see, e.g., [6, Theorem 16, p. 223]). Moreover, as a consequence of deeper algebraic arguments, it is shown in [9] that a union of fewer than continuum many translates of *finitely many* Hamel bases for $\mathbb{R}$ has Lebesgue inner measure zero and contains no second category

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subset with the Baire property. Such a result extends Muthuvel's Theorem 2 and Corollary 4 in [7].

It is to emphasize that the phrase "finitely many" above *cannot* be converted into "countably many", because of the following theorem by Erdős and Kakutani:

THEOREM 1. ([2]) *The Continuum Hypothesis holds if and only if the set of all non-zero reals is a union of countably many Hamel bases for* $\mathbb{R}$.

Transcendence bases for $\mathbb{R}$ (over $\mathbb{Q}$), i.e., maximal algebraically independent (over $\mathbb{Q}$) subsets of $\mathbb{R}$, play the same role in the theory of algebraic fields as Hamel bases for $\mathbb{R}$ do in linear algebra. While the importance of Hamel bases needs no further explanation, let us remark that also transcendence bases turn out to be useful in real analysis. As an illuminating example, let us mention Ciesielski's proof [1] about the non-existence of a maximal isometrically invariant extension of the Lebesgue measure in $\mathbb{R}^n$.

Then the following question arises naturally: to what extent (from a set-theoretical point of view, at least) does the analogy between Hamel and transcendence bases hold? Adapting Sierpiński's original arguments [8], it is proved in [10] that no transcendence basis for $\mathbb{R}$ can be an analytic set. On the other hand, it is easy to show that transcendence bases for $\mathbb{R}$ being Lebesgue measurable or Marczewski measurable or with the Baire property do exist: take a Hamel basis that is Lebesgue zero or Marczewski zero or of first category, and then extract from it a transcendence basis (it is well known that this can always be done).

The aim of this note is to prove the field-theoretic analogue of Erdős and Kakutani's theorem:

THEOREM 2. *The Continuum Hypothesis holds if and only if the set of all transcendental reals is a union of countably many transcendence bases for* $\mathbb{R}$.

The proof of Theorem 2 is differed to the last section, since it requires some preliminary, purely algebraic facts. There are, indeed, certain additional difficulties to tackle when dealing with transcendence bases instead of linear ones: as a first (rough) instance, note that the linear space over $\mathbb{Q}$ generated by a Hamel basis for $\mathbb{R}$ coincides with all of $\mathbb{R}$, while the field generated by an arbitrary transcendence basis for $\mathbb{R}$ is *properly* contained in $\mathbb{R}$ (for $\sqrt{2}$ and, more generally, every irrational algebraic number cannot belong to it).

For the reader's convenience, in the next section some standard facts and terminology taken from the theory of algebraic fields are reviewed; then, three lemmata essential for proving Theorem 2 are verified.

2. Algebraic preliminaries

Let $F \subseteq E$ be a field extension. An element $x \in E$ is said to be *algebraic over* F if there exists some non-zero polynomial $f \in F[X]$ such that $f(x) = 0$. An element of E that is not algebraic is then *transcendental over* F. By $\mathrm{alg}_E F$ we mean the set consisting of all the elements in E which are algebraic over F. It turns out that $\mathrm{alg}_E F$ is a subfield of E, and that $\mathrm{alg}_E(\mathrm{alg}_E F) = \mathrm{alg}_E F$ (see [3, Theorem (17.5)] and [3, Corollary (17.7)]).

Let $F \subseteq E$ be a field extension and let $S \subseteq E$. The subfield of E generated by F and S is denoted by $F(S)$. The set S is said to be *algebraically dependent over* F if there exist $n \in \mathbb{N}$, and n distinct elements $s_1, s_2, \dots, s_n \in S$ such that for some non-zero $f \in F[X_1, X_2, \dots, X_n]$ it holds $f(s_1, s_2, \dots, s_n) = 0$. Equivalently, S is algebraically dependent over F if and only if there exists $s \in S$ such that $s \in \mathrm{alg}_E F(S \setminus \{s\})$. A set that is not algebraically dependent over F is said to be *algebraically independent over* F.

A subset T of E is a *transcendence basis for* E *over* F if T is a maximal subset of E with the property of being algebraically independent over F. Equivalently, T is a transcendence basis for E over F if T is algebraically independent over F and $E = \mathrm{alg}_E F(T)$. It is worth recalling, by virtue of the Axiom of Choice, that transcendence bases for E over F exist and that any subset of E algebraically independent over F is contained in a transcendence basis for E over F (see [3, Lemma (24.4)]).

