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THE INTERIOR OPERATION IN f-DENSITY TOPOLOGY

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ABSTRACT. We give a characterization of the interior for an arbitrary subset A of the real line in f-density topology generated by the operator Φ_f, for which Lebesgue Density Theorem does not hold. We prove that for an arbitrary set $A \subset \mathbb{R}$ its interior in f-density topology equals $A \cap \Phi_f^{\beta}(B)$, where B is a measurable kernel of A and β is some countable ordinal. Moreover, each natural number n realizes the interior of A for some measurable A.

M. Filipczak and T. Filipczak in [FF1] have introduced a notion of f-density point of a measurable subset of the real line, which is a generalization of a well-known notion of a Lebesgue density point and a ψ-density point considered by S. J. Taylor in [T].

It is well-known that the interior of an arbitrary set $A \subset \mathbb{R}$ in a density topology equals $A \cap \Phi(B)$, where B is a measurable kernel of A and $\Phi(B)$ is a set of all density points of B. The proof of this fact depends essentially on the Lebesgue Density Theorem. But the operator Φ_f connected with f-density need not have the Lebesgue property, so in f-density topology the situation is more complicated and analogous to the ψ-density topology on the real line considered by M. Terepeta, E. Wagner–Bojakowska and W. Wilczyński in [T W-B] and [W-B W]. The aim of this paper is to find a formula for the interior of an arbitrary subset of a real line in terms of operator Φ_f.

We shall use the standard notation. Throughout the paper $\mathbb{N}$ will denote the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}$—the set of real numbers, $\mathcal{L}$—the σ-algebra of Lebesgue measurable sets and m—the Lebesgue measure on the real line. Let A' stands for $\mathbb{R} \setminus A$, $A + x = \{a + x \colon a \in A\}$ and $-A = \{-a \colon a \in A\}$. We shall say that sets $A \subset \mathbb{R}$, $B \subset \mathbb{R}$ are equivalent $(A \sim B)$ if and only if $m(A \triangle B) = 0$, where $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Let $\mathcal{A}$ be a family of all functions $f \colon (0, \infty) \to (0, \infty)$ such that

(A1) $\lim\limits_{x \to 0^+} f(x) = 0$,

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(A2) $\liminf\limits_{x\to 0+} \frac{f(x)}{x} < \infty$,

(A3) f is nondecreasing.

Let $f \in \mathcal{A}$, $A \in \mathcal{L}$ and $x \in \mathbb{R}$. We say that x is a right-hand f-density point of A if

$$\lim_{h\to 0+} \frac{m\big(A' \cap [x, x+h]\big)}{f(h)} = 0\,.$$

By $\Phi_f^+(A)$ denote the set of all right-hand f-density points of A. Analogously we can introduce a notion of a left-hand f-density point of A and a set $\Phi_f^-(A)$. We say that x is an f-density point of A if x is a right- and a left-hand f-density point of A. It is easy to see ([FF1], Proposition 1) that x is an f-density point of A if and only if

$$\lim_{\substack{h\to 0, k\to 0\\ h\geqslant 0, k\geqslant 0, h+k>0}} \frac{m\big(A' \cap [x-k, x+h]\big)}{f(h+k)} = 0\,.$$

Let $\Phi_f(A)$ denote the set of all f-density points of A. For each function $f \in \mathcal{A}$ we have

Theorem A. ([FF1], Th. 5) *For each $A \in \mathcal{L}$, $B \in \mathcal{L}$*

(a) *if $A \subset B$ then $\Phi_f(A) \subset \Phi_f(B)$;*

(b) *if $A \sim B$ then $\Phi_f(A) = \Phi_f(B)$;*

(c) $\Phi_f(\emptyset) = \emptyset$, $\Phi_f(\mathbb{R}) = \mathbb{R}$;

(d) $\Phi_f(A \cap B) = \Phi_f(A) \cap \Phi_f(B)$.

Put

$$\mathcal{T}_f = \big\{A \in \mathcal{L} \colon A \subset \Phi_f(A)\big\}\,.$$

Theorem B. ([FF1], Th. 7) *The family $\mathcal{T}_f$ is a topology essentially stronger than the Euclidean topology on the real line.*

In [FF2] the authors have divided the family $\mathcal{A}$ onto two subfamilies

$$\mathcal{A}^0 = \left\{f \in \mathcal{A} \colon \liminf_{x\to 0+} \frac{f(x)}{x} = 0\right\}$$

and

$$\mathcal{A}^1 = \left\{f \in \mathcal{A} \colon \liminf_{x\to 0+} \frac{f(x)}{x} > 0\right\}$$

and observed, that if $f \in \mathcal{A}^1$ then the operator Φ_f has properties analogous to the classical operator Φ, together with the Lebesgue Density Theorem ($\Phi_f(A) \sim A$ for $A \in \mathcal{L}$). Consequently, for each $A \subset \mathbb{R}$

$$\mathrm{Int}_{\mathcal{T}_f}(A) = A \cap \Phi_f(B)\,,$$

where $B \subset A$ is a measurable kernel of A, and $\mathrm{Int}_{\mathcal{T}_f}(A)$ denotes the interior of A in the topology $\mathcal{T}_f$.

