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ON THE OSCILLATION OF SOLUTIONS OF SYSTEMS OF NEUTRAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we study systems of differential equations of neutral type. We present sufficient conditions for the oscillatory and asymptotic properties of solutions of these systems.

1. Introduction

We consider systems of neutral differential equations of the form:

$$\begin{aligned} \big(y_1(t) - p_1\, y_1(t-\tau_1)\big)' &= a_1(t)\, f_1\big(y_2(h_2(t))\big), \\ \big(y_2(t) - p_2\, y_2(t-\tau_2)\big)' &= a_2(t)\, f_2\big(y_3(h_3(t))\big), \\ y_3'(t) &= -a_3(t)\, f_3\big(y_1(h_1(t))\big), \ t \in R_+ = [0,\infty). \end{aligned} \tag{1}$$

The following conditions are assumed to hold without further mention:

(a) $\tau_i > 0$, $0 < p_i < 1$, $i = 1,2$;

(b) $a_i \in C(R_+, R_+)$, $i = 1,2,3$, a_3 is not identically zero on any subinterval $[T,\infty) \subset R_+$ and

$$\int^{\infty} a_j(t)\,\mathrm{d}t = \infty, \qquad j = 1,2;$$

(c) $h_i \in C(R_+, R)$ and

$$\lim_{t\to\infty} h_i(t) = \infty, \qquad i = 1,2,3;$$

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(d) $f_i \in C(R,R)$ are nondecreasing functions, $uf_i(u) > 0$ for $u \neq 0$, $i = 1,2,3$.

For any $y_i(t)$ we define $u_i(t)$ by

$$u_i(t) = y_i(t) - p_i\, y_i(t-\tau_i), \qquad i = 1,2. \tag{2}$$

Let $t_0 \geq 0$ be such that

$$t_1 = \min\left\{t_0 - \tau_i,\ \inf_{t \geq t_0} h_j(t),\ i = 1,2,\ j = 1,2,3\right\} \geq 0.$$

A vector function $\mathbf{y} = (y_1, y_2, y_3)$ is a solution of the system (1) if there exists a $t_0 \geq 0$ such that $\mathbf{y}$ is continuous on $[t_1, \infty)$, u_i, $i = 1,2$ and y_3 are continuously differentiable on $[t_0, \infty)$ and $\mathbf{y}$ satisfies the system (1) on $[t_0, \infty)$.

Denote by W the set of all solutions $\mathbf{y} = (y_1, y_2, y_3)$ of the system (1) which exist on some ray $[T_y, \infty) \subset R_+$ and satisfy

$$\sup\left\{\sum_{i=1}^{3} |\, y_i(t)| \,:\, t \geq T\right\} > 0 \qquad \text{for all} \quad T \geq T_y.$$

A solution $\mathbf{y} \in W$ is defined to be non-oscillatory if there exists a $T_y \geq 0$ such that its every component is different from zero for all $t \geq T_y$. Otherwise a solution $\mathbf{y} \in W$ is said to be oscillatory.

The oscillatory and asymptotic theory of systems of neutral differential equations has been developed by many authors (for example [1]–[8]). The purpose of this article is to obtain some oscillation criteria for the three-dimensional functional differential systems of neutral type (1).

We start with a preliminary analysis of the asymptotic behavior of possible non-oscillatory solution of (1).

Let $\mathbf{y} \in W$ be a non-oscillatory solution of the system (1). Then from (1), u_i, $i = 1,2$ are eventually monotone, so that u_i have to be of constant sign. Therefore, either

$$y_i(t)\, u_i(t) > 0, \qquad i \in \{1,2\} \tag{3}$$

or

$$y_i(t)\, u_i(t) < 0, \qquad i \in \{1,2\} \tag{4}$$

for all sufficiently large t. Denote by N^+ [or N^-] the set of components y_i, $i = 1,2$, of all non-oscillatory solutions $\mathbf{y} = (y_1, y_2, y_3)$ of the system (1) such that (3) [or (4)] is satisfied.

For the components y_i, $i = 1,2$, of the non-oscillatory solutions hold the following lemmas.

Lemma 1. ([4, Lemma 3]) *Let the condition* (a) *hold and* $y_i \in N^-$, $i \in \{1,2\}$. *Then* $\lim\limits_{t\to\infty} y_i(t) = 0$, $\lim\limits_{t\to\infty} u_i(t) = 0$, $i \in \{1,2\}$.

