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IMPLICIT SINGULAR INTEGRODIFFERENTIAL EQUATIONS OF FREDHOLM TYPE

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ABSTRACT. The singular initial value problem of implicit integro-differential equations of Fredholm type and continuous dependence of solutions are the subject of the paper. There are derived asymptotic estimates of solutions of special forms of Fredholm's integrodifferential equations as well.

1. Introduction

In the past three decades, singular initial value problems for differential and integro-differential equations were studied under various conditions on the nonlinearity and the kernel (see, e.g., [1]–[9]). The fundamental tools used in the existence proofs of all mentioned works are essentially Schauder-Tychonoff's fixed point theorem, Banach contraction principle and Wazewki's topological method.

The paper generalized some results obtained for explicit systems of integrodifferential equations of Volterra type [7]–[9] and, moreover, there are determined asymptotic estimations of solutions of some classes of implicit integrodifferential equations of Fredholm type.

Consider the following problem:

$$y'(t) = \mathcal{F}\left(t, y(t), y'(t), \int\limits_{0^+}^{1} K\big(t, s, y(t), y(s), y'(t), y'(s)\big)\,\mathrm{d}s, \mu\right),$$

$$y^{(i)}(0^+, \mu) = 0, \quad i = 0, 1, \tag{1}$$

where

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(I) $\mathcal{F}\colon \Omega \to R^n$, $\mathcal{F} \in C^0(\Omega)$, $\Omega = \big\{(t, u_1, u_2, u_3, \mu) \in J \times (R^n)^3 \times R\colon\ |u_1| \le \phi_1(t),\ |u_2| \le \phi_2(t),\ |u_3(t)| \le \psi(t)\big\}$, $J = (0,1]$, $0 < \phi_i(t) \in C^0(J)$, $\phi_1(0^+) = 0$, $0 < \psi(t) \in C^0(J)$, $\int\limits_{0^+}^{t} \phi_2(s)\,\mathrm{d}s \le \phi_1(t)$, $|.|$ denotes the usual norm in R^n,
$|\mathcal{F}(t, \overline{u}_1, \overline{u}_2, \overline{u}_3, \mu) - \mathcal{F}(t, \overline{\overline{u}}_1, \overline{\overline{u}}_2, \overline{\overline{u}}_3, \mu| \le \sum\limits_{i=1}^{3} M_i |\overline{u}_i - \overline{\overline{u}}_i|$
for all $(t, \overline{u}_1, \overline{u}_2, \overline{u}_3, \mu)$, $(t, \overline{\overline{u}}_1, \overline{\overline{u}}_2, \overline{\overline{u}}_3, \mu) \in \Omega$, $M_i \ge 0$, $i = 1, 2, 3$.

(II) $K\colon \Omega^1 \to R^n$, $K \in C^0(\Omega^1)$,
$\Omega^1 = \big\{(t, s, v_1, v_2, w_1, w_2) \in J \times J \times (R^n)^4 : |v_1| \le \phi_1(t), |v_2| \le \phi_1(t), |w_1| \le \phi_2(t), |w_2| \le \phi_2(t)\big\}$,
$|K(t, s, \overline{v}_1, \overline{v}_2, \overline{w}_1, \overline{w}_2) - K(t, s, \overline{\overline{v}}_1, \overline{\overline{v}}_2, \overline{\overline{w}}_1, \overline{\overline{w}}_2| \le \sum\limits_{j=1}^{2} N_j |\overline{v}_j - \overline{\overline{v}}_j| + L_1 e^{\lambda(t-2s)} |\overline{w}_1 - \overline{\overline{w}}_1| + L_2 e^{\lambda(t-2s)} |\overline{w}_2 - \overline{\overline{w}}_2|$
for all $(t, s, \overline{v}_1, \overline{v}_2, \overline{w}_1, \overline{w}_2)$, $(t, s, \overline{\overline{v}}_1, \overline{\overline{v}}_2, \overline{\overline{w}}_1, \overline{\overline{v}}_2) \in \Omega_1$, $M_i, N_j, L_1, L_2 \in R^+$, $i = 1, 2$, $j = 1, 2$, $M_2 < 1$, $\lambda > 0$ is a sufficiently large constant such that

$$\left(\frac{M_1}{\lambda} + M_2 + \frac{M_3 \sum\limits_{i=1}^{2} (N_i + L_i)}{\lambda}\right) < 1\,.$$