If $f \in \mathcal{A}^0$ then the situation is quite different.

Theorem C. ([FF2], Th. 2) *For any function $f \in \mathcal{A}^0$ there exists a nowhere dense closed set $F \subset [0,1]$ of positive measure such that no point of F is an f-density point of F.*

Consequently, the operator Φ_f is not so called lower density, because it does not fulfil condition $A \sim \Phi_f(A)$ for arbitrary $A \in \mathcal{L}$. However, since $\Phi_f(A) \subset \mathbb{R} \setminus \Phi(\mathbb{R} \setminus A)$ for each $A \in \mathcal{L}$,

$$m\big(\Phi_f(A) \setminus A\big) = 0 \tag{1}$$

for $A \in \mathcal{L}$ ([FF1], Cor. 1).

Now we shall establish a formula for the interior of a measurable set in the topology $\mathcal{T}_f$. Let $A \in \mathcal{L}$. Put

$$\Phi_f^1(A) = \Phi_f(A).$$

If α is an ordinal number, $1 < \alpha < \Omega$, we define operator Φ_f^α as follows:

1^o if α has the predecessor, $\alpha = \beta + 1$, where $1 \leqslant \beta < \Omega$, then

$$\Phi_f^\alpha(A) = \Phi_f\Big(\Phi_f^\beta(A)\Big);$$

2^o if α is a limit number, then

$$\Phi_f^\alpha(A) = \bigcap_{1 \leqslant \beta < \alpha} \Phi_f^\beta(A).$$

Lemma 1. *For each $A \in \mathcal{L}$ and arbitrary ordinals α, β such that $1 \leqslant \beta < \alpha < \Omega$ we have*

$$\Phi_f^\alpha(A) \subset \Phi_f^\beta(A).$$

The proof using transfinite induction, conditions (b) and (d) from Theorem A and the fact that $m\big(\Phi_f(A) \setminus A\big) = 0$ for $A \in \mathcal{L}$, is analogous to the proof of Theorem 2 in [W-B W].

Lemma 2. *For each $A \in \mathcal{L}$ and each countable ordinal α greater than zero*

$$\mathrm{Int}_{\mathcal{T}_f} A \subset A \cap \Phi_f^\alpha(A).$$

There exists an ordinal β, $1 \leqslant \beta < \Omega$, such that

$$\mathrm{Int}_{\mathcal{T}_f} A = A \cap \Phi_f^\beta(A).$$

The proof using transfinite induction, Theorem A (a), (b), (d) and Lemma 1 is analogous to the proof of Lemma 3 in [W-B W].

THEOREM 3. *For arbitrary set $A \subset \mathbb{R}$ we have*

$$\operatorname{Int}_{\mathcal{T}_f} A = A \cap \Phi_f^{\beta}(B)\,,$$

where $B \subset A$ is a measurable kernel of A and β is some countable ordinal greater or equal to 1.

The proof is analogous to the proof of Theorem 4 in [W-BW].

It is easy to see that for an arbitrary set $A \subset \mathbb{R}$, if B is a measurable kernel of A, then the set

$$A \cap \Phi_f^{\beta}(B)$$

is measurable, for each countable ordinal β. Indeed,

$$A \cap \Phi_f(B) = \big(A \cap B \cap \Phi_f(B)\big) \cup \big(A \cap \Phi_f(B) \setminus B\big) = \big(B \cap \Phi_f(B)\big) \cup \big(A \cap \Phi_f(B) \setminus B\big)\,.$$

Since B is measurable, from [FF3], Theorem 4 it follows that the set $\Phi_f(B)$ is also measurable, and the second summand is measurable by virtue of (1). For $\beta > 1$ from Lemma 1 we obtain

$$A \cap \Phi_f^{\beta}(B) = \big(A \cap \Phi_f(B)\big) \cap \Phi_f^{\beta}(B)$$

and the intersection of two measurable sets is obviously measurable.

