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ON GENERALIZATION OF THE NOTION OF CONTINUITY IN BITOPOLOGICAL SPACES

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ABSTRACT. In this paper we consider relations between quasi-continuity, semi-continuity and somewhat continuity of a function $f : X \to Y$, where X is a bitopological space.

A space X, on which two (arbitrary) topologies τ_1, τ_2 are defined, is called a *bitopological space* and denoted by (X, τ_1, τ_2). A subset A of a bitopological space X is said to be τ_1 *semi-open with respect to* τ_2 if $A \subset \mathrm{cl}_{\tau_2}(\mathrm{int}_{\tau_1}(A))$, where cl_τ, int_τ denote the closure and interior operator in topological space (X, τ), respectively. The notion of semi-open sets was introduced by N. Levine [LE], but the notion of τ_1 semi-open with respect to τ_2 sets was introduced by T. Lipski [L11], [L12]. It is easy to observe that a set A is τ_1 semi-open with respect to τ_2 if and only if there exists a set $U \in \tau_1$ such that $U \subset A \subset \mathrm{cl}_{\tau_2}(U)$ and that for any collection $\{A_\alpha\}_{\alpha \in \Delta}$ of τ_1 semi-open with respect to τ_2 sets, the union $\bigcup_{\alpha \in \Delta} A_\alpha$ is again τ_1 semi-open set with respect to τ_2.

Let X be a bitopological space equipped with two topologies τ_1, τ_2, and Y be a topological space with topology τ. A function $f\colon X \to Y$ is said to be τ_1*-inclusion quasi-continuous with respect to* τ_2 (shortly: $\subset QC_{\tau_2}^{\tau_1}$) at $x \in X$ if for any open set $G \subset Y, f(x) \in G$ and for any set $V \in \tau_2, x \in V$ there exists a non-empty set $U \in \tau_1$ such that $U \subset V$ and $f(U) \subset G$. A function $f\colon X \to Y$ is said to be τ_1*-intersection quasi-continuous with respect to* τ_2 ($\cap QC_{\tau_2}^{\tau_1}$, in short) at $x \in X$ if for any open set $G \subset Y, f(x) \in G$ and for any set $V \in \tau_2, x \in V$ there exists a set $U \in \tau_1$ such that $U \cap V \neq \emptyset$ and $f(U) \subset G$.

A function f is τ_1-inclusion (intersection) quasi-continuous function with respect to τ_2 if it is τ_1-inclusion (intersection) quasi-continuous function with respect to τ_2 at any point of its domain.

Let us observe that if a function $f\colon X \to Y$ is τ_1-inclusion quasi-continuous function with respect to τ_2 then f is τ_1-intersection quasi-continuous function with respect to τ_2.

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If $\tau_1 = \tau_2$, then bitopological space is in fact a (usual) topological space, τ_1-inclusion quasi-continuity with respect to τ_2 and τ_1-intersection quasi-continuity with respect to τ_2 are equivalent to quasi-continuity (in the usual sense). Not every τ_1-intersection quasi-continuous function with respect to τ_2 is τ_1-inclusion quasi-continuous function with respect to τ_2. Let us consider the following example:

EXAMPLE 1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function defined by formula $f(x) = x$ for any $x \in \mathbb{R}$, and let τ_1 denote a natural topology of the real line $\mathbb{R}$, τ_2 denotes a discrete topology. It is easy to check, that f is τ_1-intersection quasi-continuous function with respect to τ_2 but it is not τ_1-inclusion quasi-continuous function with respect to τ_2.

A function f is said to be τ_1 *semi-continuous with respect to* τ_2 if for any open set $G \subset Y$ the set $f^{-1}(G)$ is a τ_1-semi-open set with respect to τ_2.

