Tatra Mt. Math. Publ. **38** (2007), 273–281

TATRA MOUNTAINS
Mathematical Publications

BOUNDED SOLUTIONS OF DYNAMIC EQUATIONS ON COMBINED TIME SCALES

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ABSTRACT. In this paper we discuss asymptotic behavior of solutions of a dynamic equation

$$u^{\Delta}(t) = f\big(t, u(t)\big),$$

where $f\colon \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ and $\mathbb{T}$ is a combined time scale with zero and nonzero graininess function. We formulate a principle which gives the guarantee that the graph of at least one solution of above mentioned equation stays in the prescribed domain. This principle uses the idea of the retraction method.

1. Introduction

1.1. The time scale calculus

Stefan Hilger initiated in [7], [8] the calculus of time scales in order to create a theory that unifies discrete and continuous analysis. We give some necessary definitions and theorems of the time scales calculus at first. This theoretical background is taken from [1].

Definition 1. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers.

Definition 2. Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$ we define the *forward jump operator* $\sigma\colon \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} \colon s > t\},$$

while the *backward jump operator* $\rho\colon \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} \colon s < t\}.$$

2000 Mathematics Subject Classification: 39A10, 39A11.
Keywords: time scale, asymptotic behavior of solution, retract, retraction.
The research was supported by Grant No 1/3238/06 of the Grant Agency of Slovak Republic (VEGA).

Definition 3. A point t is called *right-scattered* if $\sigma(t) > t$, while if $\rho(t) < t$ we say that t is *left-scattered.* Points that are right-scattered and left-scattered at the same time are called *isolated.*

Definition 4. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then a point t is called *right-dense* and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called *left-dense.* Points that are right-dense and left-dense at the same time are called *dense.*

Definition 5. The function $\mu\colon \mathbb{T} \to [0, \infty)$ which is defined by $\mu(t) := \sigma(t) - t$ is called the *graininess function.*

We will need the set $\mathbb{T}^\kappa$ which is derived from the time scale $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} \setminus \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. It means

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if} \quad \sup \mathbb{T} < \infty, \\ \mathbb{T} & \text{if} \quad \sup \mathbb{T} = \infty. \end{cases}$$

Definition 6. Assume $u\colon \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. We say that u is *delta differentiable* on $\mathbb{T}^\kappa$ if limit

$$u^\Delta(t) = \lim_{s \to t,\, s \in \mathbb{T}} \frac{u\big(\sigma(t)\big) - u(s)}{\sigma(t) - s}$$

exists for all $t \in \mathbb{T}^\kappa$. We call $u^\Delta(t)$ the *delta derivative* of u at t.

1.2. Class of admissible time-scales $\mathbb{T}^*$

We use the following notation: for integers s, q, $s \le q$ we define $\mathbb{Z}_s^q := \{s, s+1, \dots, q\}$, where $s = -\infty$ or $q = \infty$ are admitted, too. Throughout this paper, using notation $\mathbb{Z}_s^q$, perhaps with other sub- or superscript, we suppose $s \le q$.

Now we define a class of time scales suitable for our investigation. The reason for such restriction is that the method presented does not work on arbitrary time scales. More exactly, our approach cannot be applied, e.g., to time scales $\mathbb{T}$ containing points t^* such that the intersection $(t^* - \delta, t^*) \cap \mathbb{T}$ is (for a positive δ) an increasing sequence with limit t^* and $\mu(t^*) = 0$.

Let us consider an increasing sequence of real numbers $\{t_n\}$, $n \in \mathbb{Z}_0^\infty$. We define a set $\mathcal{T}$ as empty set, or as a finite set of elements of the sequence of $\{t_n\}$ or as a subsequence of $\{t_n\}$, i.e.,

$$\mathcal{T} := \begin{cases} \emptyset, \text{ or} \\ \{t_{n_1}, t_{n_2}, \dots, t_{n_{k^*}}\}, k^* \in \mathbb{Z}_0^\infty \text{ with } t_{n_k} \in \{t_n\}, k \in \mathbb{Z}_0^{k^*}, n_1 < n_2 < \dots \\ \qquad\qquad \dots < n_{k^*}, \text{ or a subsequence } \{t_{n_k}\}, k \in \mathbb{Z}_0^\infty \text{ of } \{t_n\}. \end{cases}$$