Lemma 2. ([3, Lemma 2]) *Let the condition (a) hold and* $y_i \in N^+,\ i \in \{1,2\}$. *If* $\lim\limits_{t\to\infty} u_i(t) = 0$, *then* $\lim\limits_{t\to\infty} y_i(t) = 0$, $i \in \{1,2\}$.

2. Main results

Theorem 1. *Suppose that the following assumptions hold:*

(A1) $h_3\big(h_2(h_1(t))\big) \le t$, h_i *are nondecreasing functions for* $i = 2,3$,

(A2) $x\,y\,f_i(x\,y) \ge K_i\,x\,y\,f_i(x)\,f_i(y)$, $0 < K_i =$*const.*, $i = 1,2,3$,

(A3)

$$\int\limits_0^{\alpha} \frac{\mathrm{d}u}{f_3\big(f_1(f_2(u))\big)} < \infty, \quad \int\limits_0^{-\alpha} \frac{\mathrm{d}u}{f_3\big(f_1(f_2(u)\big)} < \infty \quad \textit{for every constant} \quad \alpha > 0.$$

If

(A4)

$$\int\limits_{t_0}^{\infty} a_3(t)\,f_3\left(\int\limits_{t_0}^{h_1(t)} a_1(s)\,f_1\left(\int\limits_{t_0}^{h_2(s)} a_2(x)\,\mathrm{d}x\right)\mathrm{d}s\right)\mathrm{d}t \;=\; \infty,$$

(A5)

$$\int\limits_{t_0}^{\infty} a_2(t)\,f_2\left(\int\limits_{h_3(t)}^{\infty} a_3(s)\,\mathrm{d}s\right)\mathrm{d}t \;=\; \infty,$$

then every solution $\mathbf{y} \in W$ *of* (1) *is either oscillatory or* y_i, *i=1,2,3, tend to zero as* $t \to \infty$.

P r o o f. Let $\mathbf{y} \in W$ be a non-oscillatory solution of (1). Without loss of generality we may suppose that $y_1(t) > 0$ for $t \ge t_0$. Using the assumptions (b), (c) and (d), the third equation of (1) implies that y_3 is a nonincreasing function for $t \ge t_1$, where $t_1 \ge t_0$ is sufficiently large.

I. Let $y_3(t) < 0, t \ge t_2 \ge t_1$. From the second equation of (1) with regard to (b), (c), (d) we obtain that u_2 is a nonincreasing function for $t \ge t_3 \ge t_2$. In view of the monotonicity of y_3 we have $y_3(t) \le y_3(t_3)$, $t \ge t_3$. Using the conditions (c) and (d) we get $f_2\big(y_3(h_3(t))\big) \le f_2\big(y_3(t_3)\big) < 0$, $t \ge t_4 \ge t_3$. Multiplying the last inequality by $a_2(t) \ge 0$ and integrating the second equation of (1) from t_4 to t we are led to

$$u_2(t) - u_2(t_4) \le f_2\big(y_3(t_3)\big) \int_{t_4}^{t} a_2(s)\,\mathrm{d}s, \qquad t \ge t_4.$$

Hence in view of (b) we get $\lim_{t\to\infty} u_2(t) = -\infty$. That means $u_2(t) < 0$ for $t \ge t_5 \ge t_4$ and Lemma 1 implies that $y_2 \in N^+$.

Because $u_2(t) < 0$ and u_2 is a nonincreasing function, there exists a constant $A < 0$ such that $u_2(t) \le A$, $t \ge t_5$. From (2) we get $y_2(t) \le u_2(t)$, $t \ge t_6 \ge t_5$ and then $y_2(t) \le A$ for $t \ge t_6$. Using the assumptions (c), (d) and integrating the first equation of (1) from $t_7 \ge t_6$ to t we are led to

$$u_1(t) - u_1(t_7) \le f_1(A) \int_{t_7}^{t} a_1(s)\,\mathrm{d}s, \qquad t \ge t_7.$$

For $t \to \infty$ we obtain $u_1(t) \to -\infty$. That means $u_1(t) < 0$ for $t \ge t_8 \ge t_7$ and $y_1 \in N^-$ which is a contradiction with Lemma 1.

II. Let $y_3(t) > 0$, $t \ge t_2 \ge t_1$. Then u_2 is a nondecreasing function for $t \ge t_3 \ge t_2$ and either $u_2(t) > 0$ or $u_2(t) < 0$, $t \ge t_4 \ge t_3$.

