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SOLVABILITY CONDITIONS OF BOUNDARY VALUE PROBLEMS FOR SECOND ORDER NONLINEAR DIFFERENTIAL SYSTEMS

Martina Langerová

ABSTRACT. Weakly nonlinear boundary value problems for second order systems of ordinary differential equations with a small parameter are considered. A sufficient condition for existence of at least one solution of such problems is established and a convergent iterative process for finding such solutions is suggested.

1. Statement of the problem

We consider boundary value problem for second order nonlinear systems of ordinary differential equations containing a small nonnegative parameter ε of the form

$$\begin{aligned} \ddot{x} + A(t)x &= f(t) + \varepsilon X(x, t, \varepsilon)\,, \\ lx &= \alpha + \varepsilon J\big(x(\cdot, \varepsilon), \varepsilon\big), \qquad t \in [a, b]\,, \end{aligned} \tag{1}$$

where $A(t)$ is $n \times n$ matrix whose columns are real functions continuous on $[a, b]$, $f(t) = \operatorname{col}\big(f_1(t), f_2(t), \dots, f_n(t)\big)$ is n-dimensional column vector whose components are real functions continuous on $[a, b]$, $X(x, t, \varepsilon)$ is an n-dimensional vector function nonlinear as a function of x, continuously differentiable with respect to x in a certain neighborhood of the generating solution, and continuous as a function of t and ε, i.e., $X(\cdot, t, \varepsilon) \in C^1[x]$, $\|x - x_0\| \le q$, $X(x, \cdot, \varepsilon) \in C[t]$, $X(x, t, \cdot) \in C[\varepsilon]$, $\varepsilon \in [0, \varepsilon_0]$, q and ε_0 are sufficiently small constants; $J\big(x(\cdot, \varepsilon), \varepsilon\big)$ is a nonlinear bounded m-dimensional vector functional continuously differentiable with

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respect to x (in the sense of Fréchet) and continuous as a function of ε in a certain neighborhood of the generating solution; α is a constant column vector from m-dimensional real Euclidean vector space $\mathbb{R}^m$, $l = \mathrm{col}\,(l_1, l_2, \dots, l_m)$ is an m-dimensional linear vector functional defined in the space of functions continuous on $[a,b]$ [1], [2].

Our aim is to establish conditions for the existence and develop an algorithm for the construction of a solution $x = x(t,\varepsilon) = \mathrm{col}\,\big(x_1(t,\varepsilon), x_2(t,\varepsilon), \dots, x_n(t,\varepsilon)\big)$ of boundary value problem (1) such that $x(\cdot,\varepsilon) \in C^2[a,b]$ and $x(t,\cdot) \in C[\varepsilon]$ which turns into the solution of unperturbed boundary value problem

$$\ddot{x}(t) + A(t)x(t) = f(t)\,, \tag{2}$$

$$lx = \alpha\,, \qquad t \in [a,b]\,. \tag{3}$$

Problem (2)–(3) is obtained from (1) by setting $\varepsilon = 0$ and is called generating problem for (1).

2. Preliminaries

Denote by $X(t) = \big[U_1(t), U_2(t)\big]$ an $n \times 2n$ fundamental matrix of the homogeneous system (2), where $U_1(t)$, $U_2(t)$ are $n \times n$ matrix functions whose columns are linearly independent solutions of the homogeneous system corresponding to (2), and by $Q = lX(\cdot)$ an $m \times 2n$ constant matrix obtained as a result of substitution of the matrix $X(t)$ into the boundary condition (3), $\mathrm{rank}\, Q = n_1$.

