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COPS, ROBBERS AND GRAPHS

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ABSTRACT. Cops-and-robbers games are played on graphs: from their initial positions on some vertices the cops try to capture (occupy the same vertices as) the robbers, initially on some other vertices, by moving along the edges of a given graph. When do the cops have a winning strategy? How many cops are needed to win? There are many versions of the game, but we only briefly survey the basic one, giving references to some other variants.

1. Introduction

We use semi-standard conventions for graphs and digraphs, defining when in doubt. In particular, if G is an undirected graph, it will have a set of *edges* (denoted by $[u,v]$), $E(G)$, while if it is a digraph, it will have a set of *arcs* (denoted by (u,v)), $A(G)$. Multiple edges or arcs are not allowed but loops could be. If there are no loops, the (di)graph is *simple*, if every vertex has a loop, the (di)graph is *reflexive*. As usual, we denote by $N(u)$ the neighbourhood of u in the undirected graph G and by $N[u] = N(u) \cup \{u\}$ the closed neighbourhood of u. Of course, in a reflexive graph the two coincide.

Cop(s) and robber(s) games have been studied for quite a while as they model some natural real-life problems (this has not prevented people to play on infinite graphs). In all cases, the object of the game is to capture an intruder (robber), or several ones, in an object modeled by a graph. There are two main variants, *sweeping*, where the players could be anywhere on the graph (at a vertex, somewhere on an edge), and *searching*, in which case only the vertices are possible locations for the players and the assumption is made that all players move infinitely fast between adjacent vertices. Sweeping, therefore, is concerned with continuous moves, while in searching the moves are discrete. In this survey we are only concerned with searching. For more on sweeping and its roots see [19] and [2]. There is more of them, of course, and the bibliography contains

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many references that do not appear in the text. Even so, we lay no claim to exhaustiveness of this short survey.

This searching game was first described for undirected reflexive graphs with one cop and one robber by Nowakowski and Winkler [36], and Quilliot[1] [39], [40], independently. In both cases the authors characterize finite cop-win graphs in the same manner, very similar to the simplicial decomposition of chordal graphs (and so the cop-winness of triangulated graphs follows). Nowakowski and Winkler also give a second characterization of cop-win graphs that is also valid for infinite graphs and that can be successfully exploited to get a polynomial time algorithm for recognizing cop-win graphs and digraphs better than the algorithm based on decomposition.

Following these results, the most often considered problem is that of determining the minimum number of cops needed to catch a robber on a given graph, and the complexity of determining this number. Special classes of graphs have been considered as well as some generalization. Further studies, mostly by Nowakowski and his students, deal with some generalizations/restrictions of the game, always on undirected graphs. Indeed, not much is known about cops and robbers on directed graphs.

2. Basics

Let us describe the basic model first. Let $G = (V, E)$ be a reflexive undirected graph. The game played on G is a complete information game for two players, a *cop* and a *robber*, who move from vertex to vertex at alternate ticks of a common clock. The object of the game is for the cop to occupy the same vertex as the robber; if this happens, the robber is *captured* or *caught*. The robber, of course, tries to avoid the capture. Since one or the other must happen, one of the players has a winning strategy, by von Neumann's theorem. It is often convenient to think of the game as played in *rounds*, each of which consists of a cop's move followed by a robber's move. At round zero, the cop chooses a vertex, then the robber chooses a vertex; we say, each *is at the vertex* they chose. At each subsequent round, each player chooses a vertex adjacent to the one at which s/he currently is and goes there; this is her/his *move*. The robber cannot move if both s/he and the cop are at the same vertex—in this case the game is over and the cop wins. Note that the cop can win in two ways, on her/his move (and then the final round remains incomplete), or on the robber's move. When the cop wins on a reflexive graph and the robber plays optimally (that is, s/he tries to avoid the capture as long as possible), the cop always

[1]This survey seems to be the very first paper where the references to Quilliot's work are complete.

wins on her/his move. Graphs on which the cop has a winning strategy (that is, a mapping $s\colon V(G)\times V(G)\longrightarrow V(G)$ with the interpretation that if the cop is at u and the robber at v, the cop's next move is to $s(u,v)$; note that $\bigl[u,s(u,v)\bigr]\in E(G)$) are called *cop-win*; all other graphs are *robber-win*. Since a disconnected graph cannot be cop-win, we shall assume throughout the paper that the graphs in question are connected. We shall also assume that all players play the best strategy available to them, that is, the cop(s) will play to end the game as soon as possible, while the robber(s) will try to prolong the game.