II.1 Let $u_2(t) > 0$, $t \ge t_4$. Then there exists a constant $B > 0$ such that $B \le u_2(t)$, $t \ge t_4$. Lemma 1 implies that $y_2 \in N^+$ and from (2) we have $u_2(t) \le y_2(t)$, $t \ge t_5 \ge t_4$. Then $y_2(t) \ge B$, $t \ge t_5$. Using the assumptions (c), (d) and integrating the first equation of (1) from $t_6 \ge t_5$ to t we are led to

$$u_1(t) - u_1(t_6) \ge f_1(B) \int_{t_6}^{t} a_1(s)\,\mathrm{d}s, \qquad t \ge t_6.$$

Hence in view of (b) we get $\lim_{t\to\infty} u_1(t) = \infty$. That means $u_1(t) > 0$ for $t \ge t_7 \ge t_6$ and $y_1 \in N^+$.

Integrating the second equation of the system (1) from t_7 to t and using the inequalities $u_2(t) \le y_2(t)$, $f_2\big(y_3(h_3(s))\big) \ge f_2\big(y_3(h_3(t))\big)$ for $t_7 \le s \le t$ we have

$$y_2(t) \ge f_2\big(y_3(h_3(t))\big) \int_{t_7}^{t} a_2(s)\,\mathrm{d}s, \qquad t \ge t_7.$$

In view of assumptions (c), (d) and (A2) we obtain

$$f_1\bigl(y_2(h_2(t))\bigr) \geq K_1\, f_1\bigl(f_2(y_3(h_3(h_2(t)))))\bigr)\, f_1\left(\int\limits_{t_7}^{h_2(t)} a_2(s)\,\mathrm{d}s\right),$$

where $K_1 > 0$, $t \geq t_8 \geq t_7$. Multiplying by $a_1(t) \geq 0$, integrating from t_8 to t and using the inequality $y_1(t) \geq u_1(t) \geq u_1(t) - u_1(t_8)$ we get

$$y_1(t) \geq K_1 \int\limits_{t_8}^{t} a_1(u)\, f_1\bigl(f_2(y_3(h_3(h_2(u)))))\bigr)\, f_1\left(\int\limits_{t_7}^{h_2(u)} a_2(s)\,\mathrm{d}s\right)\mathrm{d}u, \qquad t \geq t_9 \geq t_8.$$

In regard to (A1), monotonicity of y_3 and (d) the inequality $u \leq t$ may be rewritten as $f_1\bigl(f_2(y_3(h_3(h_2(u)))))\bigr) \geq f_1\bigl(f_2(y_3(h_3(h_2(t)))))\bigr)$, $t \geq t_8$. Then

$$y_1(t) \geq K_1\, f_1\bigl(f_2(y_3(h_3(h_2(t)))))\bigr) \int\limits_{t_8}^{t} a_1(u)\, f_1\left(\int\limits_{t_7}^{h_2(u)} a_2(s)\,\mathrm{d}s\right)\mathrm{d}u, \qquad t \geq t_9$$

and in view of (c), monotonicity of f_3 and (A2) we are led to

$$\begin{aligned} & f_3\bigl(y_1(h_1(t))\bigr) \\ \geq\quad & K_4 f_3\bigl(f_1(f_2(y_3(h_3(h_2(h_1(t)))))))\bigr) f_3\left(\int\limits_{t_8}^{h_1(t)} a_1(u) f_1\left(\int\limits_{t_7}^{h_2(u)} a_2(s)\,\mathrm{d}s\right)\mathrm{d}u\right), \end{aligned}$$

where $t \geq t_{10} \geq t_9$, $K_4 = K_3^2 f_3(K_1) > 0$. Taking into account assumption (A1) and the monotonicity of functions y_3, f_i, $i = 1, 2, 3$ we have

$$f_3\bigl(f_1(f_2(y_3(h_3(h_2(h_1(t)))))))\bigr) \geq f_3\bigl(f_1(f_2(y_3(t)))\bigr), \qquad t \geq t_{10}.$$

Therefore

$$f_3\bigl(y_1(h_1(t))\bigr) \geq K_4 f_3\bigl(f_1(f_2(y_3(t)))\bigr)\, f_3\left(\int\limits_{t_8}^{h_1(t)} a_1(u) f_1\left(\int\limits_{t_7}^{h_2(u)} a_2(s)\mathrm{d}s\right)\mathrm{d}u\right) \tag{5}$$

for $t \geq t_{10}$. Multiplying by $\frac{a_3(t)}{f_3\big(f_1(f_2(y_3(t)))\big)} \geq 0$ and in regard to third equation of (1) we get

$$\frac{-y_3'(t)}{f_3\big(f_1(f_2(y_3(t)))\big)} \geq K_4\, a_3(t) f_3\left(\int\limits_{t_8}^{h_1(t)} a_1(u) f_1\left(\int\limits_{t_7}^{h_2(u)} a_2(s)\mathrm{d}s\right)\mathrm{d}u\right), \qquad t \geq t_{10}.$$