Let P_Q and P_{Q^*} be the $2n \times 2n$ and $m \times m$ matrix-orthoprojectors $P_Q : \mathbb{R}^{2n} \to N(Q)$, $P_{Q^*} : \mathbb{R}^m \to N(Q^*)$ (the symbol * denotes the operation of transposition). Also let Q^+ be a $2n \times m$ matrix which is a Moore-Penrose pseudoinverse to Q. Since

$$\mathrm{rank}\, P_Q = 2n - \mathrm{rank}\, Q = 2n - n_1 = r$$

and

$$\mathrm{rank}\, P_{Q^*} = m - \mathrm{rank}\, Q = m - n_1 = d\,,$$

we denote by $P_{Q_d^*}$ the $d \times m$ matrix composed of a complete system of d linearly independent rows of the matrix P_{Q^*} and by P_{Q_r} the $2n \times r$ matrix composed of a complete system of r linearly independent columns of the matrix P_Q [3],[4].

Theorem 2.1. *If* $\mathrm{rank}\, Q = n_1 \le \min\{2n, m\}$, *then homogeneous* $\big(f(t) = 0, \alpha = 0\big)$ *boundary value problem* (2)–(3) *has* r ($r = 2n - n_1$) *and only* r *linear* *independent solutions. The nonhomogeneous boundary value problem* (2)–(3) *is solvable if and only if* $f(t) \in C[a,b]$ *and* $\alpha \in \mathbb{R}^m$ *satisfy the condition*

$$P_{Q_d^*}\left\{\alpha - \int_a^b lK(\cdot,\tau)f(\tau)\,\mathrm{d}\tau\right\} = 0 \qquad (d = m - n_1)\,, \tag{4}$$

where $K(t,\tau)$ *is an* $n \times n$ *Cauchy matrix of the problem* (2)*.*

In this case, the problem possesses an r*-parameter family of solutions*

$$x = x_0(t, c_r) = X_r(t)c_r + (Gf)(t) + X(t)Q^+\alpha, \qquad c_r \in \mathbb{R}^r\,, \tag{5}$$

where $X_r(t)$ *is an* $n \times r$ *matrix* $X_r(t) = X(t)P_{Q_r}$ *and* $(Gf)(t)$ *is a generalized Green operator acting upon an arbitrary vector* $f(t)$ *from* $C[a,b]$ *as follows:*

$$(Gf)(t) = \int_a^b K(t,\tau)f(\tau)\,\mathrm{d}\tau - X(t)Q^+\int_a^b lK(\cdot,\tau)f(\tau)\,\mathrm{d}\tau\,.$$

3. Main result

We consider the boundary value problem (1) in the critical case [1, p. 102], where $\operatorname{rank} Q = n_1 < m$. Thus, by virtue of Theorem 2.1, the generating boundary value problem (2)–(3) is solvable if and only if $f(t) \in C[a,b]$ and $\alpha \in \mathbb{R}^m$ satisfy (4). The problem possesses an r-parameter family of solutions $x_0(t, c_r)$ given by the relation (5) which are called the generating solutions for the problem (1). Our aim is to establish conditions for the existence and propose an algorithm for construction of a solution $x(t,\varepsilon)$ of the boundary value problem (1) continuous as a function of ε, $\varepsilon \in [0, \varepsilon_0]$ which turns into the generating solution $x_0(t, c_r)$ given by (5) with a constant $c_r = c_r^0$ for $\varepsilon = 0$.
By the change of variables

$$x(t,\varepsilon) = x_0\left(t, c_r^0\right) + z(t,\varepsilon)\,, \qquad c_r^0 \in \mathbb{R}^r$$

in the boundary value problem (1), we arrive at the following problem: It is necessary to establish condition for the existence and an algorithm for the construction of a solution $z = z(t,\varepsilon)$ of the boundary value problem

$$\begin{aligned} \ddot{z} + A(t)z &= \varepsilon X\left(x_0\left(t, c_r^0\right) + z(t,\varepsilon), t, \varepsilon\right)\,, \\ lz &= \varepsilon J\left(x_0\left(\cdot, c_r^0\right) + z(\cdot,\varepsilon), \varepsilon\right)\,, \qquad t \in [a,b]\,, \end{aligned} \tag{6}$$

such that $z(\cdot,\varepsilon) \in C^2[a,b]$ and $z(t,\cdot) \in C[\varepsilon]$, $\varepsilon \in [0,\varepsilon_0]$, vanishing for $\varepsilon = 0$ [1], [5].