We construct the closed interval $I_{n_k} = [t_{n_k}, t_{n_k+1}]$ for every $t_{n_k} \in \mathcal{T}$ and we define a combined time-scale

$$\mathbb{T}^* := \{t_n\} \cup \left(\bigcup_{t_{n_k} \in \mathcal{T}} I_{n_k} \right).$$

We illustrate possible forms of $\mathbb{T}^*$ on three examples.

Example 1. Let $\{t_n\} = \{n\} = \{0, 1, 2, \dots\}$, $n \in \mathbb{Z}_0^\infty$ and $n_k = 2^k$, $k \in \mathbb{Z}_0^\infty$. We set $\mathcal{T} = \{t_{n_k}\} = \{2^k\} = \{1, 2, 4, 8, 16, 32, \dots, 2^k, \dots\}$. Then the combined time scale

$$\mathbb{T}^* = \{0\} \cup [1, 3] \cup [4, 5] \cup \{6\} \cup \{7\} \cup [8, 9] \cup \{10\} \cup \{11\} \cup \dots$$

is a time-scale with zero and nonzero graininess function.

Example 2. Let $\{t_n\} = \{2n\} = \{0, 2, 4, 6, 8, \dots\}$, $n \in \mathbb{Z}_0^\infty$ and $\mathcal{T} = \emptyset$. Then the combined time scale

$$\mathbb{T}^* = \{t_n\} = \{0, 2, 4, 6, 8, \dots\}.$$

This time scale contains only isolated points, it means the graininess function is nonzero always.

Example 3. Let $\{t_n\} = \{4n\} = \{0, 4, 8, 12, 16, \dots\}$, $n \in \mathbb{Z}_0^\infty$ and $n_k = 4k$, $k \in \mathbb{Z}_0^\infty$. We set $\mathcal{T} = \{t_{n_k}\} = \{t_k\} = \{0, 4, 8, 12, 16 \dots\}$. The combined time scale $\mathbb{T}^* = [0, \infty)$. The graininess function of this time scale is zero always.

1.3. Problem under consideration

Let us consider the equation

$$u^\Delta = f(t, u), \tag{1}$$

where $f \colon \mathbb{T}^* \times \mathbb{R} \to \mathbb{R}$. With respect to f we suppose that it is continuous with respect to the second argument at every isolated point $t \in \mathbb{T}^*$ and is continuous and satisfies local Lipschitz condition in a neighbourhod of every point $(t, u) \in \mathbb{T}^* \times \mathbb{R}$ with non-isolated point t (we consider right or left neighbourhoods if t is right or left boundary point of a closed interval of $\mathbb{T}^*$). The equation (1) is changed to the differential equation

$$u' = f(t, u),$$

if $t \in \mathbb{T}^*$ and $\mu(t) = 0$, and to the discrete equation

$$\Delta u = f(t, u)$$

if $t \in \mathbb{T}^*$ is isolated or left-dense and right-scattered point.

We consider the initial problem (1), (2), where

$$u(t_0) = u^0 \in \mathbb{R}. \tag{2}$$

Due to above conditions there exists a unique solution of this problem. This solution is continuable forward and continuously depends on initial value u^0. Similarly, initial problem (1), (3), where

$$u(t_s) = u^s \in \mathbb{R}, \tag{3}$$

$s \in \mathbb{Z}_0^\infty$ is fixed and $t_s \in \mathbb{T}^*$ has a unique solution which is continuable forward and continuously depends on u^s.

