Tatra Mt. Math. Publ. **38** (2007), 243–254

TATRA MOUNTAINS
Mathematical Publications

ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A CLASS OF FOURTH ORDER NONLINEAR NEUTRAL DIFFERENCE EQUATIONS WITH QUASIDIFFERENCES

Ewa Schmeidel — Jędrzej Schmeidel

ABSTRACT. We consider a class of fourth order nonlinear neutral difference equations with quasidifferences. We will give properties of nonoscillatory solutions of this equation. Necessary and sufficient conditions are obtained for the difference equation to admit the existence of minimal and maximal solutions.

1. Introduction

Consider the difference equation

$$\Delta\Big(a_n^{(3)}\Delta\big(a_n^{(2)}\Delta\big(a_n^{(1)}\Delta(y_n+p_ny_{n-\tau})\big)\big)\Big)+f(n,y_n)=0, \qquad n\in\mathbb{N}_\tau, \qquad (E)$$

where $\mathbb{N}_\tau=\{1+\tau,2+\tau,\dots\}$, Δ is the forward difference operator defined by $\Delta x_n=x_{n+1}-x_n$, $(a_n^{(i)})$, $i=1,2,3$, (p_n) are sequences of positive real numbers and τ is a positive integer. The function $f\colon\mathbb{N}\times\mathbb{R}\to\mathbb{R}$. By a solution of equation (E) we mean a sequence (y_n) which satisfies equation (E) for n sufficiently large. We consider only such solutions which are nontrivial for all large n. A solution of equation (E) is called nonoscillatory if it is eventually positive or eventually negative. Otherwise it is called oscillatory.

Asymptotic behavior of solutions of neutral difference equations has been intensively studied in recent years (see, for example, [7] and [13]).

In the last few years there has been an increasing interest in the study of oscillatory and asymptotic behavior of solutions of fourth order difference equations (see, for example, [2], [3], [6], [8]–[12] and [14]) and the references cited

2000 Mathematics Subject Classification: 39A10.
Keywords: nonlinear, neutral, difference equation, minimal solution, maximal solution, nonoscillatory solution, quasidifferences, fourth order.

therein. The results in this paper have been motivated by the results obtained by Migda, Musielak and Schmeidel in [6].

The purpose of this paper is to establish some necessary and sufficient conditions for the existence of solutions of equation (E) with special asymptotic properties.

Denote quasidifferences $L_i(x_n)$, where $i = 0, 1, 2, 3, 4$, of a sequence (x_n) as follows $L_0(x_n) = x_n$, $L_i(x_n) = a_n^{(i)}\Delta\big(L_{i-1}(x_n)\big)$, for $i = 1, 2, 3, 4$, where $a_n^{(4)} \equiv 1$. Hence equation (E) takes the form

$$L_4(z_n) + f(n, y_n) = 0, n \in \mathbb{N}_\tau, \tag{1}$$

where (z_n) is a sequence defined by

$$z_n = y_n + p_n y_{n-\tau}. \tag{2}$$

We introduce the notation

$$H_0(n) = 1, H_i(n) = (-1)^i \prod_{k=0}^{i-1} p_{n-k\tau},$$

and

$$P_{n,N} = \sum_{k=N+2}^{n} \frac{1}{a_k^{(3)}} \sum_{j=N+2}^{k} \frac{1}{a_j^{(2)}} \sum_{i=N+2}^{j} \frac{1}{a_i^{(1)}},$$

$$Q_{n,N} = \sum_{k=N}^{n-1} \frac{1}{a_k^{(1)}} \sum_{j=N}^{k-1} \frac{1}{a_j^{(2)}} \sum_{i=N}^{j-1} \frac{1}{a_i^{(3)}} = \sum_{i=N}^{n-1} \frac{1}{a_i^{(3)}} \sum_{j=i+1}^{n-1} \frac{1}{a_j^{(2)}} \sum_{k=j+1}^{n-1} \frac{1}{a_k^{(1)}}.$$