Integrating the last inequality from t_{10} to t and using the substitution $w = y_3(z)$ we are led to

$$\int\limits_{y_3(t)}^{y_3(t_{10})} \frac{\mathrm{d}w}{f_3\big(f_1(f_2(w))\big)} \geq K_4 \int\limits_{t_{10}}^{t} a_3(z) f_3\left(\int\limits_{t_8}^{h_1(z)} a_1(u) f_1\left(\int\limits_{t_7}^{h_2(u)} a_2(s)\,\mathrm{d}s\right)\mathrm{d}u\right)\mathrm{d}z$$

for $t \geq t_{10}$. Hence in view of (A3) and (A4) we get a contradiction for $t \to \infty$.

II.2 We assume that $u_2(t) < 0,\ t \geq t_4$ and we shall consider the next subcases:

a) Let $y_2 \in N^-,\ y_1 \in N^-$. Then Lemma 1 implies that $\lim\limits_{t\to\infty} y_i(t) = 0$, $i = 1, 2$.

The function y_3 is nonincreasing and positive so there exists $\lim\limits_{t\to\infty} y_3(t) = C \geq 0$. We shall show that $C = 0$. Suppose the contrary. Then $y_3(t) \geq C$ and using the conditions (c), (d) we get $f_2\big(y_3(h_3(t))\big) \geq f_2(C)$ for $t \geq t_4$. Multiplying the last inequality by $a_2(t) \geq 0$ and integrating the second equation of (1) from t_4 to t we have

$$u_2(t) - u_2(t_4) \geq f_2(C) \int\limits_{t_4}^{t} a_2(s)\,\mathrm{d}s, \qquad t \geq t_4.$$

Hence in view of (b) we get $\lim\limits_{t\to\infty} u_2(t) = \infty$ which contradicts the negativity of u_2. Therefore $\lim\limits_{t\to\infty} y_3(t) = 0$.

b) Let $y_2 \in N^-,\ y_1 \in N^+$. Then $u_1(t) > 0$ and using the assumptions (b), (c) and (d), the first equation of (1) implies that u_1 is a nondecreasing function for $t \geq t_5 \geq t_4$. This means that there exists a constant $D > 0$ such that $u_1(t) \geq D$ for $t \geq t_6 \geq t_5$ and from (2) we obtain $u_1(t) \leq y_1(t),\ t \geq t_7 \geq t_6$. Then $y_1(t) \geq D,\ t \geq t_7$ and by virtue of (c), (d) we get $f_3\big(y_1(h_1(t))\big) \geq f_3(D)$ for $t \geq t_8 \geq t_7$. Multiplying the last inequality by $-a_3(t) \leq 0$ and integrating the third equation of (1) from $h_3(t)$ to ∞, we are led to

$$y_3\big(h_3(t)\big) \geq f_3(D) \int\limits_{h_3(t)}^{\infty} a_3(s)\,\mathrm{d}s, \qquad t \geq t_9 \geq t_8.$$

In view of (d) and (A2) we get

$$f_2\big(y_3(h_3(t))\big) \geq K_5\, f_2\left(\int\limits_{h_3(t)}^{\infty} a_3(s)\,\mathrm{d}s\right),$$

where $0 < K_5 = K_2 f_2\big(f_3(D)\big),\ t \geq t_9$. Multiplying the last inequality by $a_2(t) \geq 0$ and integrating the second equation of (1) from t_9 to t we obtain

$$u_2(t) - u_2(t_9) \geq K_5 \int\limits_{t_9}^{t} a_2(u)\, f_2\left(\int\limits_{h_3(u)}^{\infty} a_3(s)\,\mathrm{d}s\right)\mathrm{d}u, \qquad t \geq t_9.$$

Hence in regard to (A5) we get $\lim\limits_{t\to\infty} u_2(t) = \infty$ which contradicts the negativity of u_2. This case cannot occur.

c) Let $y_2 \in N^+$. The assumptions (b), (c), (d) and the first equation of (1) imply that u_1 is a nonincreasing function for $t \geq t_5 \geq t_4$. Then in view of Lemma 1 we have that $y_1 \in N^+$. On the interval $[t_5, \infty)$, there holds:

- u_1 is a nonincreasing function and $u_1(t) > 0$,
- u_2 is a nondecreasing function and $u_2(t) < 0$,
- y_3 is a nonincreasing function and $y_3(t) > 0$.