Let $b, c\colon \mathbb{T}^* \to \mathbb{R}$ be delta differentiable on $\mathbb{T}^*$ functions such that $b(t) < c(t)$ for each $t \in \mathbb{T}^*$. We define a subset $\omega \subset \mathbb{T}^* \times \mathbb{R}$ as

$$\omega := \big\{(t,u)\colon t \in \mathbb{T}^*, b(t) < u < c(t)\big\},$$

the closure $\bar\omega$ is defined as

$$\bar\omega := \big\{(t,u)\colon t \in \mathbb{T}^*, b(t) \le u \le c(t)\big\}$$

and the boundary $\partial\omega$ as

$$\partial\omega := \big\{(t,u)\colon t \in \mathbb{T}^*, u = b(t) \quad \text{or} \quad u = c(t)\big\}.$$

Our aim is to establish the sufficient conditions to the right-hand side of equation (1) in order to guarantee the existence of at least one solution of (1) defined on $\mathbb{T}^*$ such that $\big(t, u(t)\big) \in \omega$ for each $t \in \mathbb{T}^*$.

1.4. Points of strict egress

We define two auxiliary functions

$$\begin{aligned} B(t,u) &:= -u + b(t),\\ C(t,u) &:= u - c(t) \end{aligned}$$

on $\mathbb{T}^* \times \mathbb{R}$ and divide the boundary $\partial\omega$ into two subsets $\omega_B \subset \mathbb{T}^* \times \mathbb{R}$, $\omega_C \subset \mathbb{T}^* \times \mathbb{R}$, where

$$\begin{aligned} \omega_B &:= \big\{(t,u) : t \in \mathbb{T}^*, B(t,u) = 0\big\},\\ \omega_C &:= \big\{(t,u) : t \in \mathbb{T}^*, C(t,u) = 0\big\}. \end{aligned}$$

Definition 7. The point $M \in \partial\omega$ is called the point of strict egress for the set ω with respect to the equation (1) if $B^\Delta(M) > 0$ in the case $M \in \omega_B$ and $C^\Delta(M) > 0$ in the case $M \in \omega_C$.

Remark 1. The geometrical sense of the notion of a point of strict egress is evident. If a point $(\widetilde{t}, u) \in \omega_B$ is a point of strict egress for the set ω with respect to (1), then $\big(t, u(t)\big) \notin \overline{\omega}$ for $t \in (\widetilde{t}, \widetilde{t} + \delta)$ with a small positive δ if $\widetilde{t}$ is right-dense point and for $t = \sigma(\widetilde{t})$ if $\widetilde{t}$ is right-scattered. If $(\widetilde{t}, u) \in \omega_C$ is a point of strict egress for the set ω with respect to (1), then $\big(t, u(t)\big) \notin \overline{\omega}$ for $t \in (\widetilde{t}, \widetilde{t} + \delta)$ with a small positive δ if $\widetilde{t}$ is right-dense point and for $t = \sigma(\widetilde{t})$ if $\widetilde{t}$ is right-scattered.

2. Main Result

The idea of the proof of the following theorem is so called the retract idea which is well-known in the field of the ordinary differential equations (see, e.g., [10]). For the readers convenience we recall the definition of a retract and a retraction.

Definition 8. [9] If $A \subset B$ are subsets of a topological space and $\pi : B \to A$ is a continuous mapping from B onto A such that $\pi(p) = p$ for every $p \in A$, then π is said to be a *retraction* of B onto A. When there exists a retraction of B onto A, A is called a *retract* of B.

An analogy of the retract method in the case of discrete equations with unit graininess function is given, e.g., in [4]–[6]. We suppose that the statement of below formulated theorem is not valid. In this situation we prove that there exists a retraction of a segment $[\alpha, \beta]$ with $\alpha < \beta$ onto two-point set α, β. But it is well known that the boundary of n-dimensional ball can not be its retract (see [3]). So, in our case it is such a retractive mapping which cannot exist because it is incompatible with continuity. Below we suppose that the function f except the indicated conditions satisfies all assumptions given in Part 1.3.

