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A CLASS OF STEINER SYSTEMS S(2,4,v) WITH ARCS OF EXTREMAL SIZE

Lorenzo Milazzo* — Zsolt Tuza†

ABSTRACT. An *arc* in a Steiner system $S = \mathrm{S}(2,4,v)$ of order v is a point set that contains at most two elements from each block. The largest size of an arc in S is denoted by $\alpha_2(S)$. The upper bound $\alpha_2 \leqslant \frac{v+2}{3}$ is always valid; an arc with $\frac{v+2}{3}$ points is called a *maximum arc*. We present a recursive construction with the following properties:

(i) From two systems $S' = \mathrm{S}(2,4,v')$ and $S'' = \mathrm{S}(2,4,v'')$ of respective orders v' and v'', an $S = \mathrm{S}(2,4,v)$ of order $v = \frac{1}{3}(v'-1)(v''-1)+1$ is obtained.

(ii) $\alpha_2(S) - 1 \geqslant (\alpha_2(S') - 1)(\alpha_2(S'') - 1)$; in particular, if both S' and S'' contain maximum arcs, then so does S, too.

(iii) If each of S' and S'' can be covered with three maximum arcs incident with a common point, then so does S, too.

1. Introduction

Motivated by the theory of mixed hypergraphs [12], in this note we study––within a particular class of balanced incomplete block designs—the maximum possible size of sets that meet each block in at most two points. The main result of the paper is the recursive construction of an infinite sequence of designs, in the members of which the cardinality of such "strongly independent" sets attains the maximum possible value in terms of the total number of points. Moreover, the point set is coverable with three "strongly independent" sets of maximum size.

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Steiner systems. A Steiner system $S(2,4,v)$ is a pair $(X,\mathcal{B})$, where X is a finite vertex set (often called point set) and $\mathcal{B}$ is a family of subsets of X, called blocks, satisfying the following conditions:

(1) Each pair of vertices is contained in one and only one block.

(2) Each block contains exactly four vertices.

Hanani proved in [3] that the necessary and sufficient condition for the existence of a $S(2,4,v)$ is $v \equiv 1$ or $4 \pmod{12}$. The definitions imply that every vertex is contained in precisely $\frac{v-1}{3}$ blocks, and $|\mathcal{B}| = \frac{v(v-1)}{12}$.

In the language of hypergraph theory, the $S(2,4,v)$ are precisely the *4-uniform, linear hypergraphs* with some number v of vertices and "at least" $\frac{v(v-1)}{12}$ edges. (A hypergraph is said to be *linear* if its edges are almost disjoint: no two of them share more than one vertex.) The quotation mark above refers to the fact that no 4-uniform linear hypergraph can have more than $\frac{v(v-1)}{12}$ edges, i.e., "at least" can be replaced here by "exactly" or by "the maximum number of".

Approaching these structures from the other direction, the adaptation of a known principle for the particular case of $S(2,4,v)$ yields: If $(X,\mathcal{B})$ is a set system such that each pair of vertices belongs to *at least one* block, and $|\mathcal{B}| = \frac{v(v-1)}{12}$, then (X,B) is a $S(2,4,v)$.

Maximum arcs An *arc* in a Steiner system $S(2,4,v)$—or more generally in a hypergraph $\mathcal{H}$—is a set $A \subset X$ of vertices that meets each block (edge) in at most 2 vertices. A block meeting an arc A in precisely one vertex is called a *tangent* of A. We denote by $\alpha_2(\mathcal{H})$ the maximum cardinality of an arc in a hypergraph $\mathcal{H}$.

It may be noted that for 3-uniform hypergraphs the concept of arc coincides with that of *independent set* (i.e., a vertex subset that contains no edges); but this is not the case with larger edge sizes. In $S(2,4,v)$, the independent sets may meet a block in 3 vertices, contrary to arcs.

Let A be an arc in $S(2,4,v)$, and x a vertex of A. As noted above, x is incident with precisely $\frac{v-1}{3}$ blocks. Since each of them may contain at most one further vertex of A beside x, it follows that

$$|A| \leqslant \frac{v+2}{3}.$$

De Resmini observed in [11] that the extremal situation is characterized by the nonexistence of tangents; more precisely, if an arc $A \neq \emptyset$ has no tangents, then A has the maximum possible size, $\frac{v+2}{3}$. Such A is often called *maximum arc*.