2. Main results

Theorem 2.1. *Let the functions $\mathcal{F}(t, u_1, u_2, u_3, \mu)$, $K(t, s, v_1, v_2, w_1, w_2)$ satisfy conditions* (I), (II) *and, moreover*

$$|\mathcal{F}| \le \sum_{i=1}^{3} g_i(t)|u_i|, \quad 0 < g_i(t) \in C^0(J),\ i = 1, 2, 3, \quad \int\limits_{0^+}^{t} g_1(t)\phi_2(s)\,\mathrm{d}s \le \alpha_1 \phi_2(t)\,,$$
$$g_2(t)\phi_2(t) \le \alpha_2 \phi_2(t), \quad g_3(t)\psi(t) \le \alpha_3 \phi_2(t), \quad \alpha_i \in R^+,\ \sum_{i=1}^{3} \alpha_i = 1\,,$$

then the initial problem (1) *has a unique solution $y(t, \mu)$ for each $\mu \in R$, $t \in J$.*

Proof. Put
$$y(t) = \int\limits_{0^+}^{t} r(s)\,\mathrm{d}s\,,$$
where $r(t) \in C^0(J)$ is an unknown function. Then y'(t) = r(t) and the system (1) is equivalent to the system of integral equations

$$r(t) = \mathcal{F}\left(t, \int_{0+}^{t} r(s)\,\mathrm{d}s\,, r(t), \int_{0+}^{1} K\left(t, s, \int_{0+}^{t} r(s)\,\mathrm{d}s\,, \int_{0+}^{s} r(\tau)\,\mathrm{d}\tau, r(t), r(s)\right)\mathrm{d}s, \mu\right). \quad (2)$$

Denote by H the Banach space of continuous vector-valued functions $h\colon J_0 \to R^n$, $J_0 = [0,1]$, $|h(t)| \le \phi_2(t)$ for each $t \in J_0$ with the norm

$$||h(t)||_\lambda = \max_{t\in J_0}\left\{e^{-\lambda t}|h(t)|\right\}.$$

Define the operator T by the right-hand side of (2)

$$\begin{aligned} &T(h)\\ &= \mathcal{F}\left(t, \int_{0+}^{t} h(s)\,\mathrm{d}s, h(t), \int_{0+}^{1} K\left(t, s, \int_{0+}^{t} h(s)\,\mathrm{d}s, \int_{0+}^{s} h(\tau)\,\mathrm{d}\tau\,, h(t), h(s)\right)\mathrm{d}s, \mu\right), \end{aligned}$$

where $h \in H$. Let $\mu \in R$ be fixed. The transformation T maps H continuously into itself because

$$\begin{aligned} &|T(h)|\\ &\le \left|\mathcal{F}\left(t, \int_{0+}^{t} h(s)\,\mathrm{d}s\,, h(t), \int_{0+}^{1} K\left(t, s, \int_{0+}^{t} h(s)\,\mathrm{d}s\,, \int_{0+}^{s} h(\tau)\,\mathrm{d}\tau, h(t), h(s)\right)\mathrm{d}s\,, \mu\right)\right|\\ &\le g_1(t)\left|\int_{0+}^{t} h(s)\,\mathrm{d}s\,\right| + g_2(t)|h(t)|\\ &\quad + g_3(t)\left|\int_{0+}^{1} K\left(t, s, \int_{0+}^{t} h(s)\,\mathrm{d}s\,, \int_{0+}^{s} h(\tau)\,\mathrm{d}\tau\,, h(t), h(s)\right)\mathrm{d}s\right|\\ &\le g_1(t)\int_{0+}^{t} \phi_2(s)\,\mathrm{d}s\, + \alpha_2\phi_2(t) + g_3(t)\psi(t) \le \phi_2(t) \end{aligned}$$

for every $h \in H$. We shall prove that

$$||T(h_2) - T(h_1)||_\lambda \le \left(\frac{M_1}{\lambda} + M_2 + \frac{M_3\sum\limits_{i=1}^{2}(N_i + L_i)}{\lambda}\right) \quad (3)$$

for all $h_1, h_2 \in H$.