Throughout the rest of our investigations, one or several of the following assumptions will be imposed:

$(H1)$ $\quad \sum_{j=1}^{\infty} \frac{1}{a_j^{(i)}} = \infty, \quad$ where $i = 1, 2, 3,$

$(H2)$ $\quad yf(n, y) > 0 \quad$ for all $\quad y \neq 0 \quad$ and $\quad n \in \mathbb{N},$

$(H3)$ $\quad \lim_{n\to\infty} p_n = 0\,.$

2. Classification of nonoscillatory solutions

The following lemma, presented by M. Migda and J. Migda in [5], will be used.

Lemma 1. *Let $y, p\colon \mathbb{N} \to \mathbb{R}$ be such that*

$$z_n = y_n + p_n y_{n+k}, n \geq \max\{0, -k\},$$

where k is an integer. Assume that (y_n) is bounded, $\lim\limits_{n\to\infty} z_n = l \in \mathbb{R}$ and $\lim\limits_{n\to\infty} p_n = p \in \mathbb{R}$. Then the following statements hold:

(i) *if $p = 1$, then $l = 0$:*

(ii) *if $|p| \neq 1$, then (y_n) is convergent and $\lim\limits_{n\to\infty} x_n = \frac{l}{1-p}$.*

Next, we prove three following lemmas.

Lemma 2. *Assume that $(H1)$ and $(H2)$ hold. Let (y_n) be nonoscillatory solution of equation (1) where the sequence (z_n) is defined by (2). If*

$$L_i(z_n)L_{i+1}(z_n) > 0,$$

for some $i = 1, 2, 3$ and sufficiently large n, then $L_{i-1}(z_n)L_i(z_n) > 0$ eventually.

Proof. Let (y_n) be a nonoscillatory solution of equation (1). Hence (z_n) is eventually of one sign. Therefore, for $i = 0, 1, 2, 3$, the sequences $\big(L_i(z_n)\big)$ are monotonic and of one sign for large n, say for $n \geqslant N_0$. Without loss of generality we may assume that $L_i(z_n) < 0$ and $L_{i+1}(z_n) < 0$. Then $L_i(z_{N_0}) = -c_1 < 0$. Hence, by the fact that $L_i(z_n)$ decreases as n increases, we get $L_i(z_n) < -c_1$, for $n \geqslant N_0$, and

$$\frac{L_i(z_n)}{a_n^{(i)}} < -\frac{c_1}{a_n^{(i)}}, \qquad \text{for} \quad n \geqslant N_0.$$

Summing both sides of the above inequality from N_0 to $n-1$, we obtain

$$L_{i-1}(z_n) - L_{i-1}(z_{N_0}) < -\sum_{j=N_0}^{n-1} \frac{c_1}{a_j^{(i)}}.$$

Hence, by $(H1)$, $L_{i-1}(z_n) \to -\infty$ as $n \to \infty$.
Then the sequence $\big(L_{i-1}(z_n)\big)$ is eventually negative. □

The next lemma gives us the classification of nonoscillatory solutions of equation (1).

Lemma 3. *Assume that $(H1)$ and $(H2)$ hold. Let (y_n) be an eventually positive solution of equation (1). Then the sequence (z_n), defined by (2), is also eventually positive and exactly one of the following statements holds:*

$$\text{(i)} \qquad L_1(z_n) > 0, \qquad L_2(z_n) > 0, \qquad L_3(z_n) > 0,$$

$$\text{(ii)} \qquad L_1(z_n) > 0, \qquad L_2(z_n) < 0, \qquad L_3(z_n) > 0,$$

for sufficiently large n.