Theorem 1. *Let $f\colon \mathbb{T}^* \times \mathbb{R} \to \mathbb{R}$. Let $b, c\colon \mathbb{T}^* \to \mathbb{R}$ be delta differentiable functions such that $b(t) < c(t)$ for each $t \in \mathbb{T}^*$. If moreover every point $M \in \omega_B \cup \omega_C$ is the point of strict egress for the set ω with respect to the equation* (1)*, then there exists a value $u^* \in \big(b(t_0), c(t_0)\big)$ such that the initial problem*

$$u(t_0) = u^*, \tag{4}$$

defines a solution $u = u^*(t)$ of equation (1) *satisfying*

$$b(t) < u^*(t) < c(t) \tag{5}$$

for every $t \in \mathbb{T}^*$.

P r o o f. At the beginning of this proof we suppose that a value u^* satisfying the inequality

$$b(t_0) < u^* < c(t_0),$$

and generating the solution

$$u = u^*(t),\ u(t_0) = u^*,$$

which satisfies (5) for any $t \in \mathbb{T}^*$, does not exist. It means that for any $\hat{u}$ satisfying the inequality

$$b(t_0) < \hat{u} < c(t_0),$$

there exists $\hat{t} \in \mathbb{T}^*$, $\hat{t} > t_0$ such that for corresponding solution $u = \hat{u}(t)$ of the initial problem $\hat{u}(t_0) = \hat{u}$ we have

$$\big(\hat{t}, \hat{u}(\hat{t}\,)\big) \notin \omega$$

and for all $\hat{t}_p \in \mathbb{T}^*$, $\hat{t}_p < \hat{t}$

$$\big(\hat{t}_p, \hat{u}(\hat{t}_p)\big) \in \omega.$$

In this situation we will prove that there exists a retraction of the set $B := \big[b(t_0), c(t_0)\big]$ onto the two-point set $A := \big\{b(t_0), c(t_0)\big\}$ (see Definition 8). But it is impossible.

Now we construct the desired retraction. Let us define auxiliary mappings P_1, P_2 and P_3:

$$P_1\colon \big[b(t_0), c(t_0)\big] \ni (t_0, u^*) \to \big(\hat{t}, \hat{u}(\hat{t}\,)\big),$$

where the value $\hat{t}$ was defined above for $u^* \in \big(b(t_0), c(t_0)\big)$. For $u^* = b(t_0)$ or $u^* = c(t_0)$ we put $\hat{t} = t_0$;

$$P_2\colon \big(\hat{t}, \hat{u}(\hat{t}\,)\big) \to \begin{cases} \big(\hat{t}, c(\hat{t}\,)\big) & \text{if} \quad \hat{u}(\hat{t}\,) \geq c(\hat{t}\,), \\ \big(\hat{t}, b(\hat{t}\,)\big) & \text{if} \quad \hat{u}(\hat{t}\,) \leq b(\hat{t}\,), \end{cases}$$

and for $(\hat{t}, \tilde{u}) \in \partial\omega$

$$P_3\colon (\hat{t}, \tilde{u}) \to \begin{cases} \big(t_0, c(t_0)\big) & \text{if} \quad \tilde{u} = c(\hat{t}\,), \\ \big(t_0, b(t_0)\big) & \text{if} \quad \tilde{u} = b(\hat{t}\,). \end{cases}$$

We will show that the composite mapping

$$P\colon (t_0, u^*) \to \big\{(t_0, b(t_0), (t_0, c(t_0))\big\},$$

where

$$P := P_3 \circ P_2 \circ P_1$$

is continuous with respect to the second coordinate of the point (t_0, u^*). In the following we use the egress property of all boundary points of ω and the property of continuous dependence of the solutions on initial data. The definition of the mapping P implies that only two resultant points are possible, namely, either $P\big(t_0, u^*\big) = \big(t_0, c(t_0)\big)$ or $P\big(t_0, u^*\big) = \big(t_0, b(t_0)\big)$.