Now we shall prove that for each function $f \in \mathcal{A}^0$ and for each $n \in \mathbb{N}$ there exists a set $A \in \mathcal{L}$ such that

$$\operatorname{Int}_{\mathcal{T}_f} A = A \cap \Phi_f^{n}(A)\,.$$

For this purpose we need some auxiliary lemmas.

LEMMA 4. *If $A \in \mathcal{L}$, $B \in \mathcal{L}$ and $A \subset B$, then for each ordinal number α, $1 \leqslant \alpha < \Omega$ we have*

$$\Phi_f^{\alpha}(A) \subset \Phi_f^{\alpha}(B)\,.$$

The proof, using transfinite induction and Theorem A (a), is analogous to the proof of Lemma 6 in [W-BW].

LEMMA 5. *For each function $f \in \mathcal{A}$ there exists a sequence $\{x_n\}_{n \in \mathbb{N}_0}$, $x_n \searrow 0$ such that for an arbitrary $A \in \mathcal{L}$ the following conditions are equivalent*

1) $\lim\limits_{n \to \infty} \dfrac{m\big(A \cap [0, x_n]\big)}{f(x_n)} = 0$,

2) $\lim\limits_{h \to 0+} \dfrac{m\big(A \cap [0, h]\big)}{f(h)} = 0$.

P r o o f. Let $f \in \mathcal{A}$. Obviously, 2) implies 1) for an arbitrary sequence $\{x_n\}_{n \in \mathbb{N}_0}$ descending to zero. We shall construct a sequence $\{x_n\}_{n \in \mathbb{N}_0}$ for which 1) implies 2).

In [FF3], Th. 1, Th. 2 and Th. 3, the authors proved that for each function $f \in \mathcal{A}$ there exists a continuous function $g \in \mathcal{A}$ such that $\Phi_f = \Phi_g$ and, consequently, $\mathcal{T}_f = \mathcal{T}_g$. This means that

$$\lim_{h \to 0+} \frac{m(A \cap [0,h])}{f(h)} = 0$$

if and only if

$$\lim_{h \to 0+} \frac{m(A \cap [0,h])}{g(h)} = 0 .$$

Let $x_0 = 1$ and $q \in (0,1)$. Suppose that we have defined $x_0, \dots, x_{n-1}$. Let $x_n \in (0, x_{n-1})$ be such that $g(x_n) = q \cdot g(x_{n-1})$. We obtain a sequence $\{x_n\}_{n \in \mathbb{N}_0}$ decreasingly tending to zero. Let $h \in (0, x_0]$. Then there exists $n \in \mathbb{N}$, $n = n(h)$, such that $h \in (x_n, x_{n-1}]$ and

$$\frac{m(A \cap [0,h])}{g(h)} \leqslant \frac{m(A \cap [0,x_{n-1}])}{g(x_n)} = \frac{m(A \cap [0,x_{n-1}])}{q \cdot g(x_{n-1})} .$$

Clearly, $n(h) \to \infty$ if $h \to 0^+$, so

$$\lim_{h \to 0+} \frac{m(A \cap [0,h])}{g(h)} = 0 ,$$

which gives 2). □

Theorem 6. *For each function $f \in \mathcal{A}^0$ and a real number c, $0 < c < 1$, there exists a perfect nowhere dense set $F \subset [0,1]$ such that $m(F) \geqslant c$ and no point of F is an f-density point of F.*

Proof. We shall construct the set F modifying the proof of Theorem 2 in [FF2]. Clearly, we can assume that $c \geqslant \frac{1}{2}$. Since $f \in \mathcal{A}^0$, there exists a decreasing sequence $\{x_n\}_{n \in \mathbb{N}}$ tending to zero such that

$$\lim_{n \to \infty} \frac{f(x_n)}{x_n} = 0 \tag{2}$$

and the sequence $\left\{\frac{f(x_n)}{x_n}\right\}_{n \in \mathbb{N}}$ is decreasing. Let $w_n = \sqrt{\frac{f(x_n)}{x_n}}$ for $n = 1, 2, \dots$. We can assume that $w_1 < \frac{1}{2}(1-c)$. Put $y_1 = \frac{1}{2}(1-c) - w_1$ and $y_n = w_{n-1} - w_n$ for $n = 2, 3, \dots$. Obviously, $y_n > 0$ and $\sum\limits_{n=1}^{\infty} y_n = \frac{1}{2}(1-c)$.