P r o o f. Assume that (y_n) is an eventually positive solution of equation (1). From $(H2)$, we have

$$L_4(z_n) < 0,$$

for sufficiently large n.
By (2), (z_n) is an eventually positive sequence. Suppose $L_3(z_n) < 0$ for large n. Using Lemma 2, this implies that $z_n < 0$. From this contradiction we get $L_3(z_n) > 0$ for large n. Then $L_2(z_n) > 0$ or $L_2(z_n) < 0$. If $L_2(z_n) > 0$ then, by Lemma 1, we get $L_1(z_n) > 0$, eventually. If $L_2(z_n) < 0$ then, by Lemma 1, it is clear that $L_1(z_n) > 0$. Hence the proof is completed. □

We note that positivity of the sequence (z_n) does not imply positivity of the sequence (y_n). For example, let $z_n = y_n + y_{n-1}$, where $y_n = \frac{1}{2} + (-1)^n$, $n > 1$. Here $z_n \equiv 1$ is a positive sequence but (y_n) is oscillatory.

From Lemma 3, we see that sequence (z_n) is monotone, but we know nothing about monotonicity of sequence (y_n).

Lemma 4. *Assume that conditions $(H1)$, $(H2)$ and $(H3)$ hold. If (y_n) is an eventually positive solution of equation (1) then there exist positive constants c_1 and c_2 and an integer N such that*

$$c_1 \leqslant y_n < c_2 Q_{n,N}, \qquad \textit{for} \quad n \geqslant N+3+\tau.$$

P r o o f. Using Lemma 3 and following an argument similar to the one used in Lemma 2.5 in [6], we obtain the existence of two positive constants c_3 and c_4 and a positive integer N such that $y_n > 0$, for $n \geqslant N+3$ and

$$\Delta z_n > 0 \quad \text{and} \quad c_3 \leqslant z_n \leqslant c_4 Q_{n,N}, \qquad \text{for} \quad n \geqslant N+3+\tau.$$

In the first place, we shall prove the existence of $c_1 > 0$ such that $y_n \geqslant c_1$, for $n \geqslant N+3+\tau$. Suppose not, and use that $y_n > 0$ for $n \geqslant N+3+\tau$ to obtain that $\liminf_{n\to\infty} y_n = 0$. Obtain from this that $\liminf_{n\to\infty} z_n = 0$, using $(H3)$ and the fact that $\Delta z_n > 0$ for $n \geqslant N+3+\tau$. This contradicts that $z_n \geqslant c_3$ for $n \geqslant N+3+\tau$. Taking $c_2 = c_4$, the other inequality follows easily from the following relations, valid for $n \geqslant N+3+\tau$

$$y_{n-\tau} > 0, p_n > 0 \quad \text{and} \quad z_n \leqslant c_4 Q_{n,N}.$$

□

We say that nonoscillatory solution (y_n) of equation (1) is asymptotically constant if there exists some positive constant α such that $\lim_{n\to\infty} y_n = \alpha$ and asymptotically $Q_{n,N}$ if there exists some positive constant β such that $\lim_{n\to\infty} \frac{y_n}{Q_{n,N}} = \beta$. According to Lemma 4 we may regard an asymptotically constant solution as a "minimal" solution, and asymptotically $Q_{n,N}$ solution as a "maximal" solution.

3. Necessary and sufficient condition

In [14] we can find the following existence theorem (some modification of Schauder's theorem) which will be used in this section.

Lemma 5. *Suppose Ω is a Banach space and K is a closed, bounded and convex subset. Suppose T is a continuous mapping such that $T(K)$ is contained in K, and suppose that $T(K)$ is uniformly Cauchy. Then T has a fixed point in K.*

Now, we present a necessary and a sufficient condition for the existence of asymptotically $Q_{n,N}$ solution.

Theorem 1. *Assume that conditions $(H1)$, $(H2)$ and $(H3)$ hold and f is a continuous and nondecreasing function in the second argument. Then a necessary and sufficient condition for equation (1) to have a solution (y_n) satisfying*

$$\lim_{n\to\infty} \frac{y_n}{Q_{n,N}} = \beta \neq 0 \tag{3}$$

is that

$$\sum_{n=1}^{\infty} |f(n, CQ_{n,N})| < \infty, \tag{4}$$

for some integer $N \geqslant 1$ and some nonzero constant C.