Using (I),(II) and the definition $||.||_\lambda$ we have

$$
\begin{aligned}
&|T(h_2) - T(h_1)| \\
&= \left| \mathcal{F}\left(t, \int_{0+}^{t} h_2(s)\,\mathrm{d}s\,, h_2(t), \int_{0+}^{1} K\left(t, s, \int_{0+}^{t} h_2(s)\,\mathrm{d}s\,, \int_{0+}^{s} h_2(\tau)\,\mathrm{d}\tau\,, h_2(t), h_(s)\right)\mathrm{d}s\,, \mu\right)\right. \\
&\quad \left. - \mathcal{F}\left(t, \int_{0+}^{t} h_1(s)\,\mathrm{d}s\,, h_1(t), \int_{0+}^{1} K\left(t, s, \int_{0+}^{t} h_1(s)\,\mathrm{d}s\,, \int_{0+}^{s} h_1(\tau)\,\mathrm{d}\tau\,, h_1(t), h_(s)\right)\mathrm{d}s\,, \mu\right)\right| \\
&\le M_1 \int_{0+}^{t} |h_2(s) - h_1(s)|\,\mathrm{d}s\, + M_2|h_2(t) - h_1(t)| \\
&\quad + M_3 \int_{0+}^{1} \left| K\left(t, s, \int_{0+}^{t} h_2(s)\,\mathrm{d}s\,, \int_{0+}^{s} h_2(\tau)\,\mathrm{d}\tau\,, h_2(t), h_2(s)\right)\right. \\
&\quad \left. - K\left(t, s, \int_{0+}^{t} h_1(s)\,\mathrm{d}s\,, \int_{0+}^{s} h_1(\tau)\,\mathrm{d}\tau\,, h_1(t), h_2(s)\right)\right| \mathrm{d}s \\
&\le M_1 ||h_2 - h_1||_\lambda \int_{0+}^{t} e^{\lambda s}\,\mathrm{d}s\, + M_2 e^{\lambda t} ||h_2 - h_1||_\lambda + M_3 \int_{0+}^{1} \left(N_1 \int_{0+}^{t} |h_2(s) - h_1(s)|\,\mathrm{d}s \right. \\
&\quad + N_2 \int_{0+}^{s} |h_2(\tau) - h_1(\tau)|\,\mathrm{d}\tau\, L_1 e^{\lambda(t-2s)} |h_2(t) - h_1(t)| \\
&\quad \left. + L_2 e^{\lambda(t-2s)} |h_2(s) - h_1(s)| \right) \mathrm{d}s \\
&\le ||h_2 - h_1||_\lambda \left(M_1 \frac{e^{\lambda t} - 1}{\lambda} + M_2 e^{\lambda t} \right. \\
&\quad \left. + (M_3 N_1 + M_3 N_2) \frac{e^{\lambda t} - 1}{\lambda} + e^{\lambda t} (M_3 L_1 + M_3 L_2) \frac{1 - e^{-\lambda}}{\lambda} \right) \\
&\le e^{\lambda t} ||h_2 - h_1||_\lambda \left(\frac{M_1}{\lambda} M_2 + \frac{M_3 \sum\limits_{i=1}^{2} (N_i + L_i)}{\lambda} \right).
\end{aligned}
$$

Thus

$$
||T(h_2) - T(h_1)||_\lambda \le \max_{t \in J_0} |T(h_2) - T(h_1)| \le q\ ||h_2 - h_1||_\lambda\,,
$$

where

$$q := \frac{M_1}{\lambda} + M_2 + \frac{M_3 \sum\limits_{i=1}^{2} (N_i + L_i)}{\lambda} < 1\,.$$

Applying the classical Banach contraction principle to T and the distance function $||h_2 - h_1||_\lambda$ we get the assertion of Theorem 2.1. □

Theorem 2.2. *Let the assumptions of* Theorem 2.1 *be satisfied and let there exist a constant $L > 0$ and the integrable function $\gamma\colon J_0 \to J_0,\ J_0 = [0,1]$ such that*

$$|\mathcal{F}(t, u_1, u_2, u_3, \mu_2) - \mathcal{F}(t, u_1, u_2, u_3, \mu_1)| \le \gamma(t)|\mu_2 - \mu_1|\,,$$

where $(t, u_1, u_2, u_3, \mu_2),\ (t, u_1, u_2, u_3, \mu_1) \in \Omega$ and

$$\max_{t \in J_0} \left\{ e^{-\lambda t} \gamma(t) \right\} \le L\,,$$

then the solution $y(t,\mu)$ of (1) is continuous with respect to the variables $(t,\mu) \in J \times R$.