Then we define two sequences of positive integers: $\{k_i\}_{i \in \mathbb{N}}$ and increasing $\{n_i\}_{i \in \mathbb{N}}$, satisfying conditions

$$\left(1 - 2\sum_{j=1}^{i} y_j\right) \frac{1}{k_1 \cdot \ldots \cdot k_i} \leqslant x_{n_i} < \frac{1}{k_1 \cdot \ldots \cdot k_i} \tag{3}$$

and

$$x_{n_i} - \frac{2y_{i+1}}{k_1 \cdot \ldots \cdot k_i} \leqslant k_{i+1} x_{n_{i+1}} < x_{n_i} - \frac{y_{i+1}}{k_1 \cdot \ldots \cdot k_i} \tag{4}$$

for $i \in \mathbb{N}$.

As $\{x_n\}_{n\in\mathbb{N}}$ tends to zero, there exists $n_1 \in \mathbb{N}$ such that

$$x_{n_1} < y_1 .$$

Hence there exists $k_1 \in \mathbb{N}$ satisfying inequalities

$$y_1 \leqslant 1 - k_1 x_{n_1} \leqslant 2y_1 .$$

Obviously, (3) is fulfilled for $i = 1$.

Suppose that we have defined the numbers $k_1, \ldots, k_i$ and $n_1, \ldots, n_i$ satisfying (3). Let n_{i+1} be an arbitrary natural number greater than n_i satisfying

$$x_{n_{i+1}} < \frac{y_{i+1}}{k_1 \cdot \ldots \cdot k_i} . \tag{5}$$

By the assumption we have $x_{n_i} > \frac{c}{k_1 \cdot \ldots \cdot k_i}$ and $2y_{i+1} < 1 - c$. Consequently,

$$x_{n_i} - \frac{2y_{i+1}}{k_1 \cdot \ldots \cdot k_i} > \frac{2c-1}{k_1 \cdot \ldots \cdot k_i} \geqslant 0 .$$

Therefore there exists $k_{i+1} \in \mathbb{N}$ such that

$$x_{n_i} - \frac{2y_{i+1}}{k_1 \cdot \ldots \cdot k_i} \leqslant k_{i+1} x_{n_{i+1}} < x_{n_i} - \frac{y_{i+1}}{k_1 \cdot \ldots \cdot k_i} \tag{6}$$

and (4) is fulfilled. Now we shall prove that (3) is fulfilled, too. From (6) and an induction hypothesis

$$k_{i+1} x_{n_{i+1}} < x_{n_i} < \frac{1}{k_1 \cdot \ldots \cdot k_i}$$

and, consequently,

$$x_{n_{i+1}} < \frac{1}{k_1 \cdot \ldots \cdot k_{i+1}} .$$

On the other hand

$$\begin{aligned} k_{i+1} x_{n_{i+1}} \geqslant x_{n_i} - \frac{2y_{i+1}}{k_1 \cdot \ldots \cdot k_i} \geqslant \left(1 - 2\sum_{j=1}^{i} y_j\right) \frac{1}{k_1 \cdot \ldots \cdot k_i} \\ - \frac{2y_{i+1}}{k_1 \cdot \ldots \cdot k_i} = \left(1 - 2\sum_{j=1}^{i+1} y_j\right) \frac{1}{k_1 \cdot \ldots \cdot k_i} , \end{aligned}$$

which completes the construction of sequences $\{k_i\}_{i\in\mathbb{N}}$ and $\{n_i\}_{i\in\mathbb{N}}$.

Now, we construct a decreasing sequence $\{F_n\}_{n\in\mathbb{N}}$ of the closed sets such that F_i consists of $k_1\cdot\ldots\cdot k_i$ disjoint closed intervals each of the length x_{n_i}. Put

$$F_1=\bigcup_{j=0}^{k_1-1}\left[\frac{j}{k_1},\frac{j}{k_1}+x_{n_1}\right].$$

Since $x_{n_1}<\frac{1}{k_1}$, the intervals $\left[\frac{j}{k_1},\frac{j}{k_1}+x_{n_1}\right]$, $j=0,\ldots k_1-1$ are disjoint.

Suppose that we have just defined a set F_i, for some $i\geqslant 1$. Denote the components of F_i by $J_p^i=[a_p^i,a_p^i+x_{n_i}]$, $p=1,\ldots,k_1\cdot\ldots\cdot k_i$. Since $x_{n_{i+1}}<\frac{x_{n_i}}{k_{i+1}}$, the set

$$F_{i+1}=\bigcup_{p=1}^{k_1\cdot\ldots\cdot k_i}\ \bigcup_{j=0}^{k_{i+1}-1}\left[a_p^i+x_{n_i}\frac{j}{k_{i+1}},a_p^i+x_{n_i}\frac{j}{k_{i+1}}+x_{n_{i+1}}\right]$$

consists of $k_1\cdot\ldots\cdot k_{i+1}$ disjoint intervals each of the length $x_{n_{i+1}}$, which finishes the definition of $\{F_i\}_{i\in\mathbb{N}}$.