Therefore there exist $\lim\limits_{t\to\infty} y_3(t) = E \geq 0$, $\lim\limits_{t\to\infty} u_2(t) = F \leq 0$ and $\lim\limits_{t\to\infty} u_1(t) = G \geq 0$. We shall show that $E = 0, F = 0$ and $G = 0$.

i) Let $E > 0$. Then $y_3(t) \geq E,\ t \geq t_5$ and with regard to (c), (d) we get $f_2\big(y_3 = (h_3(t))\big) \geq f_2(E)$ for $t \geq t_6 \geq t_5$. Multiplying the last inequality by $a_2(t) \geq 0$ and integrating the second equation of (1) from t_6 to t we have

$$u_2(t) - u_2(t_6) \geq f_2(E) \int\limits_{t_6}^{t} a_2(s)\,\mathrm{d}s, \qquad t \geq t_6.$$

Hence in view of (b) we get $\lim\limits_{t\to\infty} u_2(t) = \infty$ which contradicts the negativity of u_2. This means that $\lim\limits_{t\to\infty} y_3(t) = 0$.

ii) We assume that $F < 0$. Then $u_2(t) \leq F,\ t \geq t_5$ and from (2) we have $y_2(t) \leq u_2(t)$ for $t \geq t_6 \geq t_5$. Therefore $y_2(t) \leq F,\ t \geq t_6$ and using the conditions (c), (d) we obtain $f_1\big(y_2(h_2(t))\big) \leq f_1(F),\ t \geq t_7 \geq t_6$. Multiplying by $a_1(t) \geq 0$ and integrating the first equation of (1) from t_7

to t we have

$$u_1(t) - u_1(t_7) \le f_1(F) \int\limits_{t_7}^{t} a_1(s)\,\mathrm{d}s, \qquad t \ge t_7.$$

Letting $t \to \infty$ and with regard to (b) we get $\lim\limits_{t\to\infty} u_1(t) = -\infty$ which is a contradiction with the positivity of u_1. This means that $\lim\limits_{t\to\infty} u_2(t) = 0$ and Lemma 2 implies that $\lim\limits_{t\to\infty} y_2(t) = 0$.

iii) Let $G > 0$. Then $u_1(t) \ge G,\ t \ge t_5$ and from (2) we have $y_1(t) \ge u_1(t)$ for $t \ge t_6 \ge t_5$. Therefore $y_1(t) \ge G$, $t \ge t_6$ and using the conditions (c), (d) we obtain $f_3\big(y_1(h_1(t))\big) \ge f_3(G)$, $t \ge t_7 \ge t_6$. Multiplying the last inequality by $-a_3(t) \le 0$ and integrating the third equation of (1) from $h_3(t)$ to ∞ we are led to

$$y_3\big(h_3(t)\big) \ge f_3(G) \int\limits_{h_3(t)}^{\infty} a_3(s)\,\mathrm{d}s, \qquad t \ge t_8 \ge t_7.$$

Proceeding in the same way as in the case II.2.b we get the inequality

$$u_2(t) - u_2(t_8) \ge K_6 \int\limits_{t_8}^{t} a_2(u)\, f_2\left(\int\limits_{h_3(u)}^{\infty} a_3(s)\,\mathrm{d}s\right) \mathrm{d}u,$$

where $0 < K_6 = K_2 f_2\big(f_3(G)\big),\ t \ge t_8$. Hence in regard to (A5) we get $\lim\limits_{t\to\infty} u_2(t) = \infty$ which contradicts to the negativity of u_2. This means that $\lim\limits_{t\to\infty} u_1(t) = 0$ and Lemma 2 implies that $\lim\limits_{t\to\infty} y_1(t) = 0$. $\square$

Remark 1. For $p_2 = 0$ Theorem 1 gives a result proved in [7].

Theorem 2. *Assume that* (A1), (A2) *and* (A5) *hold and in addition*

(A6) $f_3\big(f_1(f_2(x))\big) = x$,

(A7)

$$\int\limits_{t_0}^{\infty} a_3(t) \left[f_3\left(\int\limits_{t_0}^{h_1(t)} a_1(s)\, f_1\left(\int\limits_{t_0}^{h_2(s)} a_2(x)\,\mathrm{d}x \right) \mathrm{d}s \right)\right]^{1-\epsilon} \mathrm{d}t \;=\; \infty, \quad 0 < \epsilon < 1.$$

Then the conclusion of Theorem 1 holds.