The result that could be called classic in this context has already been mentioned in the introduction; the reference easiest to find is [36]. Suppose the vertices of a finite graph G are numbered $v_1,\dots,v_n$. With this fixed order, define for a vertex $v\in V(G)$, $N_i(v)=N(v)\cap\{v_j : i\leqslant j\leqslant n\}$, and, similarly, $N_i[v]$.

Theorem 2.1. *A finite undirected reflexive graph G is cop-win if and only if its vertices can be linearly ordered $v_1,v_2,\dots,v_n$ so that for each i, $1\leqslant i<n$, there is a j, $i<j\leqslant n$ such that $N_i[v_i]\subseteq N_i[v_j]$.*

The proof depends very strongly on the reflexivity of the graph. As it is a nice application of the idea of a retract, we will prove the proposition essential to it. Recall that a *homomorphism* from a graph G to a graph H is a mapping $h\colon V(G)\longrightarrow V(H)$ such that if $[u,v]\in E(G)$ then $\bigl[h(u),h(v)\bigr]\in E(H)$. We write $h\colon G\to H$, or simply $G\to H$, to indicate that h is a homomorphism from G to H (or simply state the existence of such a homomorphism). A *retraction* is a homomorphism ρ from G onto a subgraph H of G such that ρ restricted to H is the identity on H.

Proposition 2.2. *If G is cop win and H is a retract of G, then H is cop-win.*

Proof. Consider s, the cop's winning strategy on G and a retraction ρ. Define $s'\colon V(H)\times V(H)\longrightarrow V(H)$, the cop's strategy on H, by $s'(u,v)=\rho\bigl(s(u,v)\bigr)$. If s never makes the cop move outside of H, s' is simply the restriction of s to H and is, therefore, winning. If $s(u,v)\in V(G)\setminus V(H)$ for some $u,v\in V(H)$, then $N\bigl[s(u,v)\bigr]\subseteq N\bigl[\rho\bigl(s(u,v)\bigr)\bigr]$ since ρ is a homomorphism and so moving to $\rho\bigl(s(u,v)\bigr)$ is a part in a winning strategy in this case. □

Note that the converse is not true: the path of length two is a retract of the cycle of length four and the former is cop-win while the latter is robber-win.

To complete the proof of the theorem, we need to observe that at robber's move of the one-before-last round of a game on G won by the cop, the robber is at v, the cop at u and $N[v]\subseteq N[u]$, lest the robber gets away. The graph $G-v$ is a retract of G, so it is cop win, and the argument can be repeated. This defines the ordering. The other direction also uses induction and the retractions implicitly defined by the ordering (it is much more complicated to actually specify the winning strategy!).

Similar arguments give the following theorem of [36].

THEOREM 2.3. *Retracts of Cartesian ($\Box$) products of reflexive cop-win graphs are cop-win.*

From Theorem 2.1 we obtain that some classes contain only cop-win graphs. We say that each v_i is *dominated* by the corresponding v_j. Recall that a graph is *chordal* or *triangulated* if it has no induced cycles of length greater than three. Relaxing the requirement somewhat, say a graph is *bridged* if each of its cycles of length four or more has a shortcut, that is, a pair of vertices whose distance in the graph is shorter that their distance on the cycle. It is well known that a chordal graph has a *simplicial* vertex whose neighbourhood is a clique; this leads to a *simplicial ordering* of its vertices. Similarly, a bridged graph has an *isometric* vertex whose removal does not change the distances among the remaining vertices; this leads to an *isometric ordering* of its vertices (see [7]). With this, one can deduce from the theorem that chordal graphs and bridged graphs are cop-win. In fact, Anstee and Farber prove more in [7].

THEOREM 2.4. *An undirected reflexive graph is bridged if and only if it is cop-win and contains no induced cycles of length four or five.*

From Theorem 2.1 it is also possible to obtain an algorithm for recognizing cop-win graphs: successive contractions of dominated vertices onto those that dominate them lead to a single vertex if and only if the graph is cop-win.

