Tatra Mt. Math. Publ. **38** (2007), 71–76

TATRA MOUNTAINS
Mathematical Publications

SOME NONOSCILLATORY PROPERTIES OF THIRD ORDER DIFFERENTIAL EQUATION OF NEUTRAL TYPE

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ABSTRACT. We study asymptotic properties of nonoscillatory solutions of the third order neutral differential equations. We present new sufficient conditions for all nonoscillatory solutions to be unbounded.

1. Introduction

We consider third order linear neutral functional differential equation of the form

$$\frac{\mathrm{d}^3}{\mathrm{d}t^3}\big[x(t)-p(t)x(\sigma(t))\big]+q(t)x\big(\tau(t)\big)=0, \qquad t\ge t_0, \tag{1}$$

under the standing hypotheses:

a) $p:[t_0,\infty)\to(0,\infty)$ is continuous,

b) $q,\tau:[t_0,\infty)\to R$ are continuous, $\lim\limits_{t\to\infty}\tau(t)=\infty$, $q(t)\not\equiv 0$, on any half line $[t_0,\infty)$ and τ is strictly increasing,

c) $\sigma:[t_0,\infty)\to R$ is continous and $\lim\limits_{t\to\infty}\sigma(t)=\infty$, σ is strictly increasing.

Our aim is to obtain new sufficient conditions for the nonoscillatory solutions of equation (1) to be unbounded. By a solution of equation (1) we mean a continuous function $x:[t_x,\infty)\to R$ such that $x(t)-p(t)x\big(\sigma(t)\big)$ is three times continuously differentiable and $x(t)$ satisfies Eq. (1) for all sufficiently large $t\ge t_x$. The solutions which vanish for all large t will be excluded from our consideration. A solution of (1) is called nonoscillatory if it is eventually of constant sign in $[t_x,\infty)$, otherwise it is oscillatory.

2000 Mathematics Subject Classification: 34K40, 34K25.
Keywords: asymptotic, nonoscillatory solution, neutral equation.
Supported by the Grant FPV n. 01–701–03.

2. Asymptotic behaviour

In this section we shall study the asymptotic behaviour of all nonoscillatory solutions of equation (1). $\tau^{-1}(t)$, $\sigma^{-1}(t)$ will denote the inverse functions of $\tau(t)$, $\sigma(t)$ and $\alpha : [t_0, \infty) \to R$ is continuous function. We define the function $u(t) = x(t) - p(t)\, x\big(\sigma(t)\big)$, so Eq. (1) can be written as

$$u'''(t) = -q(t)\, x\big(\tau(t)\big). \tag{1'}$$

The following lemma will be useful in the proof of the achieved result.

Lemma 1. [7] *Assume that*

$$g : [t_0, \infty) \to [0, \infty), \quad \delta : [t_0, \infty) \to R \quad \textit{are continous}, \quad \lim_{t\to\infty} \delta(t) = \infty,$$

$$\liminf_{t\to\infty} \int_t^{\delta(t)} g(s)\, \mathrm{d}s > \frac{1}{e}\,,$$

where $\delta(t) > t$ for $t \ge t_0$. Then the functional inequality

$$\dot{x}(t) - g(t)\, x\big(\delta(t)\big) \ge 0, \qquad t \ge t_0,$$

cannot have an eventually positive solution, and

$$\dot{x}(t) - g(t)\, x\big(\delta(t)\big) \le 0, \qquad t \ge t_0,$$

cannot have an eventually negative solution.