It was first proved by Denniston [1], in a more general setting, that the upper bound $\frac{v+2}{3}$ is tight for infinitely many values of v. If $v = 12\cdot 2^h + 4$, then an $\mathrm{S}(2,4,v)$ can be constructed as a Denniston arc in $\mathrm{PG}(2,4\cdot 2^h)$ containing a triple of Denniston arcs of order $\frac{v+2}{3}$ intersecting in a common point (see [2, Theorem 1] for more details).

Another infinite sequence is exhibited in the following result.

Theorem 1. [7] *For every positive integer n there exists an $\mathrm{S}(2,4,v)$ of order $v = 3\cdot 5^n + 1$ that contains an arc of size $\frac{v+2}{3}$.*

This result will be obtained from the much more general Theorem 2 below.

Comments on chronology. We proved Theorem 1 in 1998, but the corresponding manuscript [7] was not polished for publication at that time. As described in [2], until 2003 it was only Denniston's construction that appeared in print as an infinite family with maximum arcs. Then, Greig and Rosa and, independently, Ling [5] proved that an $\mathrm{S}(2,4,v)$ with an arc of size $\frac{v+2}{3}$ exists for every $v \equiv 4 \pmod{12}$. (The "1 mod 12" residue class does not admit arcs that large, cf. [10].) In this way, our Theorem 1 is superceded by the results in [2], [5], but our "old" construction still provides some new systems $\mathrm{S}(2,4,v)$. Recently, Alex Rosa kindly informed us that the recursive step in the particular case $v \to 5v-4$ seems to be known, but we do not have a related reference at hand.

Some interesting connections between maximal arcs and a certain class of graphs with circular (dihedral) symmetry can be found in the forthcoming paper [4].

Partition into arcs. Let us say that an $\mathrm{S}(2,4,v)$, $(X,\mathcal{B})$ has the *arc partition property* if X (of cardinality v) admits a partition

$$X = X_1 \cup X_2 \cup X_3 \cup \{x\},$$

where x is a specified vertex, and all the three sets $X_i \cup \{x\}$ $(i = 1,2,3)$ are arcs of size $\frac{v+2}{3}$. Greig and Rosa proved in [2] that an $\mathrm{S}(2,4,v)$ with the arc partition property exists if and only if $v \equiv 4 \pmod{12}$. In Section 2 we prove

Theorem 2. *From any $S' = \mathrm{S}(2,4,v')$ of order v' and $S'' = \mathrm{S}(2,4,v'')$ of order v'', an $S = \mathrm{S}(2,4,v)$ of order v can be constructed with the following properties:*

(i) $v = \frac{1}{3}(v'-1)(v''-1)+1$.

(ii) $\alpha_2(S) - 1 \geqslant (\alpha_2(S')-1)(\alpha_2(S'')-1)$.

(iii) *If S' and S'' have the arc partition property, then so does S, too.*

In particular, the second property preserves the existence of maximum arcs, of size $\frac{v+2}{3}$.

Mixed hypergraph coloring. Arcs, which are stronger versions of independent sets, play a role in the colorability properties of *mixed hypergraphs*. Formally, a mixed hypergraph $\mathcal{H}$ is a triple $(X, \mathcal{C}, \mathcal{D})$, where X is the vertex set, and $\mathcal{C}$ and $\mathcal{D}$ are set systems over X, the collections of the so-called *C-edges* and *D-edges*. A *coloring* of $\mathcal{H}$ is an assignment of one color to each vertex in such a way that every C-edge contains two vertices of a **c**ommon color and every D-edge contains two vertices of **d**istinct colors. The *upper chromatic number*, $\overline{\chi}(\mathcal{H})$, is the maximum number of colors that can occur in a coloring of $\mathcal{H}$.

With the help of arcs, lower bounds for $\overline{\chi}$ can be obtained in the following way. Suppose that $A \subset X$ is the largest arc in the 4-uniform hypergraph $\mathcal{H}$. We assign a distinct color to each vertex of A, and the same new color to all vertices of $X \setminus A$. This construction implies the validity of the following assertion.