THEOREM 1. *Let (X, τ_1, τ_2) be a bitopological space, and Y be a topological space. A function $f\colon X \to Y$ is $\cap QC^{\tau_1}_{\tau_2}$ function if and only if f is τ_1-semi-continuous function with respect to τ_2.*

Proof. Let f be a $\cap QC^{\tau_1}_{\tau_2}$ function, G—open subset of Y and $x_0 \in f^{-1}(G)$. If $V \in \tau_2$ is a τ_2-neighbourhood of x_0, then there exists a set $U \in \tau_1$ such that $U \cap V \neq \emptyset$ and $f(U) \subset G$. Hence $U \subset f^{-1}(G)$ and next $U \subset \operatorname{int}_{\tau_1}\left(f^{-1}(G)\right)$. Consequently,

$$\emptyset \neq U \cap V \subset \operatorname{int}_{\tau_1}\left(f^{-1}(G)\right) \cap V .$$

Since V is an arbitrary neighbourhood of x_0,

$$x_0 \in \operatorname{cl}_{\tau_2}\Big(\operatorname{int}_{\tau_1}\left(f^{-1}(G)\right)\Big) .$$

Due to arbitrary choice of x_0 we infer that $f^{-1}(G) \subset \operatorname{cl}_{\tau_2}\Big(\operatorname{int}_{\tau_1}\left(f^{-1}(G)\right)\Big)$. Thus for arbitrary open set G the set $f^{-1}(G)$ is τ_1 semi-open with respect to τ_2 and f is τ_1-semi-continuous function with respect to τ_2.

Now, let f be τ_1-semi-continuous function with respect to τ_2, $x_0 \in X$ and G be an open subset of Y, such that $f(x_0) \in G$. Let $V \in \tau_2$, $x_0 \in V$. Because $f^{-1}(G)$ is τ_1-semi-open with respect to τ_2 then $\operatorname{cl}_{\tau_2}\Big(\operatorname{int}_{\tau_1}\left(f^{-1}(G)\right)\Big) \supset f^{-1}(G)$ and, consequently, $V \cap \operatorname{int}_{\tau_1}\left(f^{-1}(G)\right) \neq \emptyset$. Thus, if $U = \operatorname{int}_{\tau_1}\left(f^{-1}(G)\right)$ then $U \in \tau_1$ and $U \cap V \neq \emptyset$. Hence f is $\cap QC^{\tau_1}_{\tau_2}$ function at x_0. Since x_0 was arbitrarily chosen, the function f is $\cap QC^{\tau_1}_{\tau_2}$ function. $\square$

A function $f\colon X \to Y$ is said to be τ_1-semi-continuous with respect to τ_2 at $x \in X$ if for any open set $G \subset Y$ such that $f(x) \in G$ there is a set U which is

τ_1-semi-open with respect to τ_2 and such that $x \in U$ and $f(U) \subset G$. It is easy to observe that the function f is τ_1 semi-open with respect to τ_2 if and only if f is τ_1 semi-open function with respect to τ_2 at x for any $x \in X$. For the case when $\tau_1 = \tau_2$ the proof of this fact is due to H. Levine [LE]. There arises a question if τ_1-semi-continuity with respect to τ_2 at a point is equivalent to the τ_1-inclusion or intersection quasi-continuity with respect to τ_2 at that point. In the case when $\tau_1 = \tau_2$ a positive answer was given by A. Neubrunnová [NA]. If the last equality does not hold, then we have the following theorem.

THEOREM 2. *Let (X, τ_1, τ_2) be a bitopological space, and Y be a topological space. A function $f\colon X \to Y$ is $\cap QC_{\tau_2}^{\tau_1}$ function at some $x \in X$ if and only if f is τ_1-semi-continuous function with respect to τ_2 at that point x.*