Theorem 1. *Suppose that*

$$0 < p(t) \le p\,,\ q(t) \ge 0\,,\ \sigma(t) < \tau(t) < t,\ \alpha(t) > t,\ t \ge t_0,\ \alpha : [t_0,\, \infty) \to R$$

is a continuous function and

(i) $$\limsup_{t\to\infty}\ \frac{1}{2} \int_t^{\tau^{-1}(t)} (\xi - t)^2 q(\xi)\, \mathrm{d}\xi > 1,$$

(ii) $$\liminf_{t\to\infty} \int_t^{\sigma^{-1}(\tau(t))} \int_v^{\alpha(v)} (\xi - v) \frac{q(\xi)}{p\big(\sigma^{-1}(\tau(\xi))\big)}\, \mathrm{d}\xi\, \mathrm{d}v > \frac{1}{e}.$$

Then every nonoscillatory solution $x(t)$ of (1) *has the property* $\lim_{t\to\infty} x(t) = \pm\infty$.

P r o o f. Assume for the sake of contradiction that $x(t)$ is an eventually positive solution of (1), then $u'''(t) \le 0$ for all large t. Then for sufficiently large t_0 we can consider two cases

1. $u''(t) < 0$,

2. $u''(t) > 0, \quad t \geq t_1 \geq t_0$.

Case **1.** If $u''(t) < 0$, then $u'(t) < 0$ and $u(t) < 0, t \geq t_2 \geq t_1$, for sufficiently large t_2. This implies $\lim_{t\to\infty} u(t) = -\infty$. For definition $u(t)$ we have $u(t) > -p(t)\, x(\sigma(t)) \geq -p\, x\big(\sigma(t)\big)$, this implies $\lim_{t\to\infty} x(t) = \infty$.

Case **2.** If $u''(t) > 0\, t \geq t_1$. In this case we have two possibilities:

(**A**) $u'(t) > 0, \quad t \geq t_2 \geq t_1,$ or (**B**) $u'(t) < 0, \quad t \geq t_2 \geq t_1$.

(**A**) If $u'(t) > 0$, then $u(t) > 0$, $t \geq t_3 \geq t_2$, for sufficiently large t_3. Since $u(t) < x(t)$ is $\lim_{t\to\infty} x(t) = \infty$.

(**B**) If $u'(t) < 0$, we get again two possibilities:

($\mathbf{B_1}$) $u(t) > 0, \quad t \geq t_3 \geq t_2,$ or ($\mathbf{B_2}$) $u(t) < 0, \quad t \geq t_3 \geq t_2$.

($\mathbf{B_1}$) From equality

$$u(t) = \sum_{i=0}^{2} (-1)^i \frac{(s-t)^i}{i\,!}\, u^{(i)}(s) - \frac{1}{2} \int_t^s (\xi - t)^2\, u'''(\xi)\, \mathrm{d}\xi, \qquad s > t,$$

then with regard to

$$\sum_{i=0}^{2} (-1)^i \frac{(s-t)^i}{i\,!}\, u^{(i)}(s) > 0,$$

we get

$$\begin{aligned} u(t) \quad &\geq \quad -\frac{1}{2} \int_t^s (\xi - t)^2\, u'''(\xi)\, \mathrm{d}\xi = \frac{1}{2} \int_t^s (\xi - t)^2\, q(\xi)\, x\big(\tau(\xi)\big)\, \mathrm{d}\xi \\ &\geq \quad \frac{1}{2} \int_t^s (\xi - t)^2\, q(\xi)\, u\big(\tau(\xi)\big)\, \mathrm{d}\xi. \end{aligned}$$

Then for $s = \tau^{-1}(t)$ we have

$$u(t) \geq \frac{1}{2} \int_t^{\tau^{-1}(t)} (\xi - t)^2\, q(\xi)\, u\big(\tau(\xi)\big)\, \mathrm{d}\xi \geq \frac{1}{2}\, u(t) \int_t^{\tau^{-1}(t)} (\xi - t)^2\, q(\xi)\, \mathrm{d}\xi,$$

which is a contradiction with the assumption (i).