P r o o f. Necessity. Let (y_n) be a nonoscillatory solution of equation (1) which satisfies (3) for some positive integer N. Without loss of generality, we may assume that $\beta > 0$. Since $Q_{n,N} > 0$, there exist positive numbers d_1 and d_2 and a positive integer k such that

$$d_1 Q_{n,N} \leqslant y_n \leqslant d_2 Q_{n,N}, \qquad n \geqslant k.$$

Now, use that f is nondecreasing in the second argument to obtain that

$$f(n, y_n) \geqslant f(n, d_1 Q_{n,N}). \tag{5}$$

On the other hand, summing equation (1) from k to $n-1$ and from Lemma 3, we get

$$0 < L_3(z_n) = L_3(z_k) - \sum_{i=k}^{n-1} f(i, y_i)$$

and

$$\sum_{i=k}^{\infty} f(i, y_n) \leqslant L_3(z_k) < \infty.$$

By (5), we have

$$\sum_{i=k}^{\infty} f(i, d_2 Q_{i,N}) < \infty.$$

Sufficiency. Assume that (4) holds with $C > 0$, a similar argument holds if $C < 0$. Since f is nondecreasing in the second argument and $Q_{n,K}$ decreases as K increases, there exists a positive integer which we keep denoting by N such that

$$\sum_{i=N-3-\tau}^{\infty} f(i, CQ_{n,N}) < \frac{1}{8}C \quad \text{and} \quad p_n < 1 \qquad \text{for} \quad n \geqslant N. \tag{6}$$

Consider the partially ordered Banach space B_N of all real sequences $z = (z_n)$ defined for $n \geqslant N + 3$ such that

$$\|z\| = \sup_{n \geqslant N+3} \frac{|z_n|}{Q^2_{n,N}} < \infty.$$

Let S be the subset of B_N defined by

$$S = \left\{ (z_n) \in B_N : \frac{C}{2} Q_{n,N} \leqslant z_n \leqslant CQ_{n,N},\ n \geqslant N + 3 + \tau \right\}.$$

It is not difficult to see that S is a bounded, convex, and closed subset of B_N.
With each (z_n) we associate a sequence $y = (y_n)$ defined by

$$y_n = \sum_{i=0}^{s} H_i(n) z_{n-i\tau}, \qquad \text{for} \quad n > N, \tag{7}$$

where s is the least positive integer such that $n - s\tau \leqslant N$.
It is easy to verify that $z_n = y_n + p_n y_{n-\tau}$, for $n \geqslant N + \tau$.
Because condition $(H3)$ holds then $p_n < 1$, for sufficiently large n. From (7), we get

$$y_n > (z_n - z_{n-\tau}) + p_n p_{n-\tau}(z_{n-2\tau} - z_{n-3\tau}) + \cdots + p_n \cdots p_{N+\tau} z_N,$$

for $n - N$ even, and

$$y_n > (z_n - z_{n-\tau}) + p_n p_{n-\tau}(z_{n-2\tau} - z_{n-3\tau}) + \cdots + p_n \cdots p_{N+2\tau}(z_{N+\tau} - z_N),$$

