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A NOTE ON THE ASYMPTOTIC PROPERTIES OF NEUTRAL DIFFERENTIAL SYSTEMS

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ABSTRACT. The behaviour of solutions to functional differential systems of neutral type is studied. Sufficient conditions for solutions either to be oscillatory or to decay to zero are found.

1. Introduction

We consider neutral functional differential systems

$$\begin{aligned} \big[y_1(t) - a(t)y_1(g(t))\big]' &= p_1(t)\, y_2(t), \\ y_i'(t) &= p_i(t)\, y_{i+1}(t), \qquad i = 2, 3, \dots, n-1, \\ y_n'(t) &= p_n(t)\, f\big(y_1(h(t))\big), \quad t \ge t_0, \end{aligned} \tag{1.1}$$

where $n \ge 3$, n is odd and the following conditions are assumed:

(a) $a\colon [t_0, \infty) \to (0, \lambda^*]$ is a continuous function, where $\lambda^* < 1$ is a constant;

(b) $g\colon [t_0, \infty) \to \mathbb{R}$ is a continuous and increasing function, $\lim_{t\to\infty} g(t) = \infty$, $g(t) < t$ for $t \ge t_0$;

(c) $p_i\colon [t_0, \infty) \to [0, \infty)$, $i = 1, 2, \dots, n$ are continuous functions; p_n not identically equal to zero in any neighbourhood of infinity, $\int^{\infty} p_j(t)\, \mathrm{d}t = \infty$, $j = 1, 2, \dots, n-1$;

(d) $h\colon [t_0, \infty) \to \mathbb{R}$ is a continuous and increasing function, $\lim_{t\to\infty} h(t) = \infty$;

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(e) $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function, $uf(u) > 0$ for $u \neq 0$ and $|f(u)| \geq K|u|$, where K is a positive constant.

For $t_1 \geq t_0$, we define

$$\widetilde{t_1} = \min\bigl\{g(t_1), h(t_1)\bigr\}\,.$$

A function $y = (y_1, \dots, y_n)$, is a solution of the system (1.1) if there exists a $t_1 \geq t_0$ such that y is continuous on $[\widetilde{t_1}, \infty)$, $y_1(t) - a(t)y_1\bigl(g(t)\bigr)$, $y_i(t)$, $i = 2, 3, \dots, n$ are continuously differentiable on $[t_1, \infty)$ and y satisfies (1.1) on $[t_1, \infty)$. Denote by W the set of all solutions $y = (y_1, \dots, y_n)$ of the system (1.1) which exist on some ray $[T_y, \infty) \subset [t_0, \infty)$ and satisfy

$$\sup\biggl\{\sum_{i=1}^{n} |y_i(t)| : t \geq T\biggr\} > 0 \qquad \text{for any} \quad T \geq T_y\,.$$

A solution $y \in W$ is considered to be non-oscillatory if there exists a $T_y \geq t_0$ such that every component is different from zero for $t \geq T_y$. Otherwise a solution $y \in W$ is said to be oscillatory.

The purpose of this article is to study asymptotic properties of solutions to the n-dimensional functional differential systems of neutral type (1.1). The asymptotic and oscillatory properties of solutions to neutral differential systems has been studied for example in the papers [1], [3], [4], [6], [7].

For a $y_1(t)$, we define

$$z_1(t) = y_1(t) - a(t)y_1\bigl(g(t)\bigr). \tag{1.2}$$

Let

$$i_k \in \{1, 2, \dots, n-1\},\ k = 1, 2, \dots, n-1 \quad \text{and} \quad t, u \in [t_0, \infty)\,.$$

We define

$$\begin{aligned} I_0(u,t) &= 1, \\ I_k(u, t; p_{i_1}, p_{i_2}, \dots, p_{i_k}) &= \int_t^u p_{i_1}(x) I_{k-1}(x, t; p_{i_2}, p_{i_3}, \dots, p_{i_k})\,\mathrm{d}x\,, \qquad (1.3) \\ & \qquad\qquad k = 1, 2, \dots, n-1. \end{aligned}$$

2. Classification of non-oscillatory solutions

Lemma 2.1 ([3, Lemma 1])**.** *Let $y \in W$ be a solution of* (1.1) *with $y_1(t) \neq 0$ on $[t_1, \infty)$, $t_1 \geq t_0$. Then y is non-oscillatory and $z_1(t), y_2(t), \dots, y_n(t)$ are monotone on some ray $[T, \infty)$, $T \geq t_1$.*

Let $y \in W$ be a non-oscillatory solution of (1.1). By (1.1) and (c) it follows that the function $z_1(t)$ from (1.2) has to be eventually of constant sign, so that either

$$y_1(t)z_1(t) > 0 \tag{2.1}$$

or

$$y_1(t)z_1(t) < 0 \tag{2.2}$$

for sufficiently large t. By [3, Lemma 4] it follows the statement in Lemma 2.2.