Obviously, $m(F_i)=k_1\cdot\ldots\cdot k_ix_{n_i}$ for $i\in\mathbb{N}$, so it is easy to see that according to (3)

$$m\big([0,1]\setminus F_i\big)=1-k_1\cdot\ldots\cdot k_ix_{n_i}\leqslant 2(y_1+\cdots+y_i) \tag{7}$$

for $i\in\mathbb{N}$.

Let i be an arbitrary positive integer. Let J^i be a component of F_i. From (4) and (3) we conclude that

$$m(J^i\setminus F_{i+1})=x_{n_i}-k_{i+1}x_{n_{i+1}}>\frac{y_{i+1}}{k_1\cdot\ldots\cdot k_i}>y_{i+1}x_{n_i}=y_{i+1}m(J^i). \tag{8}$$

We shall prove that for any $p>i$

$$m(J^i\setminus F_p)\geqslant(y_{i+1}+\cdots+y_p)m(J^i). \tag{9}$$

For $p=i+1$ it follows from (8). Suppose that (9) is true for some $p>i$. We shall prove it for $p+1$. Note that the set $J^i\cap F_p$ consists of $k_{i+1}\cdot\ldots\cdot k_p$ components. For any component J^p of this set by (8) we have

$$m(J^p\setminus F_{p+1})>\frac{y_{p+1}}{k_1\cdot\ldots\cdot k_p}.$$

Hence, using (3) we obtain

$$\begin{aligned}m\big((J^i\cap F_p)\setminus F_{p+1}\big)&\geqslant k_{i+1}\cdot\ldots\cdot k_p\frac{y_{p+1}}{k_1\cdot\ldots\cdot k_p}\\&=\frac{y_{p+1}}{k_1\cdot\ldots\cdot k_i}>y_{p+1}x_{n_i}=y_{p+1}m(J^i).\end{aligned}$$

Since (8) is true for p, we conclude that

$$\begin{aligned} m(J^i \setminus F_{p+1}) &= m(J^i \setminus F_p) + m\big((J^i \cap F_p) \setminus F_{p+1}\big) \\ &\geqslant (y_{i+1} + \dots + y_p)m(J^i) + y_{p+1}m(J^i) = (y_{i+1} + \dots + y_{p+1})m(J^i) . \end{aligned}$$

Now let

$$F = \bigcap_{i=1}^{\infty} F_i .$$

Clearly, F is perfect and, as $\frac{1}{k_1 \cdot \ldots \cdot k_i} \underset{i\to\infty}{\longrightarrow} 0$, F is nowhere dense. From (7) we have

$$m\big([0,1] \setminus F\big) \leqslant 2\sum_{i=1}^{\infty} y_i = 1 - c .$$

Consequently, $m(F) \geqslant c$.

Now we shall prove that no point of F is its f-density point. Let $x \in F$. By the definition of F it follows that for any $i \in \mathbb{N}$ there exists a component J^i of F_i such that $x \in J^i$. Therefore

$$\{x\} = \bigcap_{i=1}^{\infty} J^i .$$

From (9) we conclude that

$$m(J^i \setminus F) \geqslant \left(\sum_{p=i+1}^{\infty} y_p\right) m(J^i) = w_i \cdot m(J^i) .$$

Consequently,

$$\begin{aligned} \frac{m(J^i \setminus F)}{f\big(m(J^i)\big)} &= \frac{m(J^i \setminus F)}{m(J^i)} \cdot \frac{m(J^i)}{f\big(m(J^i)\big)} \\ &\geqslant w_i \frac{x_{n_i}}{f(x_{n_i})} > w_{n_i} \frac{x_{n_i}}{f(x_{n_i})} = \sqrt{\frac{x_{n_i}}{f(x_{n_i})}} \underset{i\to\infty}{\longrightarrow} \infty . \end{aligned}$$

It means that x is not an f-density point of F. Since F is closed, $\Phi_f(F) \subset F$ and, consequently, $\Phi_f(F) = \emptyset$. □

Lemma 7. *Let $f \in \mathcal{A}^0$. There exists a perfect nowhere dense set $F \subset [0,1)$ such that 0 is a right-hand f-density point of F and $\Phi_f(F) = \emptyset$.*