By using the fact that the vector function $X(x,t,\varepsilon)$ and vector functional $J\big(x(\cdot,\varepsilon),\varepsilon\big)$ are continuously differentiable with respect to x in a neighborhood of the point $\varepsilon=0$, we can separate a term linear as a function of z and terms of zero order with respect to ε from the vector function $X(x_0+z,t,\varepsilon)$ and the vector functional $J\big(x_0\left(\cdot,c_r^0\right)+z(\cdot,\varepsilon),\varepsilon\big)$. As a result, we get the following expansions:

$$X\left(x_0\left(t,c_r^0\right)+z,t,\varepsilon\right)=X\left(x_0\left(t,c_r^0\right),t,0\right)+A_1(t)z+R(z,t,\varepsilon)\,,$$
$$A_1(t)=\left.\frac{\partial X(x,t,0)}{\partial x}\right|_{x=x_0(t,c_r^0)},$$
$$J\left(x_0\left(\cdot,c_r^0\right)+z(\cdot,\varepsilon),\varepsilon\right)=J\left(x_0\left(\cdot,c_r^0\right),0\right)+l_1z(\cdot,\varepsilon)+R_0\big(z(\cdot,\varepsilon),\varepsilon\big)\,,$$

where $l_1z(\cdot,\varepsilon)$ is linear part of the vector functional $J\big(x_0(\cdot,c_r^0)+z(\cdot,\varepsilon),\varepsilon\big)$. The nonlinear vector function $R\big(z(t,\varepsilon),t,\varepsilon\big)$ belongs to the class $C^1[z],C[t],C[\varepsilon]$ in the region $\|z\|\le q, t\in[a,b],\varepsilon\in[0,\varepsilon_0]$ and, moreover,

$$R(0,t,0)=0,\quad \frac{\partial R(0,t,0)}{\partial z}=0,\quad R_0(0,0)=0,\quad \frac{\partial R_0(0,0)}{\partial z}=0\,.$$

Sufficient condition for existence of the solutions of problem (1) is given by the following theorem:

Theorem 3.1. *Assume that the nonlinear boundary value problem* (1) *satisfies the indicated conditions in the critical case* ($\operatorname{rank} Q=n_1<m$) *and the corresponding generating boundary value problem* (2)–(3) *possesses an r-parameter* ($r=n-n_1$) *family of the generating solutions*

$$x_0(t,c_r)=X_r(t)c_r+(Gf)(t)+X(t)Q^+\alpha\,,\qquad c_r\in\mathbb{R}^r. \tag{7}$$

Then, if the arbitrary vector $c_r=c_r^0\in\mathbb{R}^r$ satisfies the equation

$$F\left(c_r^0\right)=P_{Q_d^*}\left\{J\left(x_0\left(\cdot,c_r^0\right),0\right)-l\int_a^b K(\cdot,\tau)X\left(x_0\left(\tau,c_r^0\right),\tau,0\right)\mathrm{d}\tau\right\}=0 \qquad (d=m-n_1) \tag{8}$$

and if the equivalent relations

$$\operatorname{rank}\left[B_0=P_{Q_d^*}\left\{l_1X_r(\cdot)-\int_a^b lK(\cdot,\tau)A_1(\tau)X_r(\tau)\,\mathrm{d}\tau\right\}\right]=d\Longleftrightarrow P_{B_0^*}=0\,, \tag{9}$$

hold, the boundary value problem (1) *possesses at least one solution $x(t,\varepsilon)$: $x(\cdot,\varepsilon)\in C^2[a,b]$, $x(t,\cdot)\in C[0,\varepsilon_0]$, which for $\varepsilon=0$ turns into the generating* *solution* $x_0(t, c_r^0)$ *given by the relation* (7) *with constant vector* $c_r = c_r^0 \in \mathbb{R}^r$ *satisfying* (8) *and can be obtained as a result of the iterative process*