Lemma 2.2. *Let $y = (y_1, \dots, y_n) \in W$ be a non-oscillatory solution of* (1.1) *on $[t_1, \infty)$. Then there exists an integer $l \in \{2, 4, \dots, n-1, n\}$ and a $t_2 \ge t_1$ such that for $t \ge t_2$ either*

$$\begin{aligned} y_i(t)z_1(t) &> 0 \qquad i = 1, 2, \dots, l\,, \\ (-1)^i y_i(t)z_1(t) &> 0 \qquad i = l+1, \dots, n\,, \end{aligned} \tag{2.3}$$

or

$$\begin{aligned} y_1(t)z_1(t) &< 0, \\ (-1)^i y_i(t)z_1(t) &< 0 \qquad i = 2, \dots, n\,. \end{aligned} \tag{2.4}$$

Denote by N_l^+ the set of non-oscillatory solutions of (1.1) satisfying (2.3), and by N_1^- the set of non-oscillatory solutions of (1.1) satisfying (2.4). Denote by N the set of all non-oscillatory solutions of (1.1). Then by Lemma 2.2 we have

$$N = N_2^+ \cup N_4^+ \cup \dots \cup N_{n-1}^+ \cup N_n^+ \cup N_1^-\,. \tag{2.5}$$

The next lemma can be proved similarly as Lemma 2 in [3].

Lemma 2.3. *Let $y = (y_1, y_2, \dots, y_n) \in W$ be a non-oscillatory solution of* (1.1) *on $[t_1, \infty)$, $t_1 \ge t_0$ and let $\lim\limits_{t\to\infty} |z_1(t)| = L_1$, $\lim\limits_{t\to\infty} |y_k(t)| = L_k$, $k = 2, \dots, n$. Then*

$$L_1 < \infty \quad \Longrightarrow \quad L_k = 0, \quad k = 2, \dots, n\,.$$

Lemma 2.4. *Let $y = (y_1, y_2, \dots, y_n) \in W$ be a non-oscillatory solution of* (1.1) *on $[t_1, \infty)$ with $y_1(t)$ bounded on $[t_1, \infty)$, $t_1 > t_0$. Then*

$$y \in N_2^+ \cup N_1^-\,.$$

P r o o f. According to (2.5) it is enough to prove that

$$N_4^+ \cup \dots \cup N_{n-1}^+ \cup N_n^+$$

is an empty set. We can prove it indirectly. Let $y = (y_1, y_2, \dots, y_n) \in W$ be a non-oscillatory solution of (1.1) on $[t_1, \infty)$ with $y_1(t)$ bounded and

$$y \in N_4^+ \cup \dots \cup N_{n-1}^+ \cup N_n^+\,.$$

Without loss of generality we may suppose that $y_1(t) > 0$ on $[t_1, \infty)$. Because $a(t)$ and $y_1(t)$ are bounded $z_1(t)$ is bounded on $[t_2, \infty)$, where $t_2 \geq t_1$ is sufficiently large. If

$$y \in N_4^+ \cup \cdots \cup N_{n-1}^+ \cup N_n^+ ,$$

then a function $y_2(t)$ is positive and nondecreasing and $y_2(t) \geq M,\ 0 < M = const.$ for $t \geq t_2$.

Integrating the first equation of (1.1) from s to t we get

$$z_1(t) - z_1(s) \geq M \int_s^t p_1(u)\,\mathrm{d}u\,, \qquad t > s \geq t_2\,. \tag{2.6}$$

From (2.6) and (c) we have $\lim\limits_{t\to\infty} z_1(t) = \infty$. This contradicts the fact that $z_1(t)$ is bounded and

$$N_4^+ \cup \cdots \cup N_{n-1}^+ \cup N_n^+ = \emptyset\,.$$

The proof is complete. □

Lemma 2.5 ([5, Lemma 4]). *Assume that* $q\colon [t_0, \infty) \to [0, \infty)$ *and* $\delta\colon [t_0, \infty) \to \mathbb{R}$ *are continuous functions,* $\lim\limits_{t\to\infty} \delta(t) = \infty$, $\ \delta(t) > t$ *for* $t \geq t_0$, *and*

$$\liminf_{t\to\infty} \int_t^{\delta(t)} q(s)\,\mathrm{d}s > \frac{1}{e}.$$

Then the functional inequality

$$x'(t) - q(t)x\big(\delta(t)\big) \geq 0\,, \qquad t \geq t_0$$

has no eventually positive solution, and

$$x'(t) - q(t)x\big(\delta(t)\big) \leq 0\,, \qquad t \geq t_0$$

has no eventually negative solution.

Lemma 2.6 ([2, Lemma 2.1]). *In addition to the conditions (a) and (b) suppose that*

$$a(t) \leq 1 \qquad \text{for} \quad t \geq t_0\,.$$

Let $y_1(t)$ *be a continuous non-oscillatory solution of the functional inequality*

$$y_1(t)\big[y_1(t) - a(t)y_1(g(t))\big] < 0$$

defined in a neighbourhood of infinity.