P r o o f. Suppose that $\mathbf{y} \in W$ is a non-oscillatory solution of (1) and $y_1(t) > 0$ for $t \geq t_0$. We can proceed exactly as in the proof of Theorem 1. We shall discuss only the possibility II.1. The proofs of cases I., II.2.a, II.2.b and II.2.c are the same.

II.1 Proceeding in the same way as in the case II.1 in the proof of the previous theorem we get the inequality (5). Hence in view of (A6) and raising to the power of $1-\epsilon$ we have

$$\bigl(f_3(y_1(h_1(t)))\bigr)^{1-\epsilon} \geq \left[K_4 y_3(t) f_3\left(\int\limits_{t_8}^{h_1(t)} a_1(u) f_1\left(\int\limits_{t_7}^{h_2(u)} a_2(s)\,\mathrm{d}s\right)\mathrm{d}u\right)\right]^{1-\epsilon} \tag{6}$$

for $t \geq t_{10}$. From the proof of Theorem 1 we know that $y_1 \in N^+$, $y_2 \in N^+$ and such u_1 is a nondecreasing function. Then $u_1\bigl(h_1(t)\bigr) \geq u_1(t_7)$, $t \geq t_{11} \geq t_{10}$ and from (2) we have $y_1(t) \geq u_1(t)$ for $t \geq t_{12} \geq t_{11}$. Therefore $y_1\bigl(h_1(t)\bigr) \geq u_1(t_7)$, $t \geq t_{12}$ and using the condition (d) we obtain $f_3\bigl(y_1(h_1(t))\bigr) \geq f_3\bigl(u_1(t_7)\bigr)$, $t \geq t_{12}$. Hence

$$\bigl(f_3(y_1(h_1(t)))\bigr)^{1-\epsilon} \leq K_7 f_3\bigl(y_1(h_1(t))\bigr),$$

where $0 < K_7 = \bigl(f_3(u_1(t_7))\bigr)^{-\epsilon}$, $t \geq t_{12}$. Combining the last inequality and (6) we are led to

$$K_7 f_3\bigl(y_1(h_1(t))\bigr) \geq \bigl[K_4 y_3(t)\bigr]^{1-\epsilon}\left[f_3\left(\int\limits_{t_8}^{h_1(t)} a_1(u) f_1\left(\int\limits_{t_7}^{h_2(u)} a_2(s)\,\mathrm{d}s\right)\mathrm{d}u\right)\right]^{1-\epsilon}$$

for $t \geq t_{12}$. Multiplying by $-a_3(t)\bigl[K_4 y_3(t)\bigr]^{\epsilon-1}$ and using the third equation of (1) we have

$$K_8\, y_3'(t)\bigl(y_3(t)\bigr)^{\epsilon-1} \leq -a_3(t)\left[f_3\left(\int\limits_{t_8}^{h_1(t)} a_1(u) f_1\left(\int\limits_{t_7}^{h_2(u)} a_2(s)\,\mathrm{d}s\right)\mathrm{d}u\right)\right]^{1-\epsilon}, \tag{7}$$

where $K_8 = K_7 K_4^{\epsilon-1}$, $t \geq t_{12}$. Integrating (7) from t_{12} to t and letting $t \to \infty$ we get a contradiction with (A7). The proof of Theorem 2 is complete. □

Theorem 3. *Let* $(A2), (A3), (A5)$ *hold and moreover, let*

(A8) $h_2(t) \geq t$, $h_3(t) \leq t$,

(A9)

$$\int_{t_0}^{\infty} a_3(t)\, f_3\left(\int_{t_0}^{g(t)} a_1(s)\, f_1\left(\int_{t_0}^{s} a_2(x)\,\mathrm{d}x\right)\mathrm{d}s\right)\mathrm{d}t = \infty, \quad g(t) = \min\{t, h_1(t)\}.$$

Then the conclusion of Theorem 1 *holds.*

P r o o f. Assume that $\mathbf{y} \in W$ is a non-oscillatory solution of (1) and $y_1(t) > 0$ for $t \geq t_0$. Proceeding in the same way as in the proof of Theorem 2 we consider only the case II.1.