$$\begin{aligned}
c_k = &- B_0^+ P_{Q_d^*} \Big\{ l_1 z_k^{(1)}(\cdot, \varepsilon) + R_0\big(z_k(\cdot,\varepsilon), \varepsilon\big) \\
&- l \int_a^b K(\cdot,\tau) \Big[A_1(\tau) z_k^{(1)}(\tau,\varepsilon) + R\big(z_k(\tau,\varepsilon),\tau,\varepsilon\big)\Big] \,\mathrm{d}\tau \Big\}, \\
z_{k+1}^{(1)}(t,\varepsilon) = &\varepsilon \Big\{ G\left(X\left(x_0\left(\tau, c_r^0\right), \tau, 0\right)\right) + A_1(\tau)\left(X_r(\tau)c_k + z_k^{(1)}(\tau,\varepsilon)\right) \\
&+ R\big(z_k(\tau,\varepsilon),\tau,\varepsilon\big)\Big\}(t) \\
&+ \varepsilon X(t) Q^+ \Big\{ J\left(x_0\left(\cdot, c_r^0\right), 0\right) + l_1\left(X_r(\cdot)c_k + z_k^{(1)}(\cdot,\varepsilon)\right) \\
&+ R_0\left(z_k^{(1)}(\cdot,\varepsilon),\varepsilon\right)\Big\}, \\
z_{k+1}(t,\varepsilon) = &X_r(t)c_k + z_{k+1}^{(1)}(t,\varepsilon), \quad k = 0,1,2,\dots, \quad z_0(t,\varepsilon) = z_0^{(1)}(t,\varepsilon) = 0, \\
x_k(t,\varepsilon) = &x_0\left(t, c_r^0\right) + z_k(t,\varepsilon), \qquad k = 0,1,2,\dots .
\end{aligned} \tag{10}$$

The theorem is proved in [4].

4. Example

We now illustrate the statement mentioned above. Consider the boundary value problem of Mathieu equation [1], [5] of the form

$$\begin{aligned}
\ddot{x} + (1 + \varepsilon \cos 2t)x = \varphi(t), \qquad t \in [0,\pi], \\
lx = \begin{pmatrix} x(0) \\ x(\pi) \end{pmatrix} = \alpha,
\end{aligned} \tag{11}$$

where $\varphi(t) \in C[0,\pi]$ and $\alpha = \operatorname{col}(\alpha_1, \alpha_2) \in \mathbb{R}^2$. So $n = 1$ and $m = 2$.
In this case, fundamental matrix of the homogeneous generating system for (11) has the form

$$X(t) = [\cos t \ \sin t\ ],$$

matrix Q is the constant matrix:

$$Q = lX = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad Q^+ = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}.$$

Since $\operatorname{rank} Q = n_1 = 1$ ($r = 2n - n_1 = 1$ and $d = m - n_1 = 1$), we have

$$P_Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P_{Q_r} = P_{Q_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$P_{Q^*} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad P_{Q_d^*} = P_{Q_1^*} = \frac{1}{2}\begin{pmatrix} 1 & 1 \end{pmatrix}.$$

Cauchy function for this problem has the form

$$K(t,\tau) = \begin{cases} \sin t \cos\tau - \cos t \sin\tau & \text{for} \quad 0 \le \tau \le t \le \pi, \\ \qquad 0 & \text{for} \quad 0 \le t < \tau \le \pi \end{cases}$$

and

$$lK(\cdot,\tau) = \begin{pmatrix} 0 \\ \sin\tau \end{pmatrix}, \quad X_r(t) = X(t)P_{Q_r} = \sin t\,.$$