Proof. Let $x \in X$ and let f be a $\cap QC_{\tau_2}^{\tau_1}$ function at x. Let G be an open subset of Y such that $f(x) \in G$ and let $\beta_2(x)$ denote some base at x for the topology τ_2. Since f is $\cap QC_{\tau_2}^{\tau_1}$ function at x then for any $V \in \beta_2(x)$ there exists $U_V \in \tau_1$ such that $U_V \cap V \neq \emptyset$ and $f(U_V) \subset G$. Let $W = \bigcup_{V \in \beta_2(x)} U_V$. Hence $W \in \tau_1$ and $\emptyset \neq W \cap V$. Therefore $x \in \mathrm{cl}_{\tau_2}(W)$. The last condition and the relation $W \in \tau_1$ imply inclusion $W \cup \{x\} \subset \mathrm{cl}_{\tau_2}(\mathrm{int}_{\tau_1}(W \cup \{x\}))$ and, consequently, $W \cup \{x\}$ is τ_1 semi-open with respect to τ_2. Moreover,

$$f(W \cup \{x\}) = \bigcup_{V \in \beta_2(x)} f(U_V) \cup \{f(x)\} \subset G.$$

It was shown that f is τ_1 semi- continuous function with respect to τ_2 at x.

Let $x \in X$ and let f be τ_1-semi-continuous function with respect to τ_2 at x. Let G be an open subset of Y such that $f(x) \in G$ and $x \in U \in \tau_2$. Hence there exist some set V, $x \in V$ satisfying the following condition $V \subset \mathrm{cl}_{\tau_2}(\mathrm{int}_{\tau_1}(V))$ and $f(V) \subset G$. Denoting $W = \mathrm{int}_{\tau_1}(V)$ one can infer that $x \in \mathrm{cl}_{\tau_2}(W)$ and $W \in \tau_1$. Consequently, $W \cap U \neq \emptyset$ and $f(W) = f(\mathrm{int}_{\tau_1}(V)) \subset f(V) \subset G$.

Hence f is $\cap QC_{\tau_2}^{\tau_1}$ function at the point x. □

Let X, Y denote topological spaces and τ denote a topology in X. A mapping $f\colon X \to Y$ is said to be τ-somewhat continuous if for any open set $V \subset Y$ such that $f^{-1}(V) \neq \emptyset$ we have $\mathrm{int}_\tau(f^{-1}(V)) \neq \emptyset$ (see [GH]). Let A be a non-empty topological subspace of x. It is clear that a restricted function $f|_A\colon A \to Y$ is τ^A-somewhat continuous (where τ^A denotes the induced topology) if for any open set $V \subset Y$ such that $(f|_A)^{-1}(V) \neq \emptyset$ we have $\mathrm{int}_{\tau^A}((f|_A)^{-1}(V)) \neq \emptyset$. In the case of restricted function $f|_A$ we can also consider somewhat continuity with respect to topology τ. Restricted function $f|_A$ is said to be τ-somewhat continuous if $\mathrm{int}_\tau(f^{-1}(V) \cap A) \neq \emptyset$ for any open set $V \subset Y$ such that $(f|_A)^{-1}(V) \neq \emptyset$. Of course, if $f|_A$ is τ-somewhat continuous function then $f|_A$ is τ^A somewhat continuous function. If $A \in \tau$ then a converse theorem is also true. It is easy to show the following theorem.

Theorem 3. *Let (X, τ_1, τ_2) be a bitopological space, and Y be a topological space. Let $f\colon X \to Y$. The following statements are equivalent:*

(a) *Function f is $\subset QC^{\tau_1}_{\tau_2}$ function.*

(b) *Function $f|_U$ is τ_1-somewhat continuous function for any $U \in \tau_2$.*

(c) *There exists a basis β_2 of topology τ_2 such that $f|_U$ is τ_1-somewhat continuous function for any $U \in \beta_2$.*

P r o o f. (a) $\Rightarrow$ (b).

Let f be $\subset QC^{\tau_1}_{\tau_2}$ function and let $U \in \tau_2$. Let $V \subset Y$ be an open set such that $(f|_U)^{-1}(V) \neq \emptyset$. Therefore there exists $x_0 \in U$ such that $f(x_0) \in V$. Because f is $\subset QC^{\tau_1}_{\tau_2}$ function then there exists a non-empty set $W \in \tau_1$, $W \subset U$ such that $f(W) \subset V$. It implies that $\mathrm{int}_{\tau_1}\big(f^{-1}(V) \cap U\big) \neq \emptyset$. Since V was arbitrarily chosen the function $f|_U$ is τ-somewhat continuous function.

