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NOTE ON CONTROLLED CARATHÉODORY EXTENSIONS

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ABSTRACT. In this paper we present a theorem on controlled extensions of Carathéodory functions defined on a subset, which is a rectangle. We obtain our result by means of the Marczewski function.

1. Introduction

Let us start by giving the definition of Carathéodory functions. Let $(T, \mathcal{M})$ be a measurable space, let X, Y be topological spaces. We consider Borel σ-algebra $\mathcal{B}(X)$ on X and product σ-algebra $\mathcal{M} \otimes \mathcal{B}(X)$ on $T \times X$. A function f from $T \times X$ to Y is called Carathéodory if, for each $t \in T$, the function $f(t, \cdot)$ is continuous and $f(\cdot, x)$ is $\mathcal{M}$-measurable, for every x.

We can find a theorem in [3] (see Theorem 1, p. 186), that states the form of inverse image of open sets for Carathéodory function $f\colon T \times F \to Y$, where $F \subset X$. If we assume that X has a countable base and Y is perfectly normal, then for each open $V \subset Y$, there exist $M_n^f(V) \in \mathcal{M}$ and $U_n^f(V) \subset X$ open in F such that $f^{-1}(V) = \bigcup\limits_{n \in N} M_n^f(V) \times U_n^f(V)$.

In the paper by Frantz (see [1], Theorem 3, p. 60), there is a result on continuous extensions of real valued functions f, g defined on a closed subset A of a compact metric space X, such that $f(a) \cdot g(a) = u(a), a \in A$, where u is a given continuous function on X. The extensions of f, g fulfil the multiplicative condition over the whole space X. J. Malešič, D. Repovš and P. Semenov in [4] proved a more general theorem on controlled extensions. Before formulating their result, we introduce the necessary notation and definitions. Let us denote by $\mathbb{R}^2_{+0}$ the set $\{(x, y)\colon y > 0\} \cup \{(0, 0)\}$. We have to extend the defini-

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tion of (E,u)-controlled extensions introduced in [4] (see Definition 0.2, p. 63) in order to distinguish cases of continuous functions and Carathéodory ones.

Definition 1.1. Let X, Y be nonempty sets, let A be a subset of X, and let $E\colon Y\times Y\to Y$, $u\colon X\to Y$ be given functions. Let $f,g\colon A\to Y$ be functions satisfying the equality

$$E\bigl(f(a),g(a)\bigr)=u(a),\qquad a\in A.$$

Then the functions $\bar f\colon X\to Y$ and $\bar g\colon X\to Y$ are called (E,u)-*controlled extensions* of f and g, if $\bar f$ extends $f, \bar g$ extends g and

$$E\bigl(\bar f(x),\bar g(x)\bigr)=u(x),\qquad x\in X.$$

If X, Y are topological spaces and $E, u, f, g, \bar f, \bar g$ are continuous functions then we speak of (E,u)-*controlled continuous extensions.* If $X=T\times Z$, where $(T,\mathcal{M})$ is a measurable space, Z is a topological space, $A=T\times F$, where $F\subset X$, E is a continuous function, $f, g, \bar f, \bar g$ and u are Carathéodory functions, then we have (E,u)-*controlled Carathéodory extensions.*

The next definition provides an example of a function E.

Definition 1.2. A function $E\colon\mathbb{R}^2\to\mathbb{R}$ is called a *pseudomultiplication* if $E(x,y)=e(x)\cdot y$, where

(i) e preserves the signs of arguments,

(ii) e is continuously differentiable,

(iii) the derivative e' is positive in some neighborhood of zero and in some neighborhood of infinity.

Now, we can present the theorem on (E,u)-controlled continuous extensions.

Theorem 1.3. (see [4], Theorem 1.5, p. 65).
Let $E\colon R^2\to R$ be a pseudomultiplication. Let $u\colon X\to R$ be a continuous function on a normal space X, and let A be a closed subset of X. Then each continuous mapping $(f,g)\colon A\to R^2_{+0}$ such that $E\bigl(f(a),g(a)\bigr)=u(a), a\in A$, admits an (E,u)-controlled continuous extension $(\bar f,\bar g)\colon X\to R^2_{+0}$.