Suppose that $g(t) < t$ *for* $t \geq t_0$. *Then* $y_1(t)$ *is bounded. If, moreover,*

$$a(t) \leq \lambda^\star < 1, \qquad t \geq t_0$$

for some positive constant $\lambda^\star$, *then* $\lim\limits_{t\to\infty} y_1(t) = 0$.

3. Oscillation theorem

Theorem 3.1. *Suppose that*

$$t < h(t) \qquad \text{for} \;\; t \ge t_0\,, \tag{3.1}$$

$$\liminf_{t\to\infty} \int\limits_t^{h(t)} Kp_1(v) \int\limits_v^{\alpha(v)} I_{n-2}(u, v; p_{n-1}, p_{n-2}, \dots, p_2)\, p_n(u)\, \mathrm{d}u\, \mathrm{d}v > \frac{1}{e}\,, \tag{3.2}$$

where $\alpha\colon [t_0, \infty) \to \mathbb{R}$ *is a continuous function,* $t < \alpha(t)$.
Then every solution $y \in W$ *of* (1.1) *with* $y_1(t)$ *bounded is oscillatory or* $\lim\limits_{t\to\infty} y_i(t) = 0$, $i = 1, 2, \dots, n$.

P r o o f. Let $y \in W$ be a non-oscillatory solution of (1.1) with $y_1(t)$ bounded. By Lemma 2.4 we have $y \in N_2^+ \cup N_1^-$ on $[t_2, \infty)$. Without loss of generality we may suppose that $y_1(t)$ is positive for $t \ge t_2$.

(I) Let $y \in N_2^+$ on $[t_2, \infty)$. In this case

$$y_1(t) > 0,\;\; z_1(t) > 0,\;\; y_2(t) > 0,\;\; y_3(t) < 0,\;\; y_4(t) > 0, \dots,\;\; y_n(t) < 0 \qquad \text{for} \quad t \ge t_2\,. \tag{3.3}$$

Integrating $\int\limits_t^s I_{n-2}(u, t; p_{n-1}, p_{n-2}, \dots, p_2) y_n'(u)\, \mathrm{d}u$ by parts we derive the integral identity

$$\begin{aligned} y_2(t) = y_2(s) - I_1(s, t; p_2) y_3(s) + I_2(s, t; p_3, p_2) y_4(s) - \dots \\ \dots - I_{n-2}(s, t; p_{n-1}, p_{n-2}, \dots, p_2) y_n(s) \\ + \int\limits_t^s I_{n-2}(u, t; p_{n-1}, p_{n-2}, \dots, p_2) y_n'(u)\, \mathrm{d}u\,, \qquad s > t \ge t_2\,. \end{aligned} \tag{3.4}$$

By (3.4) with regard to (3.3) and the last equation of (1.1) we get

$$y_2(t) \ge \int\limits_t^s I_{n-2}(u, t; p_{n-1}, p_{n-2}, \dots, p_2) p_n(u) f\big(y_1(h(u))\big)\, \mathrm{d}u\,, \qquad s > t \ge t_2\,. \tag{3.5}$$

Because $z_1(t) \le y_1(t)$ for $t \ge t_2$ it follows

$$z_1\big(h(t)\big) \le y_1\big(h(t)\big) \qquad \text{for} \quad t \ge t_2. \tag{3.6}$$

Multiplying (3.5) by $p_1(t)$ and then using (e) and (3.6) we have

$$p_1(t)y_2(t) \geq K p_1(t) \int_t^s I_{n-2}(u,t;p_{n-1},p_{n-2},\dots,p_2)p_n(u)z_1\big(h(u)\big)\,\mathrm{d}u\,, \qquad s > t \geq t_2\,.$$

Using the first equation of (1.1) and the monotonicity of $z_1\big(h(t)\big)$ by the last inequality we get

$$z_1'(t) \geq \left[K p_1(t) \int_t^s I_{n-2}(u,t;p_{n-1},p_{n-2},\dots,p_2)p_n(u)\,\mathrm{d}u\right] z_1\big(h(t)\big), \qquad s > t \geq t_2\,.$$

Let $s = \alpha(t)$ and so

$$z_1'(t) - \left[K p_1(t) \int_t^{\alpha(t)} I_{n-2}(u,t;p_{n-1},p_{n-2},\dots,p_2)p_n(u)\,\mathrm{d}u\right] z_1\big(h(t)\big) \geq 0\,, \qquad t \geq t_2\,. \tag{3.7}$$

By Lemma 2.5 and the condition (3.2) the last inequality has no eventually positive solution, which is a contradiction and $N_2^+ = \emptyset$.

(II) Let $y \in N_1^-$ on $[t_2,\infty)$. In this case

$$y_1(t) > 0,\ z_1(t) < 0,\ y_2(t) > 0,\ y_3(t) < 0,\ y_4(t) > 0,\ \dots,\ y_n(t) < 0 \qquad \text{for} \quad t \geq t_2\,. \tag{3.8}$$

By Lemma 2.6 and Lemma 2.3 it follows that $\lim\limits_{t\to\infty} y_i(t) = 0$, $i = 1,2,\dots,n$. The proof is complete. □

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