Proposition 3. *Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ be a mixed hypergraph such that both $\mathcal{C}$ and $\mathcal{D}$ are 4-uniform and linear, and $\mathcal{C} \neq \emptyset$. If $A \subset X$ is an arc in $\mathcal{C}$ and $X \setminus A$ is an independent set in $\mathcal{D}$, then*

$$\overline{\chi}(\mathcal{H}) \geqslant |A| + 1 .$$

In particular, if $\mathcal{C} \neq \emptyset$ and $\mathcal{D} = \emptyset$, then

$$\overline{\chi}(\mathcal{H}) \geqslant \alpha_2(\mathcal{C}) + 1 .$$

The condition $\mathcal{C} \neq \emptyset$ is needed in both assertions, because otherwise $\overline{\chi}(\mathcal{H}) = \alpha_2(\mathcal{C}) = |X|$ holds for any $\mathcal{D}$. Now, applying Theorem 1, we obtain

Corollary 4. *There exist Steiner systems $\mathrm{S}(2, 4, v)$ of arbitrarily large order v with upper chromatic number at least $\frac{v+5}{3}$.*

This lower bound is in striking contrast with Steiner triple and quadruple systems, whose upper chromatic number is as small as $O(\log v)$. As proved in [8], such a logarithmic upper bound is valid in the class of all Steiner systems $\mathrm{S}(t, k, v)$ of block size k and "intersection size" t *if and only if* $k = t + 1$: If $k \geqslant t + 2$, then $C \cdot n^c$ is a general lower bound, where $C, c > 0$ are suitable constants whenever k and t are fixed.

2. The construction

In this section we prove Theorems 1 and 2. In fact, recursive constructions are very frequent in design theory, but they are not so often adjusted for specific parameters. A method of this kind will be described below.

We begin with the case $v = 16$, $n = 1$; and then an inductive step will be applied, along which the maximum size of an arc will be shown to increase as required in Theorem 2. In particular, for Theorem 1, both $v - 1$ and $\alpha_2 - 1$ will get multiplied by 5.

Basic case for Theorem 1: $v = 16$. A Steiner system $S(2,4,16)$ is in fact a finite affine plane, of order $q = 4$, in which each line has four points. In this particular case, using standard terminology, an arc with $\frac{v+2}{3} = 6 = q+2$ points is a *hyperoval*. In Table 1 we list the blocks of $S(2,4,16)$ with a vertex labelling that will be convenient later. A maximum arc (hyperoval) is: $A_1 = \{x', x_{1,3}, x_{2,3}, x_{3,3}, x_{4,3}, x_{5,3}\}$.

TABLE 1.

$\{x', x_{1,1}, x_{1,2}, x_{1,3}\}$	$\{x_{1,2}, x_{2,1}, x_{3,2}, x_{5,2}\}$	$\{x_{1,1}, x_{2,2}, x_{4,1}, x_{5,2}\}$	$\{x_{1,3}, x_{3,1}, x_{4,3}, x_{5,2}\}$
$\{x', x_{2,1}, x_{2,2}, x_{2,3}\}$	$\{x_{1,2}, x_{2,3}, x_{4,3}, x_{5,1}\}$	$\{x_{1,2}, x_{2,2}, x_{3,1}, x_{4,2}\}$	$\{x_{2,2}, x_{3,2}, x_{4,3}, x_{5,3}\}$
$\{x', x_{3,1}, x_{3,2}, x_{3,3}\}$	$\{x_{1,1}, x_{2,3}, x_{3,1}, x_{5,3}\}$	$\{x_{1,3}, x_{2,3}, x_{3,2}, x_{4,1}\}$	$\{x_{1,1}, x_{2,1}, x_{3,3}, x_{4,3}\}$
$\{x', x_{4,1}, x_{4,2}, x_{4,3}\}$	$\{x_{1,2}, x_{3,3}, x_{4,1}, x_{5,3}\}$	$\{x_{1,3}, x_{2,1}, x_{4,2}, x_{5,3}\}$	$\{x_{1,3}, x_{2,2}, x_{3,3}, x_{5,1}\}$
$\{x', x_{5,1}, x_{5,2}, x_{5,3}\}$	$\{x_{2,1}, x_{3,1}, x_{4,1}, x_{5,1}\}$	$\{x_{1,1}, x_{3,2}, x_{4,2}, x_{5,1}\}$	$\{x_{2,3}, x_{3,3}, x_{4,2}, x_{5,2}\}$

Figure 1 shows five blocks l_i and a generic block r of the $S(2,4,16)$ in Table 1; the first indices of the $x_{i,j}$ are tied to the blocks l_i in which they appear.