P r o o f. Since $f \in \mathcal{A}^0$, there exists a sequence $\{z_k\}_{k\in\mathbb{N}}$ such that $z_k \underset{k\to\infty}{\searrow} 0$ and $f(z_k) < z_k$ for $k \in \mathbb{N}$. Let $\{x_n\}_{n\in\mathbb{N}_0}$ be a sequence associated with f

as in Lemma 5. There exists a subsequence $\{z_{k_n}\}_{n\geqslant 2}$ of $\{z_n\}_{n\in\mathbb{N}}$ such that $z_{k_n} < x_{n-1} - x_n$, $n = 2,3,\dots$ Obviously, $f(z_{k_n}) < z_{k_n} < (x_{n-1} - x_n)$ and

$$f(z_{k_n}) \leqslant f(x_{n-1} - x_n) \tag{10}$$

for $n = 2,3,\dots$ By virtue of Theorem 6 for each positive integer $n \geqslant 2$ there exists a perfect nowhere dense set $E_n \subset (x_n, x_{n-1})$ such that

$$m(E_n) \geqslant (x_{n-1} - x_n) - \frac{1}{2^n} f(z_{k_n}) \tag{11}$$

and $\Phi_f(E_n) = \emptyset$. Put

$$F = \bigcup_{n=2}^{\infty} E_n \cup \{0\}\,.$$

Since $F \subset [0,\infty)$, 0 is not an f-density point of F. Also F does not have positive f-density points because the distance between E_n and E_m is positive for $n \neq m$. We shall show that 0 is a right-hand f-density point of F. For each $n \in \mathbb{N}$ we have

$$\begin{aligned}
\frac{m(F' \cap [0,x_n])}{f(x_n)} &= \frac{\sum\limits_{p=n+1}^{\infty} m\big(F' \cap [x_p, x_{p-1}]\big)}{f(x_n)} \\
&= \frac{\sum\limits_{p=n+1}^{\infty} m\big(E_p' \cap [x_p, x_{p-1}]\big)}{f(x_n)} = \sum_{p=n+1}^{\infty} \frac{m\big(E_p' \cap [x_p, x_{p-1}]\big)}{f(x_n)} \\
&\leqslant \sum_{p=n+1}^{\infty} \frac{m\big(E_p' \cap [x_p, x_{p-1}]\big)}{f(x_{p-1} - x_p)} \leqslant \sum_{p=n+1}^{\infty} \frac{\frac{1}{2^p} f(z_{k_p})}{f(x_{p-1} - x_p)} \\
&\leqslant \sum_{p=n+1}^{\infty} \frac{1}{2^p} = \frac{1}{2^n}
\end{aligned}$$

by virtue of (11) and (10). From Lemma 5 we obtain the required result. □

Modifying condition (11) we can receive a stronger result. For this purpose for any function $f \in \mathcal{A}^0$ we define a function $\hat{f}$ as follows:

$$\hat{f}(x) = \begin{cases} x \cdot f(x) & \text{for } x \in (0,1)\,, \\ f(x) & \text{for } x \in [1,\infty)\,. \end{cases}$$

Obviously, $\hat{f} \in \mathcal{A}^0$.

LEMMA 8. *Let $f \in \mathcal{A}^0$. There exists a perfect nowhere dense set $F \subset [0,1]$ such that 0 is a right-hand $\hat{f}$-density point of F and $\Phi_f(F) = \emptyset$.*

P r o o f. It is sufficient, using Theorem 6, to take out the set $E_n \subset (x_n, x_{n-1})$ such that

$$m(E_n) \geqslant (x_{n-1} - x_n) - \frac{1}{2^n} f(z_{k_n})(x_{n-1} - x_n)$$

for $n \geqslant 2$. □

THEOREM 9. *Let $f \in \mathcal{A}^0$. For an arbitrary perfect nowhere dense set $C \subset (c,d)$ and each $\varepsilon > 0$ there exists a nowhere dense set $D \subset [c-\varepsilon, d+\varepsilon]$ such that $C \cup D$ is a perfect nowhere dense set and $\Phi_f(C \cup D) = C$.*

P r o o f. The proof is analogous to the proof of Theorem 11 in [W-BW] after necessary modifications. Let F be a set from Lemma 8. For an arbitrary interval $[a,b]$ put

$$F(a,b) = \left((F+a) \cap \left[a, \frac{a+b}{2}\right]\right) \cup \left((-F+b) \cap \left[\frac{a+b}{2}, b\right]\right).$$