We obtain our result on Carathéodory functions from the corresponding one for continuous functions by means of the M a r c z e w s k i function (see [5]). By the Marczewski function of a sequence $\mathcal{A}=(A_n)_{n\in N}$ of subsets of T we mean the mapping $\chi\colon T\to\{0,1\}^N$ defined by $\chi(t)=\bigl(\chi_n(t)\bigr)_{n\in N}$, where χ_n is the characteristic function of A_n. The space $\{0,1\}^N$ of all zero-one sequences is considered with the product topology, and the image $\chi(T)$ with the induced topology. Let us notice that the sets $\chi(A)$, for $A\in\mathcal{A}$, are closed-open in $\chi(T)$ and the function χ is measurable with respect to the σ-algebra generated by $\mathcal{A}$.

2. Main theorem

In this section, we present the theorem on controlled extensions for Carathéodory functions.

Theorem 2.1. *Let X be a separable metric space and let $F \subset X$ be a closed subset of X. Let $E\colon R^2 \to R$ be a pseudomultiplication. Let $f\colon T \times F \to R$, $g\colon T \times F \to [0,\infty)$ and $u\colon T \times X \to R$ be Carathéodory functions such that $g^{-1}(0) \subset f^{-1}(0)$ and $E\big(f(t,x), g(t,x)\big) = u(t,x)$ for $(t,x) \in T \times F$. Then there exist $\bar{f}\colon T \times X \to R$, $\bar{g}\colon T \times X \to [0,\infty)$, which are (E,u)-controlled Carathéodory extensions of f and g.*

Proof. Let $\{V_n : n \in N\}$ be a base in R, let $M_n^f(V_k), M_n^g(V_k), M_n^u(V_k)$ be measurable subsets of T and $U_n^f(V_k), U_n^g(V_k)$ be open in F and $U_n^u(V_k)$ be open subsets of X, where $n,k \in N$, such that $f^{-1}(V_k) = \bigcup_{n\in N} M_n^f(V_k) \times U_n^f(V_k)$, $g^{-1}(V_k) = \bigcup_{n\in N} M_n^g(V_k) \times U_n^g(V_k)$ $u^{-1}(V_k) = \bigcup_{n\in N} M_n^u(V_k) \times U_n^u(V_k)$. Let $\mathcal{A}$ be a sequence of subsets, which is the union of $\{M_n^f(V_k)\colon n,k \in N\}$, $\{M_n^g(V_k)\colon n,k \in N\}, \{M_n^u(V_k)\colon n,k \in N\}$, without omitting the repeating elements and let χ be the Marczewski function of $\mathcal{A}$. We define the function $h\colon T \times X \to 2^N \times X$ by the formula $h(t,x) = \big(\chi(t), x\big)$. Let Z, B be such that $Z = h(T \times X)$ and $B = h(T \times F)$. Let us notice $Z = \chi(T) \times X$ and $B = \chi(T) \times F$, which implies that B is closed in Z. The value space $2^N \times X$ of the function h is a metric space, hence it is a normal space. We define the functions $\psi\colon B \to R$ by $\psi\big(h(t,x)\big) = f(t,x)$, $\phi\colon B \to [0,\infty)$ by $\phi\big(h(t,x)\big) = g(t,x)$ and the function $\mu\colon Z \to R$ by $\mu\big(h(t,x)\big) = u(t,x)$. Now, we are going to prove that ψ is a well defined, and analogous arguments are true for ϕ and μ. If we show that $h(t,x) \neq h(t',x')$ whenever $f(t,x) \neq f(t',x')$, then the function ψ is well defined. At the very beginning, let us notice that if $x \neq x'$ then $h(t,x) \neq h(t',x')$, so it suffices to consider $(t,x), (t',x')$ such that $x = x'$. Let us assume that $f(t,x) \neq f(t',x)$, then there exists k such that $f(t,x) \in V_k$ and $f(t',x) \notin V_k$. It means that $(t,x) \in f^{-1}(V_k) = \bigcup_{n\in N} M_n^f(V_k) \times U_n^f(V_k)$ and $(t',x) \notin f^{-1}(V_k)$. There exists $A_n \in \mathcal{A}$ such that $t \in A_n$ and $t' \notin A_n$, hence $\chi(t) \neq \chi(t')$ and $h(t,x) \neq h(t',x')$. Now, we prove continuity of ψ. We first need a formula $h\big(f^{-1}(V)\big) = \psi^{-1}(V)$, for any $V \subset R$. By the definition of ψ, we have $h\big(f^{-1}(y)\big) = \psi^{-1}(y)$, for every $y \in R$. According to basic formulas for image and inverse image of the union of the family of sets we have an expression for $\psi^{-1}(V)$. Then for any element of the base V_k, we have