II.1 For $t \geq t_7$ there holds:

- y_3 is a nonincreasing function and $y_3(t) > 0$,
- u_2 is a nondecreasing function, $u_2(t) > 0$ and $y_2(t) > 0$,
- u_1 is a nondecreasing function, $u_1(t) > 0$ and $y_1(t) > 0$.

In view of (A8) we have $h_2(t) \geq t$, $t \geq t_7$. Using the monotonicity of u_2, (2) and (d) we get $f_1\big(y_2(h_2(t))\big) \geq f_1\big(u_2(t)\big)$, $t \geq t_8 \geq t_7$. Then from the first equation of (1) we obtain

$$u_1'(t) \geq a_1(t) f_1\big(u_2(t)\big), \qquad t \geq t_8.$$

In regard to (A8) we have $h_3(t) \leq t$, $t \geq t_7$. Using the monotonicity of y_3 and (d) we get $f_2\big(y_3(h_3(t))\big) \geq f_2\big(y_3(t)\big)$, $t \geq t_9 \geq t_8$. Then from the second equation of (1) we obtain

$$u_2'(t) \geq a_2(t) f_2\big(y_3(t)\big), \qquad t \geq t_9.$$

The definitions of functions g, u_1 and the monotonicity of u_1, f_3 imply that $f_3\big(y_1(h_1(t))\big) \geq f_3\big(u_1(g(t))\big)$, $t \geq t_{10} \geq t_9$. Multiplying the last inequality by $-a_3(t) \leq 0$ and from the third equation of (1) we are led to

$$y_3'(t) \leq -a_3(t) f_3\big(u_1(g(t))\big), \qquad t \geq t_{10}.$$

Then the system (1) may be rewritten into the form

$$\begin{aligned} u_1'(t) &\geq a_1(t) f_1\big(u_2(t)\big), \\ u_2'(t) &\geq a_2(t) f_2\big(y_3(t)\big), \\ y_3'(t) &\leq -a_3(t) f_3\big(u_1(g(t))\big), \qquad t \geq t_{10}. \end{aligned} \tag{8}$$

Integrating the first inequality of (8) from t_{10} to t we obtain

$$u_1(t) \geq u_1(t) - u_1(t_{10}) \geq \int_{t_{10}}^{t} a_1(s) f_1\big(u_2(s)\big)\,\mathrm{d}s, \qquad t \geq t_{10}. \tag{9}$$

Integrating the second inequality of (8) from t_{10} to s we have

$$u_2(s) \geq u_2(s) - u_2(t_{10}) \geq \int_{t_{10}}^{s} a_2(x) f_2\big(y_3(x)\big)\,\mathrm{d}x, \qquad s \geq t_{10}.$$

Using $f_2\big(y_3(x)\big) \geq f_2\big(y_3(s)\big)$, monotonicity of f_1 and (A2) we get

$$f_1\big(u_2(s)\big) \geq K_1\, f_1\big(f_2(y_3(s))\big)\, f_1\left(\int_{t_{10}}^{s} a_2(x)\,\mathrm{d}x\right), \qquad K_1 > 0, \quad s \geq t_{10}. \tag{10}$$

Combining (9) and (10) we obtain

$$u_1(t) \geq K_1 \int_{t_{10}}^{t} a_1(s) f_1\big(f_2(y_3(s))\big)\, f_1\left(\int_{t_{10}}^{s} a_2(x)\,\mathrm{d}x\right)\mathrm{d}s, \qquad t \geq t_{10}.$$

The definition of g implies that

$$u_1\big(g(t)\big) \geq K_1 \int_{t_{10}}^{g(t)} a_1(s) f_1\big(f_2(y_3(s))\big)\, f_1\left(\int_{t_{10}}^{s} a_2(x)\,\mathrm{d}x\right)\mathrm{d}s, \qquad t \geq t_{11} \geq t_{10}.$$

Combining $t_{10} \leq s \leq g(t)$ and $g(t) \leq t$, using the monotonicity of y_3 and (d) we get $f_1\big(f_2(y_3(s))\big) \geq f_1\big(f_2(y_3(t))\big)$. Therefore

$$u_1\big(g(t)\big) \geq K_1\, f_1\big(f_2(y_3(t))\big) \int_{t_{10}}^{g(t)} a_1(s)\, f_1\left(\int_{t_{10}}^{s} a_2(x)\,\mathrm{d}x\right)\mathrm{d}s, \qquad t \geq t_{11}$$