We consider the first possibility, i.e., $P\big(t_0, u^*\big) = \big(t_0, c(t_0)\big)$. Then

$$P_1(t_0, u^*) = \big(\hat{t}, \hat{u}(\hat{t}\,)\big),\ \big(\hat{t}, \hat{u}(\hat{t}\,)\big) \notin \omega \qquad \text{with} \quad \hat{u}(\hat{t}\,) \geq c(\hat{t}\,).$$

If $\sigma(\hat{t}\,) = \rho(\hat{t}\,) = \hat{t}$ then $\big(\hat{t}, \hat{u}(\hat{t}\,)\big) = \big(\hat{t}, c(\hat{t}\,)\big)$ and for solution $u = u^\varepsilon$ of a problem

$$u^\varepsilon(t_0) = u^* + \varepsilon \tag{6}$$

with a small perturbation ε of the point u^* we get

$$P_1(t_0, u^* + \varepsilon) = \big(\hat{t}^\varepsilon, \hat{u}^\varepsilon(\hat{t}^\varepsilon)\big)$$

with $\sigma(\hat{t}^{\varepsilon}) = \hat{t}^{\varepsilon}$ and $\hat{u}^{\varepsilon}(\hat{t}^{\varepsilon}) = c(\hat{t}^{\varepsilon})$. Consequently $P(t_0, u^* + \varepsilon) = \big(t_0, c(t_0)\big)$.

If $\sigma(\hat{t}\,) > \hat{t}$ and $\hat{u}(\hat{t}\,) > c(\hat{t}\,)$ then for solution $u = u^{\varepsilon}$ of the problem (6) with a small perturbation ε of the point u^* we have $\hat{t}^{\varepsilon} = \hat{t}$ and $\hat{u}^{\varepsilon}(\hat{t}\,) > c(\hat{t}\,)$. Then $P(t_0, u^* + \varepsilon) = \big(t_0, c(t_0)\big)$.

If $\sigma(\hat{t}\,) > \hat{t}$ and $\hat{u}(\hat{t}\,) = c(\hat{t}\,)$ then for solution of the problem (6) with a small perturbation ε of the point u^* hold $\hat{t}^{\varepsilon} = \sigma(\hat{t}\,)$, $\hat{u}^{\varepsilon}\big(\sigma(\hat{t}\,)\big) > c\big(\sigma(\hat{t}\,)\big)$ and $P(t_0, u^* + \varepsilon) = \big(t_0, c(t_0)\big)$.

If $\sigma(\hat{t}\,) = \hat{t}$, $\rho(\hat{t}\,) < \hat{t}$ and $\hat{u}(\hat{t}\,) > c(\hat{t}\,)$ then for solution $u = u^{\varepsilon}$ of the problem (6) with a small perturbation ε of the point u^* we have $\hat{t}^{\varepsilon} = \hat{t}$, $\hat{u}^{\varepsilon}(\hat{t}\,) > c(\hat{t}\,)$ and $P(t_0, u^* + \varepsilon) = \big(t_0, c(t_0)\big)$.

If $\sigma(\hat{t}\,) = \hat{t}$, $\rho(\hat{t}\,) < \hat{t}$ and $\hat{u}(\hat{t}\,) = c(\hat{t}\,)$ then for solution $u = u^{\varepsilon}$ of the problem (6) with a small perturbation ε of the point u^* we have either $\hat{t}^{\varepsilon} = \hat{t}$ or $\hat{t}^{\varepsilon} > \hat{t}$. In the first case $\hat{u}^{\varepsilon}(\hat{t}\,) \geq c(\hat{t}\,)$ and in the second one $\hat{u}^{\varepsilon}(\hat{t}^{\varepsilon}) = c(\hat{t}^{\varepsilon})$. In both cases $P(t_0, u^* + \varepsilon) = \big(t_0, c(t_0)\big)$.

Similarly we proceed if $(t_0, u^*) = \big(t_0, c(t_0)\big)$. So, mapping P is continuous in the considered case.

We proceed analogously in the case $P(t_0, u^*) = \big(t_0, b(t_0)\big)$, too.