Proof. Define as above, for $h \in H$ the transformation $T_\mu(h)$ by means of the right-hand side (2). From (3) we obtain

$$||T_\mu(h) - T_\mu(y)||_\lambda \le \left(\frac{M_1}{\lambda} + M_2 + \frac{M_3 \sum\limits_{i=1}^{2} (N_i + L_i)}{\lambda} \right) ||h - y||_\lambda\,.$$

By the assumptions of the Theorem 2.2 we get

$$\begin{aligned}
&e^{-\lambda t}|T_{\mu_2}(h) - T_{\mu_1}(h)| \\
&\le \left| \mathcal{F}\left(t, \int\limits_{0+}^{t} h(s)\,\mathrm{d}s\,, h(t), \int\limits_{0+}^{1} K\left(t, s, \int\limits_{0+}^{t} h(s)\,\mathrm{d}s\,, \int\limits_{0+}^{s} h(\tau)\,\mathrm{d}\tau\,, h(t), h_(s) \right) \mathrm{d}s\,, \mu_2 \right) \right. \\
&\quad \left. - \mathcal{F}\left(t, \int\limits_{0+}^{t} h(s)\,\mathrm{d}s\,, h(t), \int\limits_{0+}^{1} K\left(t, s, \int\limits_{0+}^{t} h(s)\,\mathrm{d}s\,, \int\limits_{0+}^{s} h(\tau)\,\mathrm{d}\tau\,, h(t), h_(s) \right) \mathrm{d}s\,, \mu_1 \right) \right| \\
&\le e^{-\lambda t}\gamma(t)|\mu_2 - \mu_1| \le L|\mu_2 - \mu_1|
\end{aligned}$$

and hence

$$||T_{\mu_2}(h) - T_{\mu_1}(h)||_\lambda \le L|\mu_2 - \mu_1|.$$

From here and by the Theorem 2.1 we obtain

$$\begin{aligned}
&\|h(t,\mu_2)-h(t,\mu_1)\|_\lambda \\
&= \big\|T_{\mu_2}\big[h(t,\mu_2)\big]-T_{\mu_2}\big[h(t,\mu_1)\big]+T_{\mu_2}\big[h(t,\mu_1)\big]-T_{\mu_1}\big[h(t,\mu_1)\big]\big\|_\lambda \\
&\le \big\|T_{\mu_2}\big[h(t,\mu_2)\big]-T_{\mu_2}\big[h(t,\mu_1)\big]\big\|_\lambda+\big\|T_{\mu_2}\big[h(t,\mu_1)\big]-T_{\mu_1}\big[h(t,\mu_1)\big]\big\|_\lambda \\
&\le \left(\frac{M_1}{\lambda}+M_2+\frac{M_3\sum\limits_{i=1}^{2}(N_i+L_i)}{\lambda}\right)\|h(t,\mu_2)-h(t,\mu_1)\|_\lambda+L|\mu_2-\mu_1|\,.
\end{aligned}$$

Hence

$$||h(t,\mu_2)-h(t,\mu_1)||_\lambda \le \left(1-\left(\frac{M_1}{\lambda}+M_2+\frac{M_3\sum\limits_{i=1}^{2}(N_i+L_i)}{\lambda}\right)\right)^{-1} L|\mu_2-\mu_1|\,.$$

Consequently the function $h(t,\mu)$ is uniformly continuous with respect to the variable $\mu\in R$; so $y(t,\mu)$ is also continuous with respect two variables (t,μ) $\in J\times R$. The proof is complete. □

Now we shall consider a special form of (1)

$$p_i(t)y_i'-y_i=\mathcal{F}_i\left(t,y(t),y'(t),\int\limits_{0^+}^{1}K\big(t,s,y(t),y(s),y'(t),y'(s)\big)\,\mathrm{d}s\,,\mu\right)\,,$$

$$y^{(j)}(0^+,\mu)=0,\ j=0,1, \tag{4}$$

$i=1,\dots,n$. A solution of (4) will be described by means of a solution of an auxiliary system

$$p_i(t)y_i'-y_i=0,\qquad i=1,\dots,n. \tag{5}$$

Put

$$\eta_i(t,C_i)=C_i\exp\left(\int\limits_1^t\frac{\mathrm{d}s}{p_i(s)}\right),\qquad i=1,\dots,n\,. \tag{6}$$