Lemma 3. *Let $F \subseteq E$ be a field extension and let $S \subseteq E$. If the element $x \in \mathrm{alg}_E F(S) \setminus \mathrm{alg}_E F$, then S is algebraically dependent over $F(x)$.*

Proof. There exist $n \in \mathbb{N}$, a polynomial

$$f \in F[X_1, X_2, \dots, X_n, X] \setminus F[X_1, X_2, \dots, X_n, \widehat{X}]$$

(as it is usual, the symbol $\widehat{X}$ means typographical deletion of X), and n distinct elements $s_1, s_2, \dots, s_n \in S$ such that $f(s_1, s_2, \dots, s_n, x) = 0$. This means that the set $\{s_1, s_2, \dots, s_n\}$ (and therefore S) is algebraically dependent over $F(x)$. $\square$

Lemma 4. *Let $F \subseteq E$ be a field extension and let $A \subseteq E$ be algebraically independent over F. If B and C are subsets of A, then*

$$\mathrm{alg}_E F(B) \cap \mathrm{alg}_E F(C) = \mathrm{alg}_E F(B \cap C).$$

Proof. Let us only verify the inclusion $\subseteq$, the reversed one being obvious. Assume there exists $x \in E$ such that

$$\begin{aligned} x &\in \big(\mathrm{alg}_E F(B) \cap \mathrm{alg}_E F(C)\big) \setminus \mathrm{alg}_E F(B \cap C) \\ &= \big(\mathrm{alg}_E F(B \cap C)(B \setminus C) \cap \mathrm{alg}_E F(B \cap C)(C \setminus B)\big) \setminus \mathrm{alg}_E F(B \cap C). \end{aligned}$$

In view of Lemma 3, this implies that $B \setminus C$ is algebraically dependent over $F(B \cap C)(x)$. In other words, there is some $b \in B \setminus C$ such that (note: below we use that $x \in \mathrm{alg}_E F(B \cap C)(C \setminus B)$)

$$b \in \mathrm{alg}_E F(B \cap C)\big((B \setminus \{b\}) \setminus C\big)(x) \subseteq \mathrm{alg}_E F(B \cap C)\big(B \setminus \{b\}\big)(C \setminus B)$$
$$= \mathrm{alg}_E F\big((B \cup C) \setminus \{b\}\big) \subseteq \mathrm{alg}_E F\big(A \setminus \{b\}\big),$$

which is obviously false, for A is supposed to be algebraically independent over F and B is a subset of A. □

Lemma 5. *Let $F \subseteq E$ be a field extension and let T be a transcendence basis for E over F. To each $x \in E$ there corresponds a unique minimal (finite) subset B of T such that $x \in \mathrm{alg}_E F(B)$.*

Proof. Let $x \in E$. As T is a transcendence basis for E over F, there surely exists a minimal (finite) subset B of T such that $x \in \mathrm{alg}_E F(B)$. To prove uniqueness, suppose that C is a subset of T such that $x \in \mathrm{alg}_E F(C)$. Then, by Lemma 4 and the minimality of B, we infer $B \subseteq C$. □

3. Proof of Theorem 2

Since each transcendence basis for $\mathbb{R}$ is contained in a Hamel basis for $\mathbb{R}$, and since $\mathrm{alg}_{\mathbb{R}} \mathbb{Q} \setminus \{0\}$ is countable, Theorem 1 immediately gives the half $\Leftarrow$ of Theorem 2. So, the proof amounts to establishing the reversed implication. To do this, we will apply the transfinite argument of Erdős and Kakutani to the field-theoretic case.

Assume the Continuum Hypothesis. Fix a transcendence basis T for $\mathbb{R}$ and well-order it: $T = \big\{t(\alpha)\colon 1 \leqslant \alpha < \omega_1\big\}$ (it is immediately seen that the cardinality of T must be that of the continuum). According to Lemma 5, to each $x \in \mathbb{R} \setminus \mathrm{alg}_{\mathbb{R}} \mathbb{Q}$ there uniquely correspond $n(x) \in \mathbb{N}$ and ordinals $1 \leqslant \alpha_1(x) < \alpha_2(x) < \cdots < \alpha_{n(x)}(x) < \omega_1$ such that

$$x \in \mathrm{alg}_{\mathbb{R}} \mathbb{Q}\Big(t\big(\alpha_1(x)\big), \dots, t\big(\alpha_{n(x)}(x)\big)\Big) \Big\backslash \bigcup_{i=1}^{n(x)} \mathrm{alg}_{\mathbb{R}} \mathbb{Q}\Big(T \setminus \big\{t\big(\alpha_i(x)\big)\big\}\Big). \tag{1}$$

