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LINEAR DIFFERENCE OPERATORS IN THE SPACE OF STRICTLY MONOTONIC FUNCTIONS

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Dedicated to my father, Professor Miroslav Laitoch, on his 85th birthday

ABSTRACT. The classical theory of first-order linear functional and difference equations is obtained as a special case of the theory developed here for the functional equation model $f \circ \phi - Af = B$. The setting is a special space S of continuous strictly monotonic functions, where $\circ$ is a group multiplication defined on S. Iterative solution formulas are given in the two cases when A, B are constant and when A, B depend on x.

Introduction

The difference equation

$$f(x+1) - 2f(x) = 5 \tag{1}$$

and the functional equation

$$f\left(\frac{1}{1+x^2} - \frac{1}{2} + x\right) - xf(x) = 0 \tag{2}$$

are instances of equations represented in this article by a unifying abstract model

$$f \circ \phi - Af = B \tag{3}$$

where $\circ$ is a group multiplication defined on monotonic functions, defined in a later subsection.

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Within the abstract framework, we develop solution formulas for (3) and document the structure of solutions. The formulas apply equally to difference equations like (1) and functional equations like (2). For instance, the formulas apply to the equation

$$f(2x + \sin x) = 1 + f(x) \tag{4}$$

to produce approximate values for $f(x)$, and for computer-assisted visualization of solutions, as shown in Figure 1.

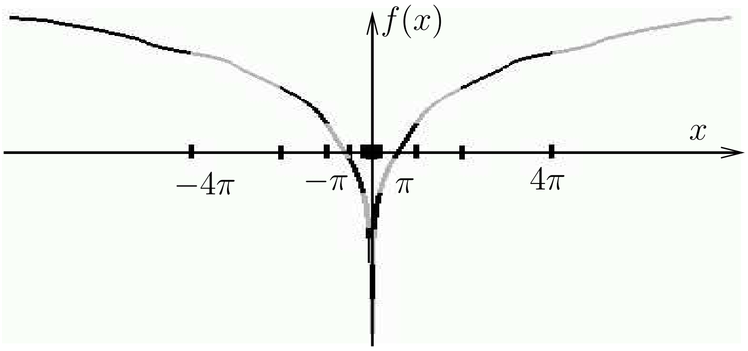

FIGURE 1. An iterative solution $y = f(x)$ for (4).
The origin $x = 0$ separates regions of increase and decrease of any solution f, because f must be infinite at fixed points of $\phi(x) = 2x + \sin x$. Odd symmetry $f(-x) = -f(x)$ is justified by the relation $\phi(-x) = -\phi(x)$. The black and grey segments of the function f represent the iterative steps used to produce the graphic.

Some basic references for the topic of the paper are [1], [2], [3], [4], [8] and [9].

Preliminaries

Notation: Let $J = (-\infty, \infty)$, $\mathbb{N}$ the set of positive integers, $\mathbb{Z}$ the integers, $\mathbb{R}$ real numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\bar{\mathbb{R}} = \langle -\infty, \infty \rangle$ and $C_0(J)$ the set of continuous functions on the interval J.

Definition 1. Let $a, b \in \bar{\mathbb{R}}$, $a < b$. The set of all functions f which satisfy $f \in C_0(J)$ and f maps one-to-one the interval J on the interval (a, b) will be denoted by symbol S and called a *space of strictly monotonic functions*.

Note 1. Thus only continuous functions which are increasing from a to b or decreasing from b to a on the interval J belong to the set S.

Definition 2. An arbitrarily chosen increasing function $X \in S$ will be called a *canonical function* in S. The inverse to the canonical function X will be denoted by X^*.

Note 2. The domain of the function X^* is the interval (a, b) and its range is $(-\infty, \infty)$.

Definition 3. Let $\alpha, \beta \in S$. Let X^* be the inverse to the canonical function $X \in S$. A function $\gamma = \gamma(x)$ defined by

$$\gamma = \alpha X^* \beta(x),$$

where the expression on the right side is a composite function $\alpha\big[X^*(\beta(x))\big]$ defined on J, will be called a *product* γ of functions $\alpha, \beta \in S$ in the class S and we write $\gamma = \alpha \circ \beta$.

Note 3. The set S with the operation of multiplication $\circ$ forms a non-commutative group; see [6].

Now we will introduce a definition of functional iteration in S, following [6].

Definition 4. Let $X \in S$ be the canonical function. Let $\phi \in S$. The *iterations* of a function ϕ in S are defined by

$$\phi^0(x) = X(x) \tag{5}$$

$$\phi^{n+1}(x) = \phi \circ \phi^n(x), \quad x \in J, \quad n = 0, 1, 2, \dots \tag{6}$$

$$\phi^{n-1}(x) = \phi^{-1} \circ \phi^n(x), \quad x \in J, \quad n = 0, -1, -2, \dots, \tag{7}$$

where $\phi^{-1} = X\phi^* X$ is the inverse element to the element ϕ in S according to the multiplication $\circ$, where ϕ^* is the inverse function to the function ϕ.

Note 4. All iterations of the function $\phi \in S$ belong to S.

Definition 5. