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MAGIC GRAPHS HAVING A SATURATED VERTEX

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ABSTRACT. A graph is called magic (supermagic) if it admits a labelling of the edges by pairwise different (and consecutive) integers such that the sum of the labels of the edges incident with a vertex is independent of the particular vertex. In the paper we characterize magic graphs with saturated vertex and we establish some conditions for magic graphs with saturated vertex to be supermagic.

1. Introduction

We consider finite undirected graphs without loops and multiple edges. If G is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and edge set of G, respectively.

Let a graph G and a mapping f from $E(G)$ into positive integers be given. The *index-mapping* of f is the mapping $f^\star$ from $V(G)$ into positive integers defined by

$$f^\star(v) = \sum_{e \in E(G)} \eta(v,e) f(e) \qquad \text{for every} \quad v \in V(G),$$

where $\eta(v,e)$ is equal to 1 when e is an edge incident with a vertex v, and 0 otherwise. An injective mapping f from $E(G)$ into positive integers is called a *magic labelling* of G for an *index* λ if its index-mapping $f^\star$ satisfies

$$f^\star(v) = \lambda \qquad \text{for all} \quad v \in V(G).$$

A magic labelling f of G is called *supermagic* if the set $\{f(e) \colon e \in E(G)\}$ consists of consecutive positive integers. We say that a graph G is *supermagic* (*magic*) if and only if there exists a supermagic (magic) labelling of G.

The concept of magic graphs was introduced by Sedláček [11]. The regular magic graphs are characterized in [2]. Two different characterizations of

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all magic graphs are given in [9] and [8]. Supermagic graphs were introduced by Stewart [12]. It is easy to see that the classic concept of a magic square of n^2 boxes corresponds to the fact that the complete bipartite graph $K_{n,n}$ is supermagic for every positive integer $n \neq 2$ (see also [12]). Stewart [13] characterized supermagic complete graphs. In [6] supermagic regular complete multipartite graphs and supermagic cubes are characterized. In [7] there are given characterizations of magic line graphs of general graphs and supermagic line graphs of regular bipartite graphs. In [10] and [1] supermagic labellings of the Möbius ladders and two special classes of 4-regular graphs are constructed. Some constructions of supermagic labellings of various classes of regular graphs are described in [5] and [6]. In [3] there are established some bounds for number of edges in supermagic graph. More comprehensive information on magic and supermagic graphs can be found in [4].

Let K_n denote the complete graph on n vertices, P_n the path on n vertices and $G \cup H$ denotes the disjoint union of graphs G and H. The join $G \oplus H$ of the disjoint graphs G and H is the graph $G \cup H$ together with all the edges joining vertices of $V(G)$ and vertices of $V(H)$. The vertex v of graph G is called *saturated vertex* if it is adjacent to all others vertices.

In this paper we will deal with magic and supermagic graphs having a saturated vertex. The graph H with saturated vertex v is isomorphic to the graph $(H - v) \oplus K_1$. So the graph with saturated vertex is also denoted by $G \oplus K_1$.

2. Magic graphs

In this section we characterize magic graphs with saturated vertex. It is obvious that the graph $K_1 \oplus K_1$, which is isomorphic to K_2, is magic. Since except for the complete graph of order 2 no graph with less than 5 vertices is magic, see [14], we will consider graphs G of order at least 4.

We say that a graph G is of *type* $\mathcal{A}$ if it has two edges e_1, e_2 such that $G - \{e_1, e_2\}$ is a balanced bipartite graph with a partition V_1, V_2, and the edge e_i joins two vertices of V_i ($i = 1, 2$). A graph G is of *type* $\mathcal{B}$ if it has two edges e_1, e_2 such that $G - \{e_1, e_2\}$ has a component H which is a balanced bipartite graph with partition V_1, V_2, and e_i joins a vertex of V_i with a vertex of $V(G) - V(H)$ ($i = 1, 2$). As usual, for $S \subset V(G)$, $\Gamma_G(S)$ denotes the set of vertices in G adjacent to a vertex in S.

In the next we will use the following characterization of magic graphs.

Proposition 1. [8] *A connected non-bipartite graph G is magic if and only if G is neither of type $\mathcal{A}$ nor of type $\mathcal{B}$, and $|\Gamma_G(S)| > |S|$ for every independent non-empty subset S of $V(G)$.*

Theorem 1. *Let G be a graph of order at least 4. The graph $G \oplus K_1$ is magic if and only if the minimum degree of graph G is at least 1, G does not contain isolated edge and $\left|\Gamma_G(S)\right| \geqslant |S|$ for every independent subset S of $V(G)$.*

Proof. Let G be a graph of order at least 4. Let G be a graph with minimum degree at least 1 such that G does not contain an isolated edge and $\left|\Gamma_G(S)\right| \geqslant |S|$ for every independent subset S of $V(G)$.

The graph $G \oplus K_1$ is non-bipartite. Suppose that v is a saturated vertex of $G \oplus K_1$.

Suppose $G \oplus K_1$ is of type $\mathcal{A}$. As the saturated vertex v is incident with all others vertices, it is easy to see that $G \oplus K_1$ has 4 vertices, thus $\left|V(G)\right| = 3$, a contradiction.

Suppose that $G \oplus K_1$ is of type $\mathcal{B}$. Let H be the component of $(G \oplus K_1) - \{e_1, e_2\}$ which is a balanced bipartite graph. Consider two cases.

Let $v \in V(H)$. It is easy to see that G is isomorphic to K_2. It means G contains an isolated edge, a contradiction.

Let $v \notin V(H)$. As saturated vertex is incident with all vertices of $V(G)$, $\left|V(G)\right| = 2$ and H is a component of G. Thus G contains an isolated edge, a contradiction.

Let S be an independent set of vertices in $V(G \oplus K_1)$. Then either $S = \{v\}$ or S is an independent set of vertices in G. If $S = \{v\}$, then $\Gamma_{G \oplus K_1}(S) = V(G)$ and $|S| = 1 < \left|V(G)\right| = \left|\Gamma_{G \oplus K_1}(S)\right|$. If S is an independent set of vertices in G, then $\left|\Gamma_{G \oplus K_1}(S)\right| = \left|\Gamma_G(S) \cup \{v\}\right| = \left|\Gamma_G(S)\right| + 1 \geqslant |S| + 1 > |S|$.

According to Proposition 1 we get that $G \oplus K_1$ is magic graph.

The converse implication is obvious. □

Note that the condition $\left|\Gamma_G(S)\right| \geqslant |S|$ for every independent subset S of $V(G)$ is equivalent to the condition that there exists a factor of graph G such that its every component is regular graph of degree 1 or 2. We call this factor a (1-2)-factor.

Let I denote the graph on Figure 1.

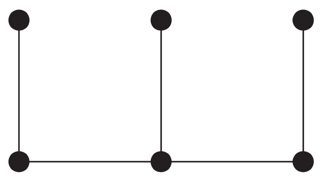

Figure 1.

From Theorem 1 we immediately obtain the bound for minimal number of edges in graph G.

Corollary 1. *Let $G \oplus K_1$ be a magic graph. Then for every nonnegative integer k it holds*

(1) *if* $|V(G)| = 5 + 4k$, *then* $|E(G)| \geqslant 5 + 3k$,
(2) *if* $|V(G)| = 6 + 4k$, *then* $|E(G)| \geqslant 5 + 3k$,
(3) *if* $|V(G)| = 7 + 4k$, *then* $|E(G)| \geqslant 6 + 3k$,
(4) *if* $|V(G)| = 8 + 4k$, *then* $|E(G)| \geqslant 6 + 3k$.

Moreover, the equality holds if and only if G is a graph isomorphic to (1) $kP_4 \cup C_5$, (2) $kP_4 \cup I$ *or* $kP_4 \cup P_6$, (3) $(k+1)P_4 \cup C_3$ *and* (4) $(k+2)P_4$.

Proof. It is known, see [14], that in a magic graph the number of vertices of degree 2 is less than the number of vertices of degree at least 3. It means that if the graph $G \oplus K_1$ is magic, then the number of vertices of degree 1 in the graph G, denoted by n_1, is less or equal to the number of vertices of degree at least 2 in G. Let $V_{\geqslant 2}$ be the set of vertices of degree at least 2 in graph G, and $n_{\geqslant 2}$ be the cardinality of $V_{\geqslant 2}$. It means

$$n_1 \leqslant n_{\geqslant 2}\,. \tag{1}$$

According to Theorem 1 G contains no isolated vertex, so

$$|V(G)| = n_1 + n_{\geqslant 2}\,. \tag{2}$$

For the number of edges we have

$$2|E(G)| = \sum_{v \in V(G)} \deg(v) = n_1 + \sum_{v \in V_{\geqslant 2}} \deg(v) \geqslant n_1 + 2n_{\geqslant 2} = 2|V(G)| - n_1\,. \tag{3}$$

Combining (1), (2) and (3) we get

$$|E(G)| \geqslant \frac{3}{4}|V(G)|\,, \tag{4}$$

which implies the mentioned bounds for number of edges in G.

Consider the case (4), i.e., $|V(G)| = 8 + 4k$. If the equality holds, and because G has no isolated vertex and no isolated edge, then every component of G has exactly 4 vertices and 3 edges. Moreover, G contains a (1-2)-factor, thus every its component is isomorphic to P_4. The other cases are analogous. □

3. Supermagic graphs

The problem of characterizing supermagic graphs having a saturated vertex seems to be difficult. It is solved in this section for regular graphs G.

Let G be an r-regular graph of order n, then Theorem 1 holds for $r \geqslant 2$. If $r = n - 1$, then G is a complete graph on n vertices and $G \oplus K_1$ is also a complete graph. Stewart [13] characterized supermagic complete graphs.

Proposition 2. [13] *A complete graph of order n is supermagic if and only if $n = 2$ or $5 < n \not\equiv 0 \pmod 4$.*

Using this proposition we have that the graph $K_n \oplus K_1$ is supermagic if and only if $n = 1$ or $4 < n \not\equiv 3 \pmod 4$.

In this section we will establish the bound for the number of vertices n in a supermagic graph $G \oplus K_1$, where G is an r-regular graph. First we prove the following assertion.

Lemma 1. *Let G be an r-regular graph of order n, $2 \leqslant r < n-1$. Let $G \oplus K_1$ be a supermagic graph with supermagic labelling $f\colon E(G \oplus K_1) \to \{a, \dots, a + \frac{n(r+2)}{2} - 1\}$. Then for the positive integer a it holds*

$$a \leqslant \frac{\frac{n(r+2)^2}{2} - n^2 - r - 1}{2(n-r-1)}\,.$$

Proof. Let G be an r-regular graph of order n, $2 \leqslant r < n-1$. Let $f\colon E(G \oplus K_1) \longrightarrow \bigl\{a, \dots, a + \frac{n(r+2)}{2} - 1\bigr\}$ be some supermagic labelling of $G \oplus K_1$ for an index λ. Then it holds

$$\begin{aligned}(n+1)\lambda &= \sum_{v \in V(G\oplus K_1)} \sum_{e \in E(G \oplus K_1)} \eta(v,e) f(e)\\ &= 2 \sum_{e \in E(G\oplus K_1)} f(e) = 2\left(a + \dots + \left(a + \frac{n(r+2)}{2} - 1\right)\right)\\ &= \left(2a + \frac{n(r+2)}{2} - 1\right)\frac{n(r+2)}{2}\,.\end{aligned}$$

So

$$\lambda = \frac{\left(2a + \frac{n(r+2)}{2} - 1\right)\frac{n(r+2)}{2}}{n+1}\,. \tag{5}$$

As the degree of the saturated vertex v is n, then

$$\lambda = f^\star(v) \geqslant a + \dots + (a + n - 1)\,. \tag{6}$$

According to (5)

$$\frac{(2a + n - 1)n}{2} \leqslant \frac{\left(2a + \frac{n(r+2)}{2} - 1\right)\frac{n(r+2)}{2}}{n+1}\,. \tag{7}$$

After some manipulations we have

$$2a(n - r - 1) \leqslant \frac{n(r+2)^2}{2} - n^2 - r - 1\,. \tag{8}$$

As $r < n - 1$ we get the mentioned bound for a. □

Using the previous lemma we get the following theorem.

Theorem 2. *Let G be an r-regular graph of order n, $n \geqslant 4$, $2 \leqslant r < n-1$. If $G \oplus K_1$ is a supermagic graph, then it holds*

$$n \leqslant \frac{\frac{r^2}{2} + 2r + \sqrt{\left(\frac{r^2}{2} + 2r\right)^2 + 4(r+1)}}{2}. \tag{9}$$

Proof. Let G be an r-regular graph of order n, $n \geqslant 4$, $2 \leqslant r < n-1$. Let $G \oplus K_1$ be a supermagic graph and suppose

$$n > \frac{\frac{r^2}{2} + 2r + \sqrt{\left(\frac{r^2}{2} + 2r\right)^2 + 4(r+1)}}{2}. \tag{10}$$

Then

$$n^2 - n\left(\frac{r^2}{2} + 2r\right) - r - 1 > 0. \tag{11}$$

After some manipulations we get

$$\frac{n(r+2)^2}{2} - n^2 - r - 1 < 2(n-r-1), \tag{12}$$

It means

$$\frac{\frac{n(r+2)^2}{2} - n^2 - r - 1}{2(n-r-1)} < 1. \tag{13}$$

According to Lemma 1 we have $a < 1$, a contradiction. □

Corollary 2. *Let G be 2-regular graph. The graph $G \oplus K_1$ is supermagic if and only if G is isomorphic to C_4 or C_5.*

Proof. Let G be 2-regular graph of order n. Let $G \oplus K_1$ be a supermagic graph. According to Theorem 1 and Theorem 2 we have $4 \leqslant n \leqslant 6$.

If $n = 6$, then by Lemma 1 we have $a \leqslant \frac{3}{2}$, thus $a = 1$. In this case, by (5), we get $\lambda = \frac{(2+11)12}{7} = \frac{13 \cdot 12}{7}$. This is impossible since λ must be an integer.

If $4 \leqslant n \leqslant 5$, then G is connected, it means G is isomorphic to a cycle C_n.

On the other hand, $C_n \oplus K_1$ is isomorphic to the wheel with n spokes. Stewart [12] proved that wheel with n spokes is supermagic if and only if $n = 4$ or $n = 5$. □

Corollary 3. *Let G be 3-regular graph of order n. The graph $G \oplus K_1$ is supermagic if and only if $n = 6$ or $n = 8$.*

Proof. Let G be 3-regular graph of order n, $4 \leqslant n \equiv 0 \pmod 2$. If $G \oplus K_1$ is supermagic graph, then according to Theorem 2 we have $n \leqslant 10$.

If $n = 10$, then by Lemma 1 we have $a \leqslant \frac{7}{4}$, and thus $a = 1$. By (5) we get $\lambda = \frac{(2+24)25}{11} = \frac{26 \cdot 25}{11}$, which is not an integer, a contradiction.

If $n = 4$, then G is a complete graph and $G \oplus K_1$ is isomorphic to K_5. According to Proposition 2 K_5 is not supermagic. So we have $n = 6$ or $n = 8$.

On the other hand, there exist exactly two nonisomorphic 3-regular graphs of order 6 and exactly six nonisomorphic 3-regular graphs of order 8. Figure 2 and Figure 3 display supermagic labellings of $G \oplus K_1$ for corresponding 3-regular graphs G. In these figures the graphs G are depicted with thick lines. $\square$

FIGURE 2.

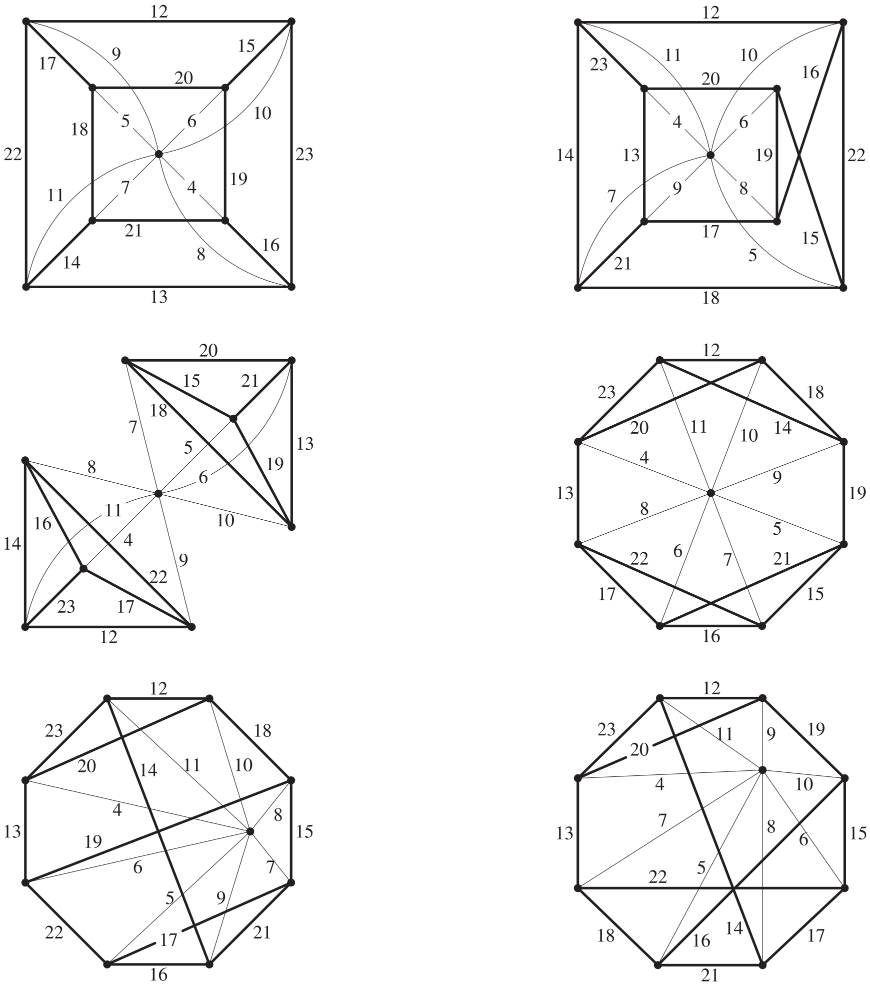

FIGURE 3.

Analogously, we can prove the following assertion.

Let G be 4-regular graph of order n. If $G \oplus K_1$ is a supermagic graph, then $5 \leqslant n \leqslant 15$.

REFERENCES

[1] BAČA, M.—HOLLÄNDER, I.—KO-WEI LIH: *Two classes of super-magic graphs*, J. Combin. Math. Combin. Comput. **23** (1997), 113–120.

[2] DOOB, M.: *Characterizations of regular magic graphs*, J. Combin. Theory, Ser. B **25** (1978), 94–104.

[3] DRAJNOVÁ, S.—IVANČO, J.—SEMANIČOVÁ, A.: *Numbers of edges in supermagic graphs*, J. Graph Theory **52** (2006), 15–26.

[4] GALLIAN, J. A.: *A dynamic survey of graph labeling*, Electron. J. Combin. **5** (1998), DS6.

[5] HARTSFIELD, N.—RINGEL, G.: *Pearls in Graph Theory*, Academic Press, Inc., Boston, 1990.

[6] IVANČO, J.: *On supermagic regular graphs*, Math. Bohem. **125** (2000), 99–114.

[7] IVANČO, J.—LASTIVKOVÁ, Z.—SEMANIČOVÁ, A.: *On magic and supermagic line graphs*, Math. Bohem. **129** (2004), 33–42.

[8] JEURISSEN, R. H.: *Magic graphs, a characterization*, Europ. J. Combin. **9** (1988), 363–368.

[9] JEZNÝ, S.—TRENKLER, M.: *Characterization of magic graphs*, Czechoslovak Math. J. **33** (1983), 435–438.

[10] SEDLÁČEK, J.: *On magic graphs*, Math. Slovaca **26** (1976), 329–335.

[11] SEDLÁČEK, J.: *Problem 27*, in: Proc. Theory Graphs Appl. Symp. Smolenice, Praha, 1963, 163–164.

[12] STEWART, B. M.: *Magic graphs, Canad. J. Math.*, Canad. J. Math. **18** (1966), 1031–1059.

[13] STEWART, B. M.: *Supermagic complete graphs*, Canad. J. Math. **19** (1967), 427–438.

[14] TRENKLER, M.: *Numbers of vertices and edges of magic graphs*, Ars Combin. **53** (2000), 93–96.

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