for $n - N$ odd. From $\Delta z_n > 0$ we have

$$y_n > 0 \qquad\qquad \text{for} \quad n \geqslant N. \tag{8}$$

Hence

$$z_n > y_n \qquad\qquad \text{for} \quad n \geqslant N. \tag{9}$$

Let us define a mapping $T\colon S \to B_N$ as follows

$$
\begin{aligned}
(Tz)_n = {} & \frac{C}{2} Q_{n,N} + Q_{n,N} \sum_{i=n-1}^{\infty} f(i-2, y_{i-2}) \\
& + \sum_{j=N}^{n-1} f(j-3, y_{j-3}) Q_{j,N} \\
& + \sum_{i=N}^{n-1} \frac{1}{a_i^{(1)}} \sum_{j=N}^{i-1} f(j-2, y_{j-2}) \sum_{k=N}^{j-1} \frac{1}{a_k^{(2)}} \sum_{s=N}^{k-1} \frac{1}{a_s^{(3)}} \\
& + \sum_{i=N}^{n-1} \frac{1}{a_i^{(1)}} \sum_{j=N}^{i-1} \frac{1}{a_j^{(2)}} \sum_{k=N}^{j-1} f(k-1, y_{k-1}) \sum_{s=N}^{k-1} \frac{1}{a_s^{(3)}} \quad \text{for} \quad n \geqslant N+3+\tau,
\end{aligned}
$$

where the sequence $y = (y_n)$ is defined by (7). We first show that $T(S) \subset S$. Each $z \in S$ is eventually positive. From Lemma 3, we get $\Delta z_n > 0$. By (8), the sequence y is positive. Hence, by $(H2)$ we have $f(n, y_n) > 0$, too. It is clear that $(Tz)_n \geqslant \frac{C}{2} Q_{n,N}$ for $n \geqslant N+3+\tau$.
Furthermore, by (9) and properties of the function f, we have

$$
\begin{aligned}
(Tz)_n \leqslant {} & \frac{C}{2} Q_{n,N} + Q_{n,N} \sum_{i=n-1}^{\infty} f(i-2, y_{i-2}) \\
& + \sum_{j=N}^{n-1} f(j-3, y_{j-3}) Q_{j,N} \\
& + \sum_{i=N}^{n-1} \frac{1}{a_i^{(1)}} \sum_{j=N}^{i-1} f(j-2, y_{j-2}) \sum_{k=N}^{j-1} \frac{1}{a_k^{(2)}} \sum_{s=N}^{k-1} \frac{1}{a_s^{(3)}} \\
& + \sum_{i=N}^{n-1} \frac{1}{a_i^{(1)}} \sum_{j=N}^{i-1} \frac{1}{a_j^{(2)}} \sum_{k=N}^{j-1} f(k-1, y_{k-1}) \sum_{s=N}^{k-1} \frac{1}{a_s^{(3)}}.
\end{aligned}
$$

Then,

$$
(Tz)_n \leqslant \frac{C}{2} Q_{n,N} + 4 Q_{n,N} \sum_{i=N-3}^{\infty} f(i, y_i) \tag{10}
$$

and by (6)

$$
(Tz_n) \leqslant C Q_{n,N}.
$$

Thus T maps S into itself.

Next we prove that T is continuous. Let $\big(y^{(m)}\big)$ be a sequence in S such that $y^{(m)} \to y$ as $m \to \infty$. Since S is closed, $y \in S$. We get

$$|(Tz^{(m)})_n - (Tz)_n| \le 4Q_{n,N} \sum_{i=N-3-\tau}^{\infty} \big|f\big(i, y_i^{(m)}\big) - f(i, y_i)\big|, \qquad n \ge N+3+\tau$$

and therefore

$$\big|\big(Tz^{(m)}\big)_n - (Tz)_n\big| \le \frac{4}{Q_{n,N}} \sum_{i=N-3-\tau}^{\infty} \big|f\big(i, y_i^{(m)}\big) - f(i, y_i)\big|.$$

Since

$$\lim_{m\to\infty} \big|f\big(i, y_i^{(m)}\big) - f(i, y_i)\big| = 0$$

and

$$\begin{aligned}
\big|f\big(i, y_i^{(m)}\big) - f(i, y_i)\big| &\le \big|f\big(i, y_i^{(m)}\big)\big| + |f(i, y_i)| \\
&\le \big|f\big(i, z_i^{(m)}\big)\big| + |f(i, z_i)| \\
&\le 2f(i, CQ_{i,N}) \qquad\qquad \text{for} \quad i \ge N+3+\tau
\end{aligned}$$

we see from Lebesgue's dominated convergence theorem that

$$\lim_{m\to\infty} \|Tz^{(m)} - Tz\| = 0.$$

This means that T is continuous.