Thus $\mathbb{R} \setminus \mathrm{alg}_{\mathbb{R}} \mathbb{Q} = \bigcup\limits_{n=1}^{\infty} A_n$, being $A_n := \big\{x \in \mathbb{R} \setminus \mathrm{alg}_{\mathbb{R}} \mathbb{Q} : n(x) = n\big\}$. For any $n \in \mathbb{N}$ and any ordinal α such that $n \leqslant \alpha < \omega_1$, let us define $A_n(\alpha) := \big\{x \in A_n\colon \alpha_n(x) = \alpha\big\}$. As α is countable, we immediately infer that also $A_n(\alpha)$ is countable. Hence, $A_n(\alpha)$ can be indexed by $\mathbb{N}$ as follows: $A_n(\alpha) = \{a_n^m(\alpha)\colon m \in \mathbb{N}\}$. Finally, for every $m \in \mathbb{N}$, put $A_n^m := \big\{a_n^m(\alpha)\colon n \leqslant \alpha < \omega_1\big\}$.

The proof is completed once we show that, for every $m, n \in \mathbb{N}$, the set $A := A_n^m$ is algebraically independent. This amounts to saying that, for any $k \in \mathbb{N}$ and any k distinct $x_1, x_2, \ldots, x_k \in A$, the set $\{x_1, x_2, \ldots, x_k\}$ is algebraically independent. Let us proceed by induction. The case $k = 1$ is, of course, true, for $A \subseteq \mathbb{R} \setminus \operatorname{alg}_{\mathbb{R}} \mathbb{Q}$. Given $k \in \mathbb{N}$, let us now verify that $k+1$ distinct numbers $x_1, x_2, \ldots, x_{k+1}$ freely chosen from A are algebraically independent, provided that any k of them are.

To achieve the aim, first notice that, if $x, y \in A$ and $x \neq y$, then $\alpha_n(x) \neq \alpha_n(y)$. Hence there exists precisely one of x_is, which can be assumed (without any restriction of generality) to coincide with x_{k+1}, such that

$$\alpha_n(x_{k+1}) > \sup\bigl\{\alpha_j(x_i) \colon i = 1, 2, \ldots, k;\ j = 1, 2, \ldots, n\bigr\}. \tag{2}$$

Suppose that $\{x_1, x_2, \ldots, x_{k+1}\}$ is algebraically dependent. This means that there is $i \in \{1, 2, \ldots, k+1\}$ such that $x_i \in \operatorname{alg} \mathbb{Q}(x_1, \ldots, \widehat{x_i}, \ldots, x_{k+1})$. The case $i = k+1$ is to exclude, since otherwise, by (2), it would be

$$\begin{aligned} x_{k+1} &\in \operatorname{alg}_{\mathbb{R}} \mathbb{Q}(x_1, x_2, \ldots, x_k) \\ &\subseteq \operatorname{alg}_{\mathbb{R}} \mathbb{Q}\Bigl(\bigl\{t\bigl(\alpha_j(x_i)\bigr) \colon i = 1, 2, \ldots, k;\ j = 1, 2, \ldots, n\bigr\}\Bigr) \\ &\subseteq \operatorname{alg}_{\mathbb{R}} \mathbb{Q}\Bigl(T \setminus \bigl\{t\bigl(\alpha_n(x_{k+1})\bigr)\bigr\}\Bigr), \end{aligned}$$

in contrast to (1). It remains to consider the case $i \in \{1, 2, \ldots, k\}$ and $x_i \in \operatorname{alg} \mathbb{Q}(x_1, \ldots, \widehat{x_i}, \ldots, x_{k+1})$. Since the set $\{x_1, x_2, \ldots, x_k\}$ is algebraically independent by hypothesis, there must exist a polynomial

$$f \in \mathbb{Q}[X_1, \ldots, X_i, \ldots, X_{k+1}] \setminus \mathbb{Q}[X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_{k+1}}]$$

such that $f(x_1, \ldots, x_i, \ldots, x_{k+1}) = 0$. This clearly leads to the (already excluded) conclusion: $x_{k+1} \in \operatorname{alg}_{\mathbb{R}} \mathbb{Q}(x_1, x_2, \ldots, x_k)$. The proof is now complete.

To my Polish friends met in Stará Lesná.

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