and in view of (d) and (A2) we are led to

$$f_3\big(u_1(g(t))\big) \geq K_9\, f_3\big(f_1(f_2(y_3(t)))\big)\, f_3\left(\int_{t_{10}}^{g(t)} a_1(s)\, f_1\left(\int_{t_{10}}^{s} a_2(x)\,\mathrm{d}x\right)\mathrm{d}s\right), \tag{11}$$

where $0 < K_9 = K_3^2 f_3(K_1)$, $t \geq t_{11}$. Multiplying by $\frac{-a_3(t)}{f_3\big(f_1(f_2(y_3(t)))\big)} \leq 0$ and in regard to the third inequality of (8) we get

$$\frac{y_3'(t)}{f_3\big(f_1(f_2(y_3(t)))\big)} \leq -K_9\, a_3(t) f_3\left(\int_{t_{10}}^{g(t)} a_1(s)\, f_1\left(\int_{t_{10}}^{s} a_2(x)\,\mathrm{d}x\right)\mathrm{d}s\right),$$
$$t \geq t_{11}.$$

Integrating the last inequality from t_{11} to t and using the substitution $w = y_3(u)$ we obtain

$$\int\limits_{y_3(t)}^{y_3(t_{11})} \frac{\mathrm{d}w}{f_3\big(f_1(f_2(w))\big)} \geq K_9 \int\limits_{t_{11}}^{t} a_3(u) f_3 \left(\int\limits_{t_{10}}^{g(u)} a_1(s)\, f_1 \left(\int\limits_{t_{10}}^{s} a_2(x)\, \mathrm{d}x \right) \mathrm{d}s \right) \mathrm{d}u, \qquad t \geq t_{11}.$$

Hence in view of (A3) and (A9) we get a contradiction for $t \to \infty$. This case cannot occur and the proof of Theorem 3 is complete. □

Theorem 4. *Suppose that* $(A2), (A5), (A6), (A8)$ *hold and in addition*

(A10)

$$\int\limits_{t_0}^{\infty} a_3(t) \left[f_3 \left(\int\limits_{t_0}^{g(t)} a_1(s)\, f_1 \left(\int\limits_{t_0}^{s} a_2(x)\, \mathrm{d}x \right) \mathrm{d}s \right) \right]^{1-\epsilon} \mathrm{d}t \;=\; \infty,$$

where $g(t) = \min\{t, h_1(t)\}$, $0 < \epsilon < 1$. *Then the conclusion of* Theorem 1 *holds.*

We can prove Theorem 4 analogously to Theorem 2 and Theorem 3.

Remark 2. Theorems 2–4 extend results of V. N. Shevelo, N. V. Varech and A. G. Gritsai in paper [6]. Since for $p_i = 0$, $i = 1, 2$, the system (1) reduces to a three-dimensional system of differential equations with deviating argument.

REFERENCES

[1] GYÖRI, I.—LADAS, G.: *Oscillation Theory Of Delay Differential Equations*. Clarendon Press, Oxford, 1991.

[2] MARUŠIAK, P.: *Oscillatory properties of functional differential systems of neutral type*, Czechoslovak Math. J. **43** (1993), 649–662.

[3] MIHALÍKOVÁ, B.: *A note on the asymptotic properties of systems of neutral differential equations*, Stud. Univ. Žilina, Math. Phys. Ser. **13** (2001), 133–139.

[4] MIHALÍKOVÁ, B.: *Some properties of neutral differential systems equations*, Boll. Unione Mat. Ital. Sez B Artic. Ric. Mat. (8) **5** (2002), 279–287.

[5] MIHÁLY, T.: *On the oscillatory and asymptotic properties of solutions of systems of neutral differential equations*, Nonlinear Anal.: Theory, Methods and Applications **66** (2007), 2053–2063.

[6] SHEVELO, V. N.—VARECH, N. V.—GRITSAI, A. G.: *Oscillatory properties of solutions of systems of differential equations with deviating arguments*, preprint no. 85.10, Institute of Mathematics of the Ukrainian Academy of Sciences, 1985. (In Russian)

[7] ŠPÁNIKOVÁ, E.: *Oscillatory properties of solutions of three-dimensional differential systems of neutral type*, Czechoslovak Math. J. **50** (2000), 879–887.

[8] ŠPÁNIKOVÁ, E.—ŠAMAJOVÁ, H.: *Asymptotic behaviour of three-dimensional functional differential systems of neutral type.* In: Proceeding of the 4th International Conference – APLIMAT '05 (M. Kováčová, ed.), Slovak University of Technology, Bratislava, 2005, pp. 147–152.

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