(b) $\Rightarrow$ (c). Implication is obvious.

(c) $\Rightarrow$ (a). Now, let $x_0 \in X$ and let G be an open subset of Y such that $f(x_0) \in G$. Let $V \in \tau_2$, $x_0 \in V$. There exists $U \in \beta_2$ such that $x_0 \in U \subset V$. Because $f(x_0) \in G$ and $x_0 \in U$ then $(f|_U)^{-1}(G) \neq \emptyset$. Under the assumptions $W = \mathrm{int}_{\tau_1}\big(f^{-1}(G) \cap U\big) \neq \emptyset$ and consequently $W \in \tau_1$, $W \subset U$. Of course, $f(W) \subset f\big(f^{-1}(G) \cap U\big) \subset f\big(f^{-1}(G)\big) = G$. Since V, G and x_0 were arbitrary chosen the function f is $\subset QC^{\tau_1}_{\tau_2}$ function. $\square$

Remark. If we replace the symbol τ_1 with the symbol τ_1^U from Theorem 3 then implication (c) $\Rightarrow$ (a) does not hold.

We know that a mapping $f\colon X \to Y$ is τ-somewhat continuous if and only if for any dense set $S \subset X$ the set $f(S)$ is dense in $f(X)$ (see [GH]).

Theorem 4. *Let (X, τ_1, τ_2), where $\tau_2 \subset \tau_1$, be a bitopological space, and Y be a topological space. A function $f\colon X \to Y$ is $\subset QC^{\tau_1}_{\tau_2}$ function if and only if for any dense set S in topology τ_1 the set $f(S \cap U)$ is dense in $f(U)$ for any $U \in \tau_2$.*

P r o o f. Let us assume that f is $\subset QC^{\tau_1}_{\tau_2}$ function. Let S be a dense set in topology τ_1 and let $U \in \tau_2$. Since $\tau_2 \subset \tau_1$ we have

$$\mathrm{cl}_{\tau_1}(S \cap U) \supset \mathrm{cl}_{\tau_1}(S) \cap U = U$$

and, consequently, $\mathrm{cl}_{\tau_1}(U) \subset \mathrm{cl}_{\tau_1}(S \cap U)$. So, $S \cap U$ is dense in U in topology τ_1. Since $f|_U$ is a τ_1-somewhat continuous function then the set $(f|_U)(S \cap U)$ is a dense subset of $(f|_U)(U)$. Therefore $f(U \cap S)$ is a dense subset of $f(U)$. Now, let us assume that for any set S dense in topology τ_1 the set $f(U \cap S)$ is a dense subset of $f(U)$ for any $U \in \tau_2$. Let $U \in \tau_2$, then $U \in \tau_1$. Let S_U be a dense subset of U in topology τ_1. Then the set $S = S_U \cup (X \setminus U)$ is a dense subset of X in topology τ_1 and $S_U = S \cap U$. Therefore $(f|_U)(S)$ is a dense subset of $(f|_U)(U)$. Since S_U was arbitrarily chosen, $f|_U$ is a τ_1 somewhat continuous and since $U \in \tau_2$ was arbitrarily chosen, f is τ_1 quasi-continuous function with respect to τ_2. □

The assumption $\tau_2 \subset \tau_1$ in Theorem 4 seems to be very strong. It seems that under this assumption any $\subset QC_{\tau_2}^{\tau_1}$ function should be a quasi-continuous function with respect to topology τ_1 or topology τ_2. However, it is not true. We can see the following example.