Let $\{a_n, b_n\}_{n\in\mathbb{N}}$ be a sequence of components of the set $(c-\varepsilon, d+\varepsilon) \setminus C$. We shall assume that $(a_1, b_1) = (c-\varepsilon, \inf C)$, $(a_2, b_2) = (\sup C, d+\varepsilon)$. Put

$$\varepsilon_n = \frac{1}{2} m\left(F' \cap \left[0, \frac{b_n - a_n}{4}\right]\right) \tag{12}$$

for $n > 2$. Obviously, $0 < \varepsilon_n < (b_n - a_n)/4$ for $n > 2$. Let I_n, for $n > 2$, be an open interval included in $\big((a_n+b_n)/2 - \varepsilon_n, (a_n+b_n)/2 + \varepsilon_n\big)$ such that

$$\inf I_n \in \left(\frac{a_n+b_n}{2} - \varepsilon_n, \frac{a_n+b_n}{2}\right) \setminus F(a_n, b_n)$$

and

$$\sup I_n \in \left(\frac{a_n+b_n}{2}, \frac{a_n+b_n}{2} + \varepsilon_n\right) \setminus F(a_n, b_n).$$

Put additionally

$$D_1 = (-F + b_1) \cap [a_1, b_1]$$

in case a_1 is a right-hand accumulation point of $-F + b_1$ and

$$D_1 = (-F + b_1) \cap (a_1, b_1]$$

in the other case. Similarly, let

$$D_2 = (F + a_2) \cap [a_2, b_2]$$

if b_2 is a left-hand accumulation point of $F+a_2$ and

$$D_2=(F+a_2)\cap[a_2,b_2)$$

if it is not. At last, let

$$D=D_1\cup D_2\cup\bigcup_{n=3}^{\infty}\big(F(a_n,b_n)\setminus I_n\big).$$

It is not difficult to observe that D is a nowhere dense set contained in $[c-\varepsilon,d+\varepsilon]$ and $C\cup D$ is a perfect nowhere dense set.

Now we shall prove that $\Phi_f(C\cup D)=C$. Let $x\notin C$. There are two cases:

1^o $x\notin D$. Then $x\notin C\cup D$ and $x\notin\Phi_f(C\cup D)$ since $C\cup D$ is closed.

2^o $x\in D$. Then $x\notin\Phi_f(C\cup D)$, since the set F from Lemma 8 has no positive f-density point. Consequently, $\Phi_f(C\cup D)\subset C$.

Now, let $x\in C$ and $h>0$. Obviously,

$$(C\cup D)'\cap(x,x+h)=C'\cap D'\cap(x,x+h).$$

Put $\bar N=\{n\in\mathbb{N}:(a_n,b_n)\cap(x,x+h)\neq\emptyset\}$. If $x+h\notin C$, then there exists $n_1\in\bar N$ such that $x+h\in(a_{n_1},b_{n_1})$. In this case we have

$$(C\cup D)'\cap(x,x+h)=\big[D'\cap(a_{n_1},b_{n_1})\cap(x,x+h)\big]\cup\bigcup_{n\in\bar N\setminus\{n_1\}}\big[D'\cap(a_n,b_n)\big].\tag{13}$$

Obviously, if $n\in\bar N\setminus\{n_1\}$ then

$$m\big(D'\cap(a_n,b_n)\big)\leqslant 2m\left(F'\cap\left(0,\frac{b_n-a_n}{2}\right)\right)+2\varepsilon_n,$$

so from (12) we obtain

$$m\big(D'\cap(a_n,b_n)\big)\leqslant 3m\big(F'\cap(0,b_n-a_n)\big)\tag{14}$$

for $n\in\bar N\setminus\{n_1\}$.

Since $(a_{n_1},b_{n_1})\cap(x,x+h)=(a_{n_1},x+h)$, so

$$\begin{aligned}D'\cap(a_{n_1},b_{n_1})\cap(x,x+h)&=D'\cap(a_{n_1},x+h)\\&\subset\begin{cases}F'(a_{n_1},b_{n_1})\cap(a_{n_1},x+h)\cup I_{n_1} & \text{if } n_1\neq 2\\ \qquad\text{and } \frac{b_{n_1}-a_{n_1}}{2}-\varepsilon_{n_1}<h, & \\ F'(a_{n_1},b_{n_1})\cap(a_{n_1},x+h) & \text{in the contrary case.}\end{cases}\end{aligned}$$

Hence

$$m\big(D' \cap (a_{n_1}, b_{n_1}) \cap (x, x+h)\big) \leqslant \begin{cases} 2m\big(F' \cap (0, x+h-a_{n_1})\big) + 2\varepsilon_{n_1} & \text{if } n_1 \neq 2 \\ \qquad \text{and } \frac{b_{n_1}-a_{n_1}}{2} - \varepsilon_{n_1} < h, \\ m\big(F' \cap (0, x+h-a_{n_1})\big) & \text{in the contrary case.} \end{cases} \tag{15}$$