It is obvious that (6) is the general solution of (5). Denote

$$Y=(Y_1,\dots,Y_n),\quad Y_i(t,\mu)=\frac{y_i(t,\mu)-\eta_i(t,C_i)}{\eta_i(t,C_i)},\ G=(G_1,\dots,G_n)\,,$$

$$G_i\left(t, y(t), y'(t), \int_{0^+}^{1} K\big(t, s, y(t), y(s), y'(t), y'(s)\big)\,\mathrm{d}s\,, \mu\right)$$
$$= \big(p_i(t)\eta_i(t, C_i)\big)^{-1}\mathcal{F}_i\left(t, y(t), y'(t), \int_{0^+}^{1} K\big(t, s, y(t), y(s), y'(t), y'(s)\big)\,\mathrm{d}s\,, \mu\right).$$

$$\eta(t)\big(1+Y(t)\big) = \big(\eta_1(t, C_1)\big(1+Y_1(t,\mu)\big), \dots, \eta_n(t, C_n)\big(1+Y_n(t,\mu)\big)\big).$$

Then

$$Y_i' = \frac{\Big(y_i'(t,\mu) - \frac{\eta_i(t,C_i)}{p_i(t)}\Big)\eta_i(t,C_i) - \big(y_i(t,\mu) - \eta_i(t,C_i)\big)\frac{\eta_i(t,C_i)}{p_i(t)}}{\eta_i(t,C_i)^2}$$
$$= \frac{p_i(t)y_i'(t,\mu) - y_i(t,\mu)}{p_i(t)\eta_i(t,C_i)}, \qquad i = 1, 2, \dots, n\,.$$

Now, we can rewrite the equation (4) in the form

$$Y_i' = G_i\Bigg(t, \eta(t)\big(1+Y(t)\big), \big(\eta(t)\big(1+Y(t)\big)\big)', \int_{0^+}^{1} K\big(t, s, \eta(t)\big(1+Y(t)\big)$$
$$\eta(s)\big(1+Y(s)\big)\ \big(\eta(t)\big(1+Y(t)\big)\big)', \big(\eta(s)\big(1+Y(s)\big)\big)'\big)\,\mathrm{d}s\,, \mu\Bigg).$$

Denote

$$\mathcal{G}_i\left(t, Y(t), Y'(t), \int_{0^+}^{1} \mathcal{K}\big(t, s, Y(t), Y(s), Y'(t), Y'(s)\big)\,\mathrm{d}s\,, \mu\right)$$
$$= G_i\Bigg(t, \eta(t)\big(1+Y(t)\big), \big(\eta(t)\big(1+Y(t)\big)\big)', \int_{0^+}^{1} K\Big(t, s, \eta(t)\big(1+Y(t)\big),$$
$$\eta(s)\big(1+Y(s)\big), \big(\eta(t)\big(1+Y(t)\big)\big)', \big(\eta(s)\big(1+Y(s)\big)\big)'\Big)\,\mathrm{d}s\,, \mu\Bigg).$$

Then

$$Y_i' = \mathcal{G}_i\left(t, Y(t), Y'(t), \int_{0^+}^{1} \mathcal{K}\big(t, s, Y(t), Y(s), Y'(t), Y'(s)\big)\,\mathrm{d}s\,, \mu\right). \tag{7}$$

We suppose:

(i) $\mathcal{G} \in C^0(\overline{\Omega})$,
$\overline{\Omega} = \big\{(t, Y, Y', u, \mu) \in J \times (R^n)^3 \times R : |Y| \le \phi_1(t),\ |Y'| \le \phi_2(t),\ |u| \le \psi(t)\big\}$,

$|\mathcal{G}(t,\overline{Y},\overline{Y}'\overline{u},\mu)-\mathcal{G}(t,\overline{\overline{Y}},\overline{\overline{Y}}',\overline{\overline{u}},\mu)| \le M_1|\overline{Y}-\overline{\overline{Y}}|+M_2|\overline{Y}'-\overline{\overline{Y}}'+|M_3|\overline{u}-\overline{\overline{u}}|$
for all $(t,\overline{Y},\overline{Y}',\overline{u},\mu),\ (t,\overline{\overline{Y}},\overline{\overline{Y}}',\overline{\overline{u}}) \in \overline{\Omega}$.