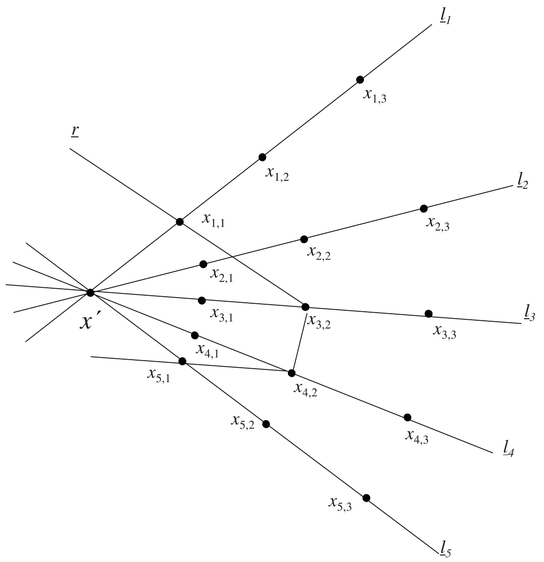

FIGURE 1.

The recursive construction. Let the systems $(X', \mathcal{B}')$ and $(X'', \mathcal{B}'')$ of orders v' and v'' be given. We also assume that $x' \in X'$ and $x'' \in X''$ are specified vertices, and that the following subsets are available:

- $X'_1 \cup X'_2 \cup X'_3 = X' \setminus \{x'\}$ and $X''_1 \cup X''_2 \cup X''_3 = X'' \setminus \{x''\}$ are partitions such that $|X'_i| = \frac{v'-1}{3}$ and $|X''_i| = \frac{v''-1}{3}$ for $i = 1,2,3$.
- $A' \cup \{x'\} \subseteq X'_3 \cup \{x'\}$ and $A'' \cup \{x''\} \subseteq X''_3 \cup \{x''\}$ are the largest arcs.
- If $(X', \mathcal{B}')$ and $(X'', \mathcal{B}'')$ have the arc partition property, then $X'_i \cup \{x'\}$ and $X''_i \cup \{x''\}$ are arcs, for all $i = 1,2,3$.

Putting $s = \frac{v'-1}{3}$, we denote by $l_1, \dots, l_s$ the blocks of $(X', \mathcal{B}')$ incident with x'. Let $l_i = \{x', x_{i,1}, x_{i,2}, x_{i,3}\}$ for $i = 1, \dots, s$.

To construct a system $(X, \mathcal{B})$, let us replace each vertex $x_{i,j}$ with a set $X_{i,j}$ that contains $\frac{v''-1}{3}$ vertices, for $1 \leqslant i \leqslant s$ and $1 \leqslant j \leqslant 3$. This scheme is shown for $v' = 16$ in Fig. 2. It contains $v = \frac{(v'-1)(v''-1)}{3} + 1$ vertices, the set of which will be denoted by X. The vertex with specific label x' also belongs to X. We denote by L_i the set resulted from the original block l_i after this operation.

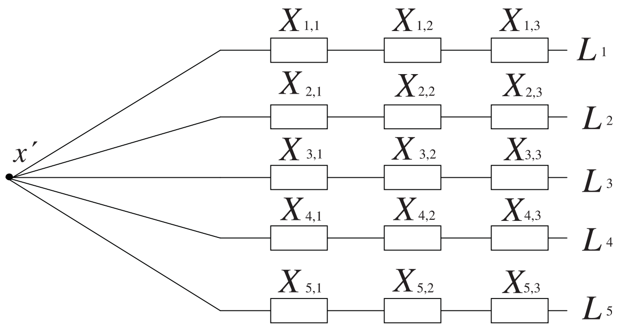

FIGURE 2.

Blocks inside L_i. For each $i = 1, \dots, s$ we take a mapping

$$f_i : X'' \to X'_{i,1} \cup X'_{i,2} \cup X'_{i,3} \cup \{x'\}$$

in such a way that

- $f_i(x'') = x' \quad (1 \leqslant i \leqslant s)$,
- $f_i(X''_j) = X_{i,j} \quad (1 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant 3)$.

These mappings induce the set of blocks inside the L_i in the natural way:

$$\bigcup_{i=1}^{s} \{f_i(B) | B \in \mathcal{B}''\}.$$

Crossing blocks among the L_i. Each generic block $\{x_{a,p}, x_{b,q}, x_{c,r}, x_{d,s}\}$ (where a, b, c, d are distinct) has become a 4-partite set $X_{a,p} \cup X_{b,q} \cup X_{c,r} \cup X_{d,s}$ after the substitution. Inside each part of these sets, the pairs are already covered with the blocks of the subsystems $f_1(\mathcal{B}''), \dots, f_s(\mathcal{B}'')$, while the other pairs are uncovered so far. They will be covered for each blown-up block of $\mathcal{B}' - \{x'\}$ separately. In this way the construction is reduced to finding an edge partition of the complete 4-partite graph $K_{t,t,t,t}$, $t = \frac{v''-1}{3}$, into edge-disjoint copies of K_4.