Suppose that $n_1 \neq 2$ and $(b_{n_1} - a_{n_1})/2 - \varepsilon_{n_1} < h$. Then from (12)

$$h > \frac{b_{n_1} - a_{n_1}}{2} - \frac{1}{2} m\left(F' \cap \left[0, \frac{b_{n_1} - a_{n_1}}{4}\right]\right) \geqslant \frac{b_{n_1} - a_{n_1}}{4}.$$

Hence

$$2\varepsilon_{n_1} = m\left(F' \cap \left[0, \frac{b_{n_1} - a_{n_1}}{4}\right]\right) \leqslant m\big(F' \cap [0, h]\big). \tag{16}$$

For the convenience assume that $\varepsilon_2 = 0$. From (13), (14), (15) and (16) we obtain

$$\begin{aligned} m\big((C \cup D)' \cap [x, x+h)\big) &\leqslant \sum_{n \in \bar{N} \setminus \{n_1\}} 3m\big(F' \cap (0, b_n - a_n)\big) \\ &\quad + 2m\big(F' \cap (0, x+h-a_{n_1})\big) + 2\varepsilon_{n_1} \\ &\leqslant \sum_{n \in \bar{N} \setminus \{n_1\}} 3m\big(F' \cap (0, b_n - a_n)\big) + 3m\big(F' \cap (0, h)\big) \end{aligned}$$

since $x + h - a_{n_1} < h$. Obviously, $b_n - a_n < h$ for $n \in \bar{N} \setminus \{n_1\}$ and f is nondecreasing, so $f(b_n - a_n) \leqslant f(h)$ for $n \in \bar{N} \setminus \{n_1\}$.

Put

$$A(h) = \sup\left\{ \frac{m\big(F' \cap [0, b_n - a_n)\big)}{(b_n - a_n) f(b_n - a_n)}, \quad n \in \bar{N} \setminus \{n_1\} \right\}.$$

We have

$$\begin{aligned} \frac{m\big((C \cup D)' \cap [x, x+h)\big)}{f(h)} &\leqslant \sum_{n \in \bar{N} \setminus \{n_1\}} \frac{3m\big(F' \cap (0, b_n - a_n)\big)}{f(b_n - a_n)} + \frac{3m\big(F' \cap (0, h)\big)}{f(h)} \\ &\leqslant 3A(h) \sum_{n \in \bar{N} \setminus \{n_1\}} (b_n - a_n) + \frac{3m\big(F' \cap (0, h)\big)}{h f(h)} \cdot h \\ &\leqslant 3A(h) \cdot h + \frac{3m\big(F' \cap (0, h)\big)}{h f(h)} \cdot h. \end{aligned}$$

Clearly, $A(h)$ is bounded and both summands tend to zero because 0 is a right-hand $\hat{f}$-density point of F. Consequently,

$$\lim_{h \to 0+} \frac{m\big((C \cup D)' \cap [x, x+h)\big)}{f(h)} = 0.$$

Analogous considerations can be carried out for an interval $(x-h,x]$ and the same result is obtained. Finally, x is an f-density point of $C\cup D$. Consequently, $C\subset\Phi_f(C\cup D)$. □

THEOREM 10. *Let $f\in\mathcal{A}^0$. For each $n\in\mathbb{N}$ there exists a perfect nowhere dense set A such that*

$$\operatorname{Int}_{\mathcal{T}_f} A = A\cap\Phi_f^n(A).$$

Proof. For $n=1$ it suffices to take the set F from Theorem 6.

Suppose that there exists a perfect nowhere dense set C such that $\operatorname{Int}_{\mathcal{T}_f} C = C\cap\Phi_f^n(C)$. From Theorem 9 it follows that there exists a nowhere dense set D such that $A=C\cup D$ is a perfect nowhere dense set and $\Phi_f(A)=\Phi_f(C\cup D)=C$. From Lemma 1 we have

$$\Phi_f^{n+1}(A)\subset\Phi_f(A)=C,$$

so using Lemma 2 we have

$$\begin{aligned}\operatorname{Int}_{\mathcal{T}_f} A\subset A\cap\Phi_f^{n+1}(A) &= A\cap C\cap\Phi_f\big(\Phi_f^n(A)\big)\\ &= C\cap\Phi_f^n\big(\Phi_f(A)\big)\\ &= C\cap\Phi_f^n(C)=\operatorname{Int}_{\mathcal{T}_f} C\subset\operatorname{Int}_{\mathcal{T}_f} A.\end{aligned}$$

□

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