($\mathbf{B_2}$) In this case $u'''(t) < 0$, $u''(t) > 0$, $u'(t) < 0$, $u(t) < 0$, for $t \ge t_3 \ge t_2$. From equality

$$u'(t) = u'(s) - (s-t)\,u''(s) + \int_t^s (\xi - t)\,u'''(\xi)\,\mathrm{d}\xi, \qquad t < s$$

we obtain

$$u'(t) \le \int_t^s (\xi - t)\,u'''(\xi)\,\mathrm{d}\xi.$$

Since for $u < 0$, $t \ge t_3 \ge t_2$ we have

$$u(t) \ge -p(t)\,x\big(\sigma(t)\big),\ -\frac{u\big(\sigma^{-1}(\tau(t))\big)}{p\big(\sigma^{-1}(\tau(t))\big)} \le x\big(\tau(t)\big),$$

and from the equality $u'''(t) = -q(t)\,x\big(\tau(t)\big)$ we get

$$u'(t) \le \int_t^s (\xi - t)\,\frac{q(\xi)}{p\big(\sigma^{-1}\big(\tau(\xi)\big)\big)}\,u\big(\sigma^{-1}(\tau(\xi))\big)\mathrm{d}\xi.$$

Let $s = \alpha(t)$, then

$$u'(t) \le \int_t^{\alpha(t)} (\xi - t)\,\frac{q(\xi)}{p\big(\sigma^{-1}(\tau(\xi))\big)}\,u\big(\sigma^{-1}(\tau(\xi))\big)\,\mathrm{d}\xi.$$

Function $u(t)$ is decreasing, we claim that

$$u'(t) \le u\big(\sigma^{-1}(\tau(t))\big) \int_t^{\alpha(t)} (\xi - t)\,\frac{q(\xi)}{p\big(\sigma^{-1}(\tau(\xi))\big)}\,\mathrm{d}\xi,$$

then

$$u'(t) - \left(\int_t^{\alpha(t)} (\xi - t)\,\frac{q(\xi)}{p\big(\sigma^{-1}(\tau(\xi))\big)}\,\mathrm{d}\xi \right) u\big(\sigma^{-1}\big(\tau(t)\big) \le 0.$$

By Lemma 1 and condition (ii) the last inequality cannot have an eventually negative solution, which is a contradiction. The proof is complete. □

Example 1.
Let

$$p(t) = \frac{37}{162}, \quad q(t) = 1, \quad \sigma(t) = \frac{1}{2}t, \quad \tau(t) = \frac{2}{3}t, \quad \alpha(t) = 2t, \quad t \ge 1.$$

Then

$$\tau^{-1}(t)=\frac{3}{2}t,\ \sigma^{-1}(t)=2t, \quad \text{and} \quad \sigma^{-1}\big(\tau(t)\big)=\sigma^{-1}\left(\frac{2}{3}t\right)=\frac{4}{3}t.$$

We have

$$\lim_{t\to\infty}\frac{1}{2}\int_{t}^{3t/2}(\xi-t)^2\,\mathrm{d}\xi=\lim_{t\to\infty}\frac{1}{2}\int_{0}^{t/2}s^2\,\mathrm{d}s=\lim_{t\to\infty}\frac{1}{48}\,t^3=\infty.$$

Then

$$\limsup_{t\to\infty}\frac{1}{2}\int_{t}^{3t/2}(\xi-t)^2\,\mathrm{d}\xi=\infty.$$

For the condition (ii) we get

$$\lim_{t\to\infty}\int_{t}^{4t/3}\int_{v}^{2v}(\xi-v)\,\mathrm{d}\xi\,\mathrm{d}v=\lim_{t\to\infty}\frac{1}{2}\int_{t}^{4t/3}v^2\,\mathrm{d}v=\lim_{t\to\infty}t^3=\infty.$$

So we have

$$\liminf_{t\to\infty}\int_{t}^{4t/3}\int_{v}^{2v}(\xi-v)\,\mathrm{d}\xi\,\mathrm{d}v=\infty.$$

All conditions of Theorem 1 are satisfied. Then every nonoscillatory solution $x(t)$ of (1) has the property $\lim\limits_{t\to\infty}x(t)=\pm\infty$.

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Received November 30, 2006

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