3. Cop number

It is a triviality that if there is a cop at every vertex of a graph G, then a robber has no chance, whether the graph is connected or not. Thus it makes sense to ask for the minimum number k of cops needed so that they can always catch the robber, that is, so that k cops can capture the robber if the cop's move is redefined to be the cops' move and they all move simultaneously, alternately with the robber. Define the cop number of a graph G to be this least k. In the literature the cop number is usually denoted by $c(G)$, or $cn(G)$, or $\mathrm{sn}(G)$ (for *search number*, see [2]). A graph on which k cops can always catch the robber is called k-cop-win. So $1 \leqslant \mathrm{sn}(G) \leqslant |V(G)|$ for any graph G, connected or not. Concentrating on connected components of a graph is sufficient for finite graphs (though not for infinite ones).

3.1 Degree bounds

It is easy to see that a cycle of length at least four is 2-cop-win or that any tree is cop-win. In general, bounds on $\mathrm{sn}(G)$ are not easy to find, although there are some interesting results. For example, the following is due to Aigner and Fromm [1].

Theorem 3.1. *If G is a finite reflexive graph with minimum degree d and if G has no cycles of length three or four, then* $\mathrm{sn}(G) \geqslant d$.

It follows from the above and the fact that for any natural number d there is a connected d-regular graph without cycles of length three or four that, for any d, there is a graph that needs at least d cops to catch a lone robber.

Recall that $\mathrm{girth}(G)$ is the length of the shortest cycle in G. Frankl improves the preceding theorem in [22].

Theorem 3.2. *If G has* $\mathrm{girth}(G) \geqslant 8t-3$ *and minimum degree $d+1$ then* $\mathrm{sn}(G) \geqslant d^t$.

There seems to be a relationship between the genus of a graph and its cop number. Indeed, Schroeder [44] gives a bound on $\mathrm{sn}(G)$ in terms of $g(G)$, its (orientable) genus, improving the previous work of Quilliot [41].

Theorem 3.3. *For a finite reflexive graph G,* $\mathrm{sn}(G) \leqslant 2g(G)+3$.

Theorem 3.4. *For a finite reflexive graph G,* $\mathrm{sn}(G) \leqslant \lfloor \frac{3}{2}g(G) \rfloor + 3$.

Corollary 3.5. *If G is planar,* $\mathrm{sn}(G) \leqslant 3$.

See Section 6 for a conjecture on this.

Last, but not least, excluding some minors can guarantee a small cop number, see Andreae [5].

Theorem 3.6. *Let H be a graph and u a vertex of H such that $H-u$ has no isolated vertices. If G is a graph without an H-minor, then* $\mathrm{sn}(G) \leqslant \left|E(H-u)\right|$.

Let M_1 be the graph on the vertex set a,b,c,d,e,f with edges set $\{[a,b],[a,c],[a,d],[a,e],[a,f],[b,c],[c,d],[d,e],[e,f],[f,b]\}$. Let M_2 have vertex set $\{a,b,c,d,e,f,g,h\}$ and edge set $\{[a,b],[a,d],[a,e],[a,f],[a,g],[a,h],[b,c],[c,d],[d,e],[e,f],[f,g],[g,h],[h,b],[c,f],[e,h]\}$.

Theorem 3.7. *Let G be a graph. If G has no M_1-minor, then* $\mathrm{sn}(G) \leqslant 2$. *If G has no M_2-minor, then* $\mathrm{sn}(G) \leqslant 3$.

Note that Corollary 3.5 also follows from Theorem 3.7 since both K_5 and $K_{3,3}$ are minors of M_2.

3.2 Cayley graphs

Recall that a Cayley (di)graph $\mathrm{Cay}(\Gamma; S)$ is defined on a group Γ by specifying a *connection set* $S \subseteq \Gamma$ and setting (a, b) to be an edge if $a^{-1}b \in S$. For our purposes we wish S to be a generating set since a Cayley graph is strongly connected if and only if S is a generating set. When $S = S^{-1}$ (i.e., $s \in S$ if and only if $s^{-1} \in S$), we can identify the two arcs (a, b) and (b, a) to obtain an undirected graph, otherwise the construction yields a directed graph. The identity e is usually not in S, but if we want loops, we can put it in. As before, this will depend on the uses and conventions.