The idea below works for every odd t. We select the following collection of quadruples of integers:

$$(g, h, g+h-1, g+2h-2), \qquad 1 \leqslant g, h \leqslant t$$

with addition taken modulo t, in order to keep all elements of the quadruples stay in the range $1, \ldots, t$. Then, each quadruple specifies a block in each blown-up generic block of $\mathcal{B}' - \{x'\}$. The 1st, 2nd, 3rd, 4th element of the quadruple tells us, which element of $X_{a,p}$, $X_{b,q}$, $X_{c,r}$, $X_{d,s}$ has to be taken to form a block in $\mathcal{B}$. (More generally, $K_{t,t,t,t}$ is known to admit a K_4-decomposition if and only if there exist two orthogonal Latin squares of order t; but this fact is not needed here.)

Verification of properties. Since any two original blocks of $\mathcal{B}'$ have at most one vertex in common, the intersection of any two $K_{t,t,t,t}$ lies inside just one L_i. Therefore, no "crossing" vertex-pair is covered twice. Conversely, to see that each pair *is* covered, it suffices to observe these:

(1) On two vertices belonging to different L_i, for each $l, m \in \{1, 2, \ldots, s\}$ and for any two "functions" $G, H \in \{g, h, g+h-1, g+2h-2\}$ there exists precisely one solution (g, h) of the equation system $l = G(g, h)$, $m = H(g, h)$.

(2) On two vertices of the same L_i, there is one block by the assumptions on $\mathcal{B}''$.

The construction also ensures that no block of $\mathcal{B}$ meets $(A' \times A'') \cup \{x'\}$ in more than two vertices; i.e., this set is an arc, and its cardinality is as large as required for Theorem 2.

Finally, to prove Theorem 1, it suffices to apply the recursion $n-1$ times, taking $v' = 16$ with the system of Table 1 in each step, and $v'' = 3 \cdot 5^i + 1$ in step i, with the system of Table 1 for $i = 1$ and the one obtained in step $i-1$ for each $i > 1$.

3. Graph decomposition

The second part of the construction, where $K_{t,t,t,t}$ is partitioned into copies of K_4, may also be considered as a component sum of factorizations. In order to illustrate this idea on an example, we take the case where $\mathrm{S}(2, 4, 3 \cdot 5^{n-1} + 1)$ is substituted into $\mathrm{S}(2, 4, 16)$ to yield $\mathrm{S}(2, 4, 3 \cdot 5^n + 1)$. Let us consider the 4-tuple $\{x_{1,1}, x_{3,2}, x_{4,2}, x_{5,1}\}$ of block r in Figure 1. Its vertices give rise to the vertex sets $X_{1,1}$, $X_{3,2}$, $X_{4,2}$ and $X_{5,1}$; let us split them into two pairs, $(X_{1,1}, X_{3,2})$ and $(X_{4,2}, X_{5,1})$. These may be viewed as two complete bipartite graphs. We decompose the first one into perfect matchings, $\{F_1, F_2, \ldots, F_{5^{n-1}}\}$, and the second one into stars, $\{G_1, G_2, \ldots, G_{5^{n-1}}\}$. If we denote the vertices with double subscripts, the first one indicating the first index of $X_{i,j}$ and the second one specifying which vertex of $X_{i,j}$ it is, the factors in our case are

$$F_h = \{\{y_{1,p}, y_{3,p+h-1} | p = 1, 2, \dots\}\}$$

and

$$G_h = \{\{y_{4,s}, y_{5,h} | s = 1, 2, \dots\}\},$$

where all symbolic indices range between 1 and 5^{n-1}. Moreover, a *cyclic shift* is applied on F_h, to obtain that

- the first pair in F_h contains the vertex $y_{1,h}$, the second one contains $y_{1,h+1}$, $\dots$ and G_h remains in the natural order,
- the first pair in G_h contains the vertex $y_{4,1}$, the second one contains $y_{4,2}$, $\dots$

(Throughout, subscript addition is taken modulo 5^{n-1}.)

After all these, for each h and m ($1 \leqslant h, m \leqslant 5^{n-1}$) a 4-element block is defined as the union of the mth pair of F_h and the mth pair of G_h.