Finally, we need to show that $T(S)$ is uniformly Cauchy. To see this, we have to show that, given any $\epsilon > 0$, there exists an integer N_1 such that
for $m > n > N_1$

$$\left|\frac{(Tz)_m}{Q_{m,N}^2} - \frac{(Tz)_n}{Q_{n,N}^2}\right| < \epsilon$$

for any $z \in S$. Indeed, by (10) and (6) we have

$$\left|\frac{(Tz)_m}{Q_{m,N}^2} - \frac{(Tz)_n}{Q_{n,N}^2}\right| \le \frac{C}{Q_{n,N}} + \frac{8}{Q_{n,N}} \sum_{i=N-3-\tau}^{\infty} f(i, y_i) \le \frac{2C}{Q_{n,N}} \to 0.$$

Therefore, by Lemma 5, there exists $z \in S$ such that $z_n = (Tz)_n$, for $n \geqslant N+3+\tau$. It is easy to see, that (y_n) is a solution of equation (1). So,

$$\begin{aligned}
z_n &= y_n + p_n y_{n-1} \\
&= \frac{C}{2} Q_{n,N} + Q_{n,N} \sum_{i=n-1}^{\infty} f(i-2, y_{i-2}) \\
&\quad + \sum_{j=N}^{n-1} f(j-3, y_{j-3}) Q_{j,N} \\
&\quad + \sum_{i=N}^{n-1} \frac{1}{a_i^{(1)}} \sum_{j=N}^{i-1} f(j-2, y_{j-2}) \sum_{k=N}^{j-1} \frac{1}{a_k^{(2)}} \sum_{s=N}^{k-2} \frac{1}{a_s^{(3)}} \\
&\quad + \sum_{i=N}^{n-1} \frac{1}{a_i^{(1)}} \sum_{j=N}^{i-1} \frac{1}{a_i^{(2)}} \sum_{k=N}^{j-1} f(k-1, y_{k-1}) \sum_{s=N}^{k-1} \frac{1}{a_s^{(3)}}, \qquad \text{for} \quad n \geqslant N+3+\tau.
\end{aligned}$$

It is easy to verify that sequence (y_n) is a solution of equation (E), so

$$\begin{aligned}
\Delta z_n = \frac{C}{2} \Delta Q_{n,N} &+ \Delta Q_{n,N} \sum_{i=n}^{\infty} f(i-2, y_{i-2}) \\
&+ \frac{1}{a_n^{(1)}} \sum_{j=N}^{n-1} f(j-2, y_{j-2}) \sum_{k=N}^{j-1} \frac{1}{a_k^{(2)}} \sum_{s=N}^{k-1} \frac{1}{a_s^{(3)}} \\
&+ \frac{1}{a_n^{(1)}} \sum_{j=N}^{n-1} \frac{1}{a_j^{(2)}} \sum_{k=N}^{j-1} f(k-1, y_{k-1}) \sum_{s=N}^{k-1} \frac{1}{a_s^{(3)}},
\end{aligned}$$

$$\begin{aligned}
a_n^{(1)} \Delta z_n = \frac{C}{2} &\sum_{j=N}^{n-1} \frac{1}{a_j^{(2)}} \sum_{k=N}^{j-1} \frac{1}{a_k^{(3)}} \\
&+ \sum_{j=N}^{n-1} \frac{1}{a_j^{(2)}} \sum_{k=N}^{j-1} \frac{1}{a_k^{(3)}} \sum_{i=n}^{\infty} f(i-2, y_{i-2}) \\
&+ \sum_{j=N}^{n-1} f(j-2, y_{j-2}) \sum_{k=N}^{j-1} \frac{1}{a_k^{(2)}} \sum_{s=N}^{k-1} \frac{1}{a_s^{(3)}} \\
&+ \sum_{j=N}^{n-1} \frac{1}{a_j^{(2)}} \sum_{k=N}^{j-1} f(k-1, y_{k-1}) \sum_{s=N}^{k-1} \frac{1}{a_s^{(3)}}
\end{aligned}$$