(ii) $\mathcal{K} \in C^0(\overline{\Omega}_1),\ \overline{\Omega}_1 = \big\{(t,s,Y(t),Y(s),Y'(t),Y'(s)) \in J\times J\times (R^n)^4 : |Y(t)| \le \phi_1(t),\ |Y(s)| \le \phi_1(t),\ |Y'(t)| \le \phi_2(t),\ |Y'(s)| \le \phi_2(t)\big\}$,
$\big|\mathcal{K}\big(t,s,\overline{Y}(t),\overline{Y}(s),\overline{Y}'(t),\overline{Y}'(s)\big)-\mathcal{K}\big(t,s,\overline{\overline{Y}}(t),\overline{\overline{Y}}(s),\overline{\overline{Y}}'(t),\overline{\overline{Y}}'(s)\big)\big|$
$\le N_1|\overline{Y}(t)-\overline{\overline{Y}}(t)|+N_2|\overline{Y}(s)-\overline{\overline{Y}}(s)|+L_3te^{-\lambda s}|\overline{Y}'(t)-\overline{\overline{Y}}'(t)|$
$+L_4e^{\lambda(t-2s)}|\overline{Y}'(s)\overline{\overline{Y}}'(s)|$
for all $\big(t,s,\overline{Y}(t),\overline{Y}(s),\overline{Y}'(t),\overline{Y}'(s)\big),\ (t,s,\overline{\overline{Y}}(t),\overline{\overline{Y}}(s),\overline{\overline{Y}}'(t),\overline{\overline{Y}},(s)) \in \overline{\Omega}_1$.

The functions $\phi(t),\psi(t)$ and the constants $M_i,N_i,\ i=1,2,\ \lambda$ have the same properties as in Theorem 2.1.

Theorem 2.3. *Let the functions* $\mathcal{G}\big(t,Y(t)Y'(t)u,\mu\big),\ \mathcal{K}\big(t,s,Y(t),Y(s),Y'(t),Y'(s)\big)$ *satisfy the conditions* (i),(ii) *and*

$$|\mathcal{G}| \le g_1(t)|Y|+g_2(t)|Y'|+g_3(t)|u|, \qquad 0<g_i(t)\in C^0(J), \qquad i=1,2,3,$$

$$\int_{0^+}^{t} g_1(t)\phi_2(s)\,\mathrm{d}s \le \alpha_1\phi_2(t)\,,$$

$$g_2(t)\phi_2(t)\le\alpha_2\phi_2(t), \quad g_3(t)\psi(t)\le\alpha_3\phi_2(t), \qquad \alpha_i\in R^+, \quad \sum_{i=1}^{3}\alpha_i=1\,.$$

$$0<p_i(t)\in C^0(J), \qquad \int_{0^+}^{1}\frac{\mathrm{d}t}{p_i(t)}=\infty, \qquad i=1,\dots,n\,.$$

Then there exists a unique solution $y(t,\mu)$ *of* (4) *satisfying inequalities*

$$|y_i(t,\mu)-\eta_i(t,C_i)| \le \phi_1(t)|\eta_i(t,C_i)|\,,$$
$$|y_i'(t,\mu)-\eta'(t,C_i)| \le \phi_1(t)|\eta_i'(t,C_i)|+\phi_2(t)|\eta_i(t,C_i)|$$

on J, $i=1,\dots,n$.

P r o o f. The equation (7) satisfies all assumptions of Theorem 2.1. Thus, there exists a unique solution $Y(t,\mu)$ of (7) such that $|Y(t,\mu| \le \phi(t)$. From here and by the notations above, we have

$$|Y_i(t,\mu)| = \left|\frac{y_i(t,\mu)-\eta_i(t,C_i)}{\eta_i(t,C_i)}\right| \le \phi(t) \Rightarrow |y_i(t,\mu)-\eta_i(t,C_i)| \le \phi(t)|\eta_i(t,C_i)|$$

and

$$Y_i' = \frac{y_i'(t) - \eta_i'(t, C_i}{\eta_i(t, C_i)} - \frac{y_i(t) - \eta_i(t, C_i)}{p_i(t)\eta_i(t, C_i)}\,.$$

Thus we get

$$\begin{aligned}|y_i'(t) - \eta_i'(t, C_i)| &\le |\eta_i(t, C)|\phi_2(t) + \frac{1}{p(t)}|\eta_i(t, C_i)|\phi_1(t)\\ &= |\eta_i(t, C)|\phi_2(t) + \eta_i'(t, C_i)|\phi_1(t).\end{aligned}$$

The proof is complete. □

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