For example, in the particular case of $n = 2$ we obtain

F_1	F_2	F_3	F_4	F_5
$y_{1,1}, y_{3,1}$	$y_{1,2}, y_{3,3}$	$y_{1,3}, y_{3,5}$	$y_{1,4}, y_{3,2}$	$y_{1,5}, y_{3,4}$
$y_{1,2}, y_{3,2}$	$y_{1,3}, y_{3,4}$	$y_{1,4}, y_{3,1}$	$y_{1,5}, y_{3,3}$	$y_{1,1}, y_{3,5}$
$y_{1,3}, y_{3,3}$	$y_{1,4}, y_{3,5}$	$y_{1,5}, y_{3,2}$	$y_{1,1}, y_{3,4}$	$y_{1,2}, y_{3,1}$
$y_{1,4}, y_{3,4}$	$y_{1,5}, y_{3,1}$	$y_{1,1}, y_{3,3}$	$y_{1,2}, y_{3,5}$	$y_{1,3}, y_{3,2}$
$y_{1,5}, y_{3,5}$	$y_{1,1}, y_{3,2}$	$y_{1,2}, y_{3,4}$	$y_{1,3}, y_{3,1}$	$y_{1,4}, y_{3,3}$

G_1	G_2	G_3	G_4	G_5
$y_{4,1}, y_{5,1}$	$y_{4,1}, y_{5,2}$	$y_{4,1}, y_{5,3}$	$y_{4,1}, y_{5,4}$	$y_{4,1}, y_{5,5}$
$y_{4,2}, y_{5,1}$	$y_{4,2}, y_{5,2}$	$y_{4,2}, y_{5,3}$	$y_{4,2}, y_{5,4}$	$y_{4,2}, y_{5,5}$
$y_{4,3}, y_{5,1}$	$y_{4,3}, y_{5,2}$	$y_{4,3}, y_{5,3}$	$y_{4,3}, y_{5,4}$	$y_{4,3}, y_{5,5}$
$y_{4,4}, y_{5,1}$	$y_{4,4}, y_{5,2}$	$y_{4,4}, y_{5,3}$	$y_{4,4}, y_{5,4}$	$y_{4,4}, y_{5,5}$
$y_{4,5}, y_{5,1}$	$y_{4,5}, y_{5,2}$	$y_{4,5}, y_{5,3}$	$y_{4,5}, y_{5,4}$	$y_{4,5}, y_{5,5}$

For instance, F_1 and G_1 generate the blocks $\{y_{1,1}, y_{3,1}, y_{4,1}, y_{5,1}\}$, $\{y_{1,2}, y_{3,2}, y_{4,2}, y_{5,1}\}$, $\{y_{1,3}, y_{3,3}, y_{4,3}, y_{5,1}\}$, $\{y_{1,4}, y_{3,4}, y_{4,4}, y_{5,1}\}$, $\{y_{1,5}, y_{3,5}, y_{4,5}, y_{5,1}\}$.

4. Concluding remarks

In the end, let us give some comments in three different directions.

Substitution. Applying the method described in Section 2, alternative constructions can be obtained for several results of [2]. This leads us one step closer to the answer to the following open question.

Problem 5. *Determine or estimate the number of non-isomorphic systems* $\mathrm{S}(2,4,v)$ *with the following property*

(a) *they contain an arc with* $\frac{v+2}{3}$ *vertices, or*

(b) *they have the arc partition property.*

Upper chromatic number. The upper chromatic number of Steiner systems was first studied in [6], and then in many subsequent papers by various authors. A recent survey on the subject can be found in [9].

One of the problems related to the present study that remains open for later research is:

Problem 6. *Investigate the tightness of Corollary 4.*

We can prove a general upper bound strictly less than $\frac{3v}{5}$ but the lower and upper estimates do not match asymptotically so far.

Independent sets. Contrary to arcs, the easy-to-prove general upper bound on the maximum cardinality of *independent sets* in $\mathrm{S}(2,4,v)$ is $\frac{2v+1}{3}$. This estimate is tight for infinitely many v, following from the fact that some $\mathrm{S}(2,4,v)$ of order v can contain a subsystem of order $\frac{v-1}{3}$ if and only if $v \equiv 4$ or $13 \pmod{36}$.

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Lorenzo Milazzo
Department of Mathematics
University of Catania
Viale A. Doria, 6
I–95125-Catania
ITALY
E-mail: milazzo@dipmat.unict.it

Zsolt Tuza
Computer and Automation Institute
Hungarian Academy of Sciences
Kende u. 13-17
H–1111 Budapest
HUNGARY

Department of Computer Science
University of Pannonia
H–8200 Veszprém
HUNGARY
E-mail: tuza@lutra.sztaki.hu