and

$$a_n^{(2)}\Delta\big(a_n^{(1)}\Delta z_n\big) = \frac{C}{2}\sum_{k=N}^{n-1}\frac{1}{a_k^{(3)}} + \sum_{k=N}^{n-1}\frac{1}{a_k^{(3)}}\sum_{i=n+1}^{\infty} f(i-2, y_{i-2}) + \sum_{j=N}^{n-1} f(j-1, y_{j-1})\sum_{s=N}^{k-1}\frac{1}{a_s^{(3)}}.$$

Hence

$$\begin{aligned} a_n^{(3)}\Delta\big(a_n^{(2)}\Delta(a_n^{(1)}\Delta z_n)\big) &= \frac{C}{2} + \sum_{i=n+2}^{\infty} f(i-2, y_{i-2}) \\ &= \frac{C}{2} + \sum_{i=n}^{\infty} f(i, y_i) \end{aligned} \tag{11}$$

and

$$\Delta\big(a_n^{(3)}\Delta\big(a_n^{(2)}\Delta(a_n^{(1)}\Delta z_n)\big)\big) = -f(n, y_n).$$

Because f is nondecreasing, relation (11) together with (9) and (4) (supposing that $y_i > 0$) implies that $\lim\limits_{n\to\infty} L_3(z_n)$ exists. Furthermore, by Stolz's theorem, we have

$$\begin{aligned} \lim_{n\to\infty}\frac{z_n}{Q_{n,N}} &= \lim_{n\to\infty}\frac{\Delta z_n}{\Delta Q_{n,N}} = \lim_{n\to\infty}\frac{L_1(z)}{a_n^{(1)}\Delta Q_{n,N}} \\ &= \lim_{n\to\infty}\frac{L_2(z_n)}{a_n^{(2)}\Delta(a_n^{(1)}\Delta Q_{n,N})} \\ &= \lim_{n\to\infty}\frac{L_3(z_n)}{a_n^{(3)}\Delta\left(\sum\limits_{i=N}^{n-1}\frac{1}{a_i^{(3)}}\right)} \\ &= \lim_{n\to\infty} L_3(z_n). \end{aligned}$$

Moreover, using (11), we obtain

$$\lim_{n\to\infty} L_3(z_n) = \lim_{n\to\infty}\left(\frac{C}{2} + \sum_{i=n}^{\infty} f(i, y_i)\right) = \frac{C}{2}.$$

Hence

$$\lim_{n\to\infty}\frac{z_n}{Q_{n,N}} = \frac{C}{2}$$

also exists. By Lemma 1 we get

$$\lim_{n\to\infty}\frac{y_n}{Q_{n,N}} = \frac{C}{2}.$$

The proof of this theorem is completed. □

Next we present a necessary and sufficient condition for the existence of an asymptotically constant solution of equation (E).

Theorem 2. *Assume that conditions $(H1)$, $(H2)$ and $(H3)$ hold and the function f is continuous and monotonic in the second argument. A necessary and sufficient condition for equation (E) to have a solution (y_n) which satisfies*

$$\lim_{n\to\infty} y_n = \alpha \neq 0 \tag{12}$$

is that

$$\sum_{i=1}^{\infty} P_{i,N}|f(i,c)| < \infty, \tag{13}$$

for some integer $N \geqslant 1$ and some nonzero constant c.

Proof. Necessity. From (12) and (2), we have $\lim\limits_{n\to\infty} z_n = \alpha \neq 0$. Using arguments similar to the ones used in the proof of necessity of Theorem 2.8 [6], we obtain (13).