By virtue of Theorem 2.1, the criterion of solvability of the generating boundary value problem for (11) takes the form

$$\frac{1}{2}\begin{pmatrix} 1 & 1 \end{pmatrix}\left\{\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} - \int_0^\pi \begin{pmatrix} 0 \\ -\sin\tau \end{pmatrix}\varphi(\tau)\,\mathrm{d}\tau\right\} = 0\,.$$

Thus, this problem is solvable if and only if $\alpha = \operatorname{col}(\alpha_1,\alpha_2) \in \mathbb{R}^2$ and $\varphi(t) \in C[0,\pi]$ satisfy the relation

$$\alpha_1 + \alpha_2 + \int_0^\pi \varphi(\tau)\sin\tau\,\mathrm{d}\tau = 0\,.$$

In this case, the generating boundary value problem for (11) has the one-parameter family ($r = 1$) of solutions

$$x_0(t,c_1) = c_1\sin t + (Gf)(t) + \frac{1}{2}\cos t\,(\alpha_1 - \alpha_2), \qquad c_1 \in \mathbb{R}\,, \tag{12}$$

where $(Gf)(t)$ is the generalized Green operator of the generating boundary value problem (11) of the form

$$\begin{aligned} (Gf)(t) &= \int_0^\pi \big[K(t,\tau) - X(t)Q^+ lK(\cdot,\tau)\big]\,\varphi(\tau)\,\mathrm{d}\tau \\ &= \int_0^t \left(\sin t\,\cos\tau - \frac{1}{2}\cos t\,\sin\tau\right)\varphi(\tau)\,\mathrm{d}\tau + \int_t^\pi \frac{1}{2}\cos t\,\sin\tau\varphi(\tau)\,\mathrm{d}\tau\,. \end{aligned}$$

The generating solution can be rewritten in the form

$$x_0(t, c_1) = c_1 \sin t + \bar{x}_0(t)\,,$$

where

$$\bar{x}_0(t) = (Gf)(t) + \frac{(\alpha_1 - \alpha_2)}{2} \cos t\,.$$

According to Theorem 3.1, the vector $c_1 = c_1^0 \in \mathbb{R}$ must satisfy the equation

$$F\left(c_1^0\right) = -\frac{1}{2} \begin{pmatrix} 1 & 1 \end{pmatrix} \int\limits_0^\pi \begin{pmatrix} 0 \\ \sin\tau \end{pmatrix} (\cos 2\tau)\, x_0\left(\tau, c_1^0\right) \,\mathrm{d}\tau = 0\,,$$

or

$$F\left(c_1^0\right) = -\frac{1}{2} \int\limits_0^\pi \sin\tau \cos 2\tau \left(c_1^0 \sin\tau + \bar{x}_0(\tau)\right) \,\mathrm{d}\tau = 0\,. \tag{13}$$

Thus, from the condition (9) we have

$$B_0 = -\frac{1}{2} \begin{pmatrix} 1 & 1 \end{pmatrix} \int\limits_0^\pi \begin{pmatrix} 0 \\ \sin\tau \end{pmatrix} \cos 2\tau \sin\tau \,\mathrm{d}\tau = \frac{\pi}{8} \neq 0\,.$$

The vector $c_1 = c_1^0 \in \mathbb{R}$ satisfying (13) has the form

$$c_1^0 = -\frac{4}{\pi} \int\limits_0^\pi \sin\tau \cos 2\tau\; \bar{x}_0(\tau) \,\mathrm{d}\tau\,.$$

Then the boundary value problem (11) possesses at least one solution $x(t,\varepsilon)$: $x(\cdot,\varepsilon) \in C^2[a,b]$, $x(t,\cdot) \in C[0,\varepsilon_0]$, which for $\varepsilon = 0$ turns into the generating solution $x_0(t, c_1^0)$ given by the relation (12) with constant c_1^0 and this solution can be found with help of the iterative process (10).

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Department of Applied Mathematics
Faculty of Science
Žilina University
Hurbanova 15
010 26 Žilina
SLOVAKIA
E-mail: martina.langerova@fpv.uniza.sk