The continuity of P was proven for $b(t_0) \leq u^* \leq c(t_0)$. So the desired retraction is realized by P because the mapping

$$\big[b(t_0), c(t_0)\big] \quad \xrightarrow{P} \quad \big\{b(t_0), c(t_0)\big\}$$

is continuous and

$$\begin{aligned} \big\{b(t_0)\big\} \quad &\xrightarrow{P} \quad \big\{b(t_0)\big\}, \\ \big\{c(t_0)\big\} \quad &\xrightarrow{P} \quad \big\{c(t_0)\big\}, \end{aligned}$$

i.e., points $\big\{b(t_0)\big\}$, $\big\{c(t_0)\big\}$ are stationary.

In regard of above mentioned fact it is impossible. Our supposition is false and there exists an initial problem (4) such that the corresponding solution $u = u^*(t)$ satisfies the inequalities (5) for every $t \in \mathbb{T}^*$. The theorem is proved. □

Theorem 1 can be generalized in the following way. As it easily follows from its proof, the assumptions with respect to the function f were used only for values $(t, u) \in \overline{\omega}$, although they were supposed to be valid on $\mathbb{T}^* \times \mathbb{R}$. Therefore we can reformulate this theorem. In view of the facts mentioned just before, we may omit the proof. We define a subset $\omega^{\varphi} \subset \mathbb{T}^* \times \mathbb{R}$ as

$$\omega^{\varphi} := \big\{(t, u) \colon t \in \mathbb{T}^*, b(t) - \varphi < u < c(t) + \varphi\big\},$$

where φ is a small positive number.

Theorem 2. *Let* $f\colon \omega^\varphi \to \mathbb{R}$ *and all assumptions of* Theorem 1*, restricted to the domain* ω^φ*, hold. Then there exists an initial problem* (4) *with* $u^* \in \big(b(t_0), c(t_0)\big)$ *such that the corresponding solution* $u = u^*(t)$ *satisfies the inequalities* (5) *for every* $t \in \mathbb{T}^*$.

Remark 2. The presented method is a suitable tool for the investigation of asymptotic behaviour of solutions of the dynamic equations. For the discrete case see, e.g., papers [4]–[6] and [2].

3. Example

Let us consider the time scale from the Example 1, i.e.,

$$\mathbb{T}^* = \{0\} \cup [1,3] \cup [4,5] \cup \{6\} \cup \{7\} \cup [8,9] \cup \{10\} \cup \{11\} \cup \dots$$

and the dynamic equation

$$u^\Delta = 2u + \cos(tu), \qquad t \in \mathbb{T}^*. \tag{7}$$

We will use Theorem 1 with

$$f(t,u) := 2u + \cos(tu),\ \ c := 1,\ \ b := -1.$$

We will verify that every point $M \in \omega_B \cup \omega_C$, where

$$\begin{aligned} \omega_B &:= \big\{(t,u) : t \in \mathbb{T}^*, -u-1 = 0\big\}, \\ \omega_C &:= \big\{(t,u) : t \in \mathbb{T}^*, u-1 = 0\big\} \end{aligned}$$

is the point of strict egress for the set

$$\omega := \{(t,u) : t \in \mathbb{T}^*, -1 < u < 1\}$$

with respect to the equation (7). We compute, for $M = (t,u) \in \omega_B$,

$$\begin{aligned} B^\Delta(t,u) &= (-u+b)^\Delta = (-u+1)^\Delta = -u^\Delta = -\big(2u + \cos(tu)\big) = \\ &\quad [\,\text{if } (t,u) \in \omega_B \ \text{ then } \ u = -1\,] = 2 - \cos t > 0 \end{aligned}$$

and similarly, for $M = (t,u) \in \omega_C$,

$$\begin{aligned} C^\Delta(t,u) &= (u-c)^\Delta = (u-1)^\Delta = u^\Delta = 2u + \cos(tu) = \\ &\quad [\,\text{if } (t,u) \in \omega_C, \ \text{ then } \ u = 1\,] = 2 + \cos t > 0. \end{aligned}$$

Therefore all suppositions of Theorem 1 hold, there exists an initial value $u^* \in (-1,1)$ such that the initial problem $u(t_0) = u(0) = u^*$ defines a solution $u = u^*(t)$ of (7) satisfying $|u^*(t)| < 1$ for every $t \in \mathbb{T}^*$.

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Received October 31, 2006

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