Example 2. Let $f\colon X \to \mathbb{R}$, where $X = \{1,2,3,4\}$ and $\mathbb{R}$ denotes the space of reals with natural topology, be a function defined by the following formula

$$f(x) = \begin{cases} 0 & \text{if } x \in \{2,4\}, \\ 2 & \text{if } x \in \{1,3\}. \end{cases}$$

Let τ_1, τ_2 be topologies in X defined as follows.

$$\tau_1 = \bigl\{\emptyset, \{1\}, \{3\}, \{4\}, \{2,3\}, \{1,4\}, \{1,3\}, \{3,4\}, \{2,3,4\}, \{1,2,3\}, \\ \{1,3,4\}, \{1,2,3,4\}\bigr\},$$

$$\tau_2 = \{\emptyset, \{1\}, \{2,3,4\}, \{1,2,3,4\}\}.$$

It is easy to show, that f is $\subset QC_{\tau_2}^{\tau_1}$ function. But f is not a quasi-continuous function with respect to topology τ_1. Indeed, let $x_0 = 2$. Then $f(x_0) = 0$. Let $V = (-1,1)$ be an open neighbourhood of the point 0 in topology τ_1, and let $U = \{1,2,3\}$ be an open neighbourhood of the point x_0 in topology τ_1. Then there exists no set $W \in \tau_1$ such that $f(W) \subset V$. We can also show that the function f is not a quasi-continuous function with respect to topology τ_2.

We know that for any quasi-continuous mapping (which is also semi-continuous) $f\colon X \to Y$, where Y is a second countable, the set of discontinuity points is of the first category (see [LE]). In case when f is $\subset QC_{\tau_2}^{\tau_1}$ function or $\cap QC_{\tau_2}^{\tau_1}$ function, we can ask about the characterization of the set of discontinuity points with respect to topology τ_1 and topology τ_2. The characterization of these sets is given by Theorem 5. Of course, we know that if f is $\subset QC_{\tau_2}^{\tau_1}$ function then it is $\cap QC_{\tau_2}^{\tau_1}$ function and, consequently, f is τ_1 semi-continuous function with respect to τ_2 (see Theorem 1).

Let D_f^{τ} be the set of all discontinuity points of the function $f\colon X \to Y$ with respect to topology τ of the space X.

Theorem 5. *Let (X, τ_1, τ_2), where $\tau_1 \subset \tau_2$, be a bitopological space, and Y be a second countable topological space. If $f\colon X \to Y$ is τ_1-semi-continuous function with respect to τ_2 then the sets $D_f^{\tau_1}$ and $D_f^{\tau_2}$ are of the first category with respect to each topologies τ_1 and τ_2.*

Proof. Since f is τ_1-semi-continuous function with respect to τ_2 then f is $\cap QC_{\tau_2}^{\tau_1}$ function (see Theorem 1). Hence, on the assumptions that $\tau_1 \subset \tau_2$, the function f is quasi-continuous function with respect to topology τ_1 and topology τ_2 simultaneously and the relation $D_f^{\tau_2} \subset D_f^{\tau_1}$ holds. So, $D_f^{\tau_2}$ is of the first category with respect to topology τ_1 and topology τ_2, and $D_f^{\tau_1}$ is of the first category with respect to topology τ_1. We shall show that $D_f^{\tau_1}$ is of the first category with respect to topology τ_2. Let $\{V_n : n \in \mathbf{N}\}$ be a countable base for topology Y. Then under the assumptions that f is τ_1 semi-continuous function with respect to τ_2 we have

$$\begin{aligned} D_f^{\tau_1} &= \bigcup_{n=1}^{+\infty} \Big(f^{-1}(V_n) \setminus \operatorname{int}_{\tau_1}\big(f^{-1}(V_n)\big)\Big) \\ &\subset \bigcup_{n=1}^{+\infty} \Big(\operatorname{cl}_{\tau_2}\big(\operatorname{int}_{\tau_1}\big(f^{-1}(V_n)\big)\big) \setminus \operatorname{int}_{\tau_1}\big(f^{-1}(V_n)\big)\Big). \end{aligned}$$

Hence, the set $D_f^{\tau_1}$ is of the first category set. □

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