A Cayley graph is *normal* (called *full* in [22]) if $a^{-1}sa \in S$ for all $s \in S$ and all $a \in \Gamma$ (see [30] for more on this fruitful idea). Frankl [22] has the following.

Theorem 3.8. *If* $\mathrm{Cay}(\Gamma; S)$ *is normal, then* $\mathrm{sn}\big(\mathrm{Cay}(\Gamma; S)\big) \leqslant |S \cup S^{-1}|$.

In the same paper it is also shown that $\mathrm{sn}(G) \in o\big(|V(G)|\big)$, not quite what Meyniel (personal communication to Frankl) conjectured, namely that $\mathrm{sn}(G) \in O\Big(\sqrt{|V(G)|}\Big)$ (the notation $f = O(g)$ is bad and easy to correct, as we do here; similarly for the other orders).

More is known about cops and robbers on Cayley graphs but as it involves generalizing the ideas, we will return to the subject in Section 5.

3.3 Complexity.

Goldstein and Reingold, [24] looked at the complexity of deciding whether a graph is k-cop-win. As mentioned earlier, it is not difficult (i.e., it is polynomial) to decide if a given graph is cop-win. Indeed, for a fixed k, it is polynomial to determine if a graph is k-cop win, [24]. Hahn and MacGillivray in [26] give a general algorithm that determines in time polynomial in the number of vertices whether k cops can capture l robbers on a given (di)graph. As in [24], k—as well as l—must be fixed as the auxiliary graphs used have n^k and n^l vertices, respectively, where n is the order of the given (di)graph.

If k is a parameter, the situation is different.

Theorem 3.9. *The problem of determining if k cops can catch a robber from given initial positions is* EXPTIME*-complete.*

The Goldstein and Reingold paper [24] provides a good survey of the work on complexity of cop-and-robber games on both undirected and directed graphs and there is no need to reproduce that work here.

4. Infinite graphs

It is perhaps surprising that infinite graphs can be cop-win. Once the trivial ones are out of the way (a K_α or a $K_{1,\alpha}$ for an infinite α), can they be characterized? Nowakowski and Winkler [36] give a characterization of cop-win graphs that applies equally well to finite and to infinite graphs. It is useful since it gives a nice polynomial algorithm to recognize cop-win graphs (and it forms a basis for the algorithm to decide if k cops can win against l robbers on a directed graph with or without loops in [26]).

Define R by first defining a sequence of relations R_α for ordinals $\alpha \leqslant |V(G)|$ (we take the view that a cardinal is the least of all equipotent ordinals). This is done recursively.

1. $R_0 = \{(u,u)\colon u \in V(G)\}$;
2. for $\alpha > 0$, $(u,v) \in R_\alpha$ if for each $x \in N[u]$ there is a $\beta < \alpha$ and a $y \in N[v]$ with $(x,y) \in R_\beta$.

Since $R_\alpha = R_{|V(G)|}$ for all $\alpha > |V(G)|$, there is a least γ such that $R_\alpha = R_\gamma$ for $\alpha > \gamma$. Observe that $R_\alpha \subseteq R_\beta$ if $\alpha \leqslant \beta$. Set $R = R_\gamma$.

Theorem 4.1. *A reflexive graph G is cop-win if and only if the binary relation R on $V(G)$ defined above is trivial.*

The least α such that $(u,v) \in R_\alpha$ determines the length of the game when the cop is at u and the robber at v as there are no infinite descending sequences of ordinals and the cop can move to the neighbour defined by the least β over its neighbourhood. This is exploited in the algorithm in [26]. Since finite chordal and bridged graphs are cop-win, it is natural to ask, as Anstee and Farber did (privately) whether infinite (or, at least, countably infinite) bridged graphs are cop-win. Surprisingly, this is not the case. Indeed, a result of [25] shows this rather strongly.

Theorem 4.2. *There are chordal graphs of diameter two of any infinite cardinality which are not cop-win.*

The result can be proved either by a direct construction, or by compactness from the following, which is of independent interest.