Sufficiency. Using arguments similar to those used in the proof of sufficiency of Theorem 2.8 [6], we obtain $\lim\limits_{n\to\infty} z_n = \alpha \neq 0$. Hence, by Lemma 1 and $(H3)$, we get (12). □

REFERENCES

[1] AGARWAL, R. P.: *Difference Equations and Inequalities. Theory, Methods and Applications*, Pure and Applied Mathematics, Marcel Dekker, New York, 1992.

[2] CHENG, S. S.: *On a class of fourth order linear recurrence equations*, Int. J. Math. Math. Sci. **71** (1984), 131–149.

[3] GRAEF, J. R.—THANDAPANI, E.: *Oscillatory and asymptotic behavior of fourth order nonlinear delay difference equations*, Fasc. Math. **31** (2001), 23–36.

[4] KELLEY, W. G.—PETERSON, A. C.: *Difference Equations: An Introduction with Applications*, Academic Press, Inc., Boston, 1991.

[5] MIGDA, M.—MIGDA, J.: *Asymptotic properties of solutions of second-order neutral difference equations*, Nonlinear Anal. 63 (2005), e789–e799.

[6] MIGDA, M.—MUSIELAK, A.—SCHMEIDEL, E.: *On a class of fourth order nonlinear difference equations*, Adv. Difference Equ. **1** (2004), 23–36.

[7] LI, W. T.—CHENG, S. S.: *Asymptotic trichotomy for positive solutions of a class of odd order nonlinear neutral difference equations,* Comput. Math. Appl. **35** (1998), 101–108.

[8] LIU, B.—Yan, J.: *Oscillatory and asymptotic behavior of fourth order nonlinear differential equations,* Acta Math. Sin., New Ser. **13** (1997), 105–115.

[9] SCHMEIDEL, E.: *Oscillation and nonoscillation theorems for fourth order difference equations,* Rocky Mountain J. Math. **33** (2003), 1083–1094.
[10] SCHMEIDEL, E.: *Nonscillation and oscillation properties for fourth order nonlinear difference equations.* In: Proceedings of the Sixth International Conference on Difference Equations, New Progress in Difference Equations, Augsburg, Germany, 2001 (B. Aulbach, S. Elaydi, G. Ladas, eds.), CRC, Boca Raton, FL, 2004, pp. 531–538.
[11] SCHMEIDEL, E.: *Asymptotic trichotomy for positive solutions of a class of fourth order nonlinear neutral difference equations with quasidifferences,* Nonlinear Anal. **63** (2005), e899–e907.
[12] THANDAPANI, E.—AROCKIASAMY, M.: *Oscillatory and asymptotic properties of solutions of nonlinear fourth order difference equations,* Glas. Mat. Ser. III **37** (2002), 119–131.
[13] THANDAPANI, E.—SUNDARAM, P.—GRAEF, J. R.—MICIANCO, A.—SPIKES, P. W.: *Classification of nonoscillatory solutions of higher order neutral type difference equations,* Arch. Math. (Brno) **31** (1995), 263–277.
[14] ZHANG, B. G.—CHENG, S. S.: *On a class of nonlinear difference equations,* J. Difference Equ. Appl. **1** (1995), 391–411.

Received October 5, 2006

Ewa Schmeidel
Institut of Mathematics
Faculty of Electrical Engineering
Poznań University of Technology
ul. Piotrowo 3a
PL-60–965 Poznań
POLAND

E-mail: ewa.schmeidel@put.poznan.pl
URL: http://www.put.poznan.pl/∼schmeide/pl/

Jędrzej Schmeidel
Institute of Solid State Physics
Surface Science Section
Leibniz University of Hannover
Appelstrasse 2
DE-30–167 Hannover
GERMANY

E-mail: schmeidel@fkp.uni-hannover.de
URL: http://www.fkp.uni-hannover.de/∼schmeidel/".