$$\psi^{-1}(V_k) = h\big(f^{-1}(V_k)\big) = \bigcup_{n\in N} \chi\big(M_n^f(V_k)\big) \times U_n^f(V_k).$$

The sets $\chi\big(M_n^f(V_k)\big)$ are closed and open in Z, and $U_n^f(V_k)$ are open in F, which completes the proof of continuity of ψ. We can apply those arguments again to obtain the result that ϕ and μ are continuous in their domains. In the case of function μ, we consider the sets $U_n^u(V_k)$, which are open in X.

In order to prove $(\psi,\phi)\colon B \to R^2_{+0}$ it is sufficient to show that $\phi^{-1}(0) \subset \psi^{-1}(0)$. It follows from

$$\phi^{-1}(0) = h\big(g^{-1}(0)\big) \subset h\big(f^{-1}(0)\big) = \psi^{-1}(0)\,.$$

Additionally, condition $E(\psi,\phi) = \mu|_B$ is fulfilled. For every $(a,x) \in B$, there exists $t \in T$ such that $h(t,x) = (a,x)$, hence we have

$$\begin{aligned} E\big(\psi(a,x),\phi(a,x)\big) &= E\Big(\psi\big(h(t,x)\big),\phi\big(h(t,x)\big)\Big) \\ &= E\big(f(t,x),g(t,x)\big) = u(t,x) = \mu\big(h(t,x)\big) = \mu(a,x)\,. \end{aligned}$$

By means of Theorem 1.3 we obtain continuous extensions $(\bar\psi,\bar\phi)\colon Z \to R^2_{+0}$ of (ψ,ϕ) such that $E(\bar\psi,\bar\phi) = \mu$. Let us define $\bar f\colon T\times X \to R$ by the formula $\bar f = \bar\psi\circ h$ and $\bar g\colon T\times X\to[0,\infty)$ by $\bar g = \bar\phi\circ h$. The function $\bar f$ is an extension of f, and it follows from calculation:

$$\bar f(t,x) = \bar\psi\big(h(t,x)\big) = \psi\big(h(t,x)\big) = f(t,x)\,,$$

for all $(t,x)\in T\times F$. The function $\bar f$ is Carathéodory function. Indeed, continuity of $\bar f(t,\cdot)$, for every $t\in T$, follows from the definition $\bar f(t,\cdot) = \bar\psi(\chi(t),\cdot)$ and continuity of $\bar\psi$. Let V be an open subset of Y, then $\bar f^{-1}(V) = h^{-1}\big(\bar\psi^{-1}(V)\big)$. Since $\bar\psi$ is continuous, h is $\sigma(\mathcal{A})\otimes\mathcal{B}(X)$-measurable and $\mathcal{A}\subset\mathcal{M}$, from this we have $\bar f$ is $\mathcal{M}\otimes\mathcal{B}(X)-$measurable. Analogously, we prove that $\bar g$ is an extension of g and is Carathéodory. Moreover, it is easy to show that $E\big(\bar f(t,x),\bar g(t,x)\big) = u(t,x)$, for all $(t,x)\in T\times X$. □

Remark 2.2. No attempt has been made here to achieve a result on controlled extensions of Carathéodory functions defined over some subset of $T\times X$, which is not a rectangle. Theorems on extensions of Carathéodory functions defined over some $B\subset T\times X$ can be found in [2] and [3].

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