Theorem 4.3. *For any natural number k there is a chordal graph of diameter two on which a cop cannot catch a robber in fewer than k rounds.*

As of this writing, no interesting classes of infinite cop-win graphs have been described. It is known (Bonato, Hahn and Tardif, work in progress 2007) that for each ordinal $\alpha < \omega$ (i.e., each non-negative integer) there are $2^{\aleph_\alpha}$ non--isomorphic trees of cardinality $\aleph_\alpha$ without rays, that is, cop-win (the construction breaks down at ω.) For related work, see also Chastand, Laviolette, Polat *On constructible graphs, bridged graphs and weakly cop-win graphs*, Discrete Math. **224** (2000), 61–78.

One of the fashionable and fascinating current subjects is the structure of web graph (the graph whose vertices are web pages and whose arcs or edges are links of these pages) and its description. Part of the problem is the ever changing nature of the graph, but even if the graph is considered at a fixed point in time, its structure is elusive. Some of the attempts to describe the web graph involve infinite graphs, either directly (the random graph, that is, the Rado graph, the unique countable homogeneous graph), or as limits of sequences of finite graphs constructed according to fixed (usually probabilistic) rules, see [12] and [14]. One suggested use of cops-and-robbers games is looking for intruders on the web (hackers, spamers). While the game approach is too simple actually to find such intruders, it is nevertheless interesting to consider. Unfortunately, all the models of the web graph studied in [14] are robber-win for any finite number of cops and one robber. In all cases, the proofs revolve around the property that for any finite disjoint subsets X and Y of such a graph there is a vertex adjacent to all vertices of X and none of Y. The work is due to Ramanampanoharana [42].

Since the models of the web graph are necessarily countable, $\aleph_0$ cops are necessary and sufficient. Therefore, it makes, sense to ask about the density of the cops in countably infinite graphs when the game is played with one robber. The cop-density of an infinite graph G can be defined using the limit of $\frac{\mathrm{sn}(H)}{|V(H)|}$ over a chain (under containment) of finite subgraphs whose union is G. The *upper density* of G is then the lim sup over all chains whose union is G of the cop-densities of the chains. It is shown in [13] that the upper density of a countably infinite graph is either zero or one and is one exactly when G has the property that for any subset S of vertices of G there is a vertex in $G - S$ not adjacent to any vertex of S. It is also proved in the same paper that for any $r \in [0, 1]$ there is a countably infinite graph G with $\mathrm{sn}(G) = 1$ and which is the union of a chain C such that the limit of the cop-densities of graphs in C is r.

5. Generalizations

There are obvious generalizations to explore: non reflexive graphs, with none or some loops, and directed graphs. Not much has been done in either case, perhaps because analysis becomes less tractable. A few special classes have been considered, notably those of directed graphs obtained from Cayley graphs. In [21] Frankl answers a question of Hamidoune [27] for Cayley graphs on commutative groups and in [3] the result is extended to Cayley graphs on dihedral groups. These results involve one further generalization that has been studied for undirected reflexive graphs, namely the restriction of moves of each player to a specific set of edges (or arcs). We begin with the latter.

In [34], Neufeld and Nowakowski consider graphs in which two disjoint (except for loops) sets of edges are specified, the cop's and the robber's. They do share the loops (as in many other papers, the game is described on an irreflexive graph with each player allowed to move along an edge, or pass). Each player may only move along edges in her set. The authors first consider the graph G and its complement $\overline{G}$, with the robber moving in G, the cop in $\overline{G}$, both with all loops. Define $\overline{c}(G)$ to be the minimum number of cops needed to capture the robber under these conditions. Let $\gamma(G)$ be the domination number of G, that is, the cardinality of a smallest set S of vertices such that every vertex of the graph is either in S, or has a neighbour in, S.

Theorem 5.1. *Let G be a finite undirected graph, $\gamma(\overline{G})-1 \leqslant \overline{c}(G) \leqslant \gamma(\overline{G})$.*

Next on the agenda of [34] is a study of strong product of two graphs. To this end the edges of the strong ($\boxtimes$) product are divided into those obtained from the Cartesian ($\square$) product and those from the categorical ($\times$)product. Note that the products are formed on loopless graphs, with the loops added to the result (or, again, one can do it like the authors: think of the game on irreflexive graphs with the option not to move for each player). The authors obtain bounds on the number of cops needed in the products in terms of the cop numbers of the components and depending on which of the players move on which set of edges (or its complement in the strong product). For example, if the cops move along the Cartesian edges and the robber along the categorical edges, the following is true (the parameter $vs(X)$ of a graph X is the minimum over all possible linear orderings $L_i = (v_{i,1}, \dots, v_{i,n})$ of $V(X)$ of $\max\limits_{1<j<n} \bigl\{\bigl|\{(l,k)\colon l < j < k, v_{i,l}v_{i,k} \in E(X)\}\bigr|\bigr\}$).

Theorem 5.2. *If G and H are finite connected graphs, then $\operatorname{sn}(G \boxtimes H) \leqslant \min\bigl\{\operatorname{sn}(H)+vs(G), \operatorname{sn}(G)+vs(H), \operatorname{sn}(G)+\bigl|V(H)\bigr|-\alpha(H)-1, \operatorname{sn}(H)+\bigl|V(G)\bigr|-\alpha(G)-1\bigr\}$, where $\alpha(X)$ is the independence number of the graph X.*

The only thing the results of [34] share with those of [21] and [3] is the specification of the sets of edges for the players to move along. Indeed, in the Cayley graph results we are about to describe, the sets are not disjoint but rather one set is a subset of the other.

Let X be a directed graph and let Y be a subgraph of X. The cops-and-robber game can be played on X with the robber's moves limited to using the arcs of Y. We call such a game a *restricted* (X,Y) game and define the (X,Y) *cop number* to be the minimum number of cops needed to guarantee a win for them in the restricted (X,Y) game. The corresponding cop number can be denoted by $\operatorname{sn}_Y(X)$. If X is a Cayley (di)graph with connection set $S = S^{-1}$

and if $T \subseteq S$, we can refer to the restricted game and cop number by S and T instead of the edges generated by each and so speak about $\mathrm{sn}_T(\mathrm{Cay}(\Gamma; S))$. The result of [21] is this.

Theorem 5.3. *If Γ is a commutative group, $S = S^{-1}$ a generating set for Γ and $T \subseteq S$, then*

$$\mathrm{sn}_T(\mathrm{Cay}(\Gamma; S)) \leqslant \left\lceil \frac{(|T|+1)}{2} \right\rceil .$$

Alspach, Hanson and Li extend this in [3] to dihedral groups. The dihedral group $D_n, n \geqslant 2$ is the group generated by two elements ρ and τ such that $\rho^n = \tau^2 = 1$ and $\tau\rho\tau = \rho^{-1}$. Note that $|D_n| = 2n$. Write $\langle\rho\rangle$ for the subgroup generated by ρ and similarly $\langle\tau\rangle$. Analogously, $D_n = \langle\rho, \tau\rangle$.

Theorem 5.4. *Let D_n be the dihedral group as defined above, S a subset of D_n and $T \subseteq S$. Let $S_1 = S \cap \langle\rho\rangle$, $S_2 = S \cap \langle\rho\rangle\tau$, $T_2 = T \cap \langle\rho\rangle\tau$, and partition $T \cap \langle\rho\rangle$ into $T_0 \cup T_1$, where $T_0 = T_0^{-1}$ and $\rho^i \in T_1$ if and only if $\rho^i \in T$ but $\rho^{-i} \notin T$. Then*

$$\mathrm{sn}_T(D_n; S) \leqslant \begin{cases} |T_2| & \text{if } T = T_2 \neq \emptyset, \\ \left\lfloor \frac{|T_0|+2|T_1|+1}{2} \right\rfloor + |T_2|, & \text{otherwise}, \end{cases}$$

provided that if $\langle S\rangle \neq D_n$, then the robber must choose a vertex in the strongly connected component containing the identity.

When $S = T$, $\langle S\rangle = D_n$, and $S = S^{-1}$, and with the same definitions of S_1 and S_2 we get the following.

Corollary 5.5. *If $\mathrm{Cay}(D_n; S)$ is a connected Cayley graph on the dihedral group D_n, then*

$$\mathrm{sn}(\mathrm{Cay}(D_n; S)) \leqslant \begin{cases} |S_2| & \text{if } S = S_2 \neq \emptyset, \\ \left\lfloor \frac{|S_1|+1}{2} \right\rfloor + |S_2|, & \text{otherwise}. \end{cases}$$

There are other generalizations. Clark and Nowakowski, [16], [17], [18] study a variant in which two cops catch a robber, but the cops must move in tandem, that is, after each move they remain on adjacent vertices. They almost have a characterization of graphs on which the cops can catch the robber; such graphs are called *tandem-win*.

Theorem 5.6. *Let G be a reflexive graph whose vertices can be linearly ordered $\{v_1, \dots, v_n\}$ so that for each i there is a $i < j \leqslant n$ with $N_i'(v_i) \subseteq N_i'(v_j)$, with $N'(v) = N(v) \setminus \{v\}$. Then G is tandem-win.*

Note that the theorem only gives a sufficient condition for a graph to be tandem-win. As with other generalizations, a complete characterization remains open. On the positive side, there is an *algorithmic* characterization of many cops-and-robbers games. More precisely, consider any such game with k cops and l robbers, k, l fixed, with conditions on the cops' and robbers' moves expressible by adjacencies in the auxiliary graph described below. Then there is an algorithm polynomial in $\left|V(G)\right|^k + \left|V(G)\right|^l$ to determine if a directed graph is cop-win, or k-cop-win, or (k,l)-cop win (k cops catch l robbers).

Let D be a directed graph. For k cops (and, analogously, for l robbers) to be on at most k vertices, it means that they are at a point in V^k. For a point $u \in V^k$ write $[u] = \{u_1, u_2, \dots, u_k\}$. Fix integers $0 < l \leqslant k$ and a digraph $D = (V, A)$ with minimum outdegree at least 1. Define $C_k(D) = (V^k, A_k)$ by setting $A_k = \{(u,v)\colon u, v \in V^k,\ v_i \in N_D^+(u_i),\ i = 1, 2, \dots, k\}$. Define $R_l(D) = (V^l, A_l)$ similarly. Thus a move by either the cops or the robbers on D corresponds to moves by one cop on $C_k(D)$ or by one robber on $R_l(D)$. It remains to connect the two. The (k,l,D) *game graph* $\mathcal{D}$ is the disjoint union of copies of $C_k(D)$ and $R_l(D)$. Formally, $\mathcal{D} = \big((V^k \times \{c\}) \cup (V^l \times \{r\}), A(c) \cup A(r)\big)$ where $A(c) = \{(u,v)\colon u, v \in V^k \times \{c\},\ (u,v) \in A_k\}$ and $A(r) = \{(u,v)\colon u, v \in V^l \times \{r\},\ (u,v) \in A_l\}$. Now the game can be played on $\mathcal{D}$: the cops move from vertex to vertex of $V^k \times \{c\}$, while the robbers move on $V^l \times \{r\}$ (the cops choose their position first, the robbers second, then they alternate). The cops win if at some stage they are at some (u,c) with the robbers at (v,r) such that $[v] \subseteq [u]$.

The algorithm then uses an analogue of Theorem 4.1 to label the vertices of a second auxiliary graph. This graph is bipartite, with each vertex a triple (c, v, i) such that the cop and the robber are at the vertices c and v, respectively, and i indicates whose move it is from this position. Optimal strategies for both players can be read from the labels when the algorithm finishes. For more on this, see [26].

6. Open problems

Open directions are many. We list just a tip of the iceberg.

1. Schroeder [44] conjectures that $\mathrm{sn}(G) \leqslant g(G) + 3$.
2. Find on trivial classes of infinite cop-win graphs.
3. Characterize k-cop-win directed graphs (start with $k = 1$).
4. Characterize k-cop-win graphs (start with $k = 2$).
5. Find bounds on the number of moves needed for the cop(s) to win on a (k-)cop-win graph, perhaps in terms of some other graph parameters (but see [25] for many likely candidates that do not work, such as the

diameter, the length of a longest path, the length of a longest chordless path).

6. Can the cop number of a directed Cayley graph be bounded in some way similar to Theorem 5.3? What about non-abelian groups other than the dihedral group? What if the Cayley graph is directed?
7. What other graph parameters is the cop number related to in any of the variants? It is known that there is a strong relationship with treewidth and pathwidth, see [10], [45], [43] for more.

Added in proof. News on Problem 5 above: Gavenčiak proves that no more than $n-3$ moves are needed for the cop to capture the robber on a cop-win graph on n vertices and characterizes cop-win graphs that require this number of moves (*Cop-win graphs with maximal capture-time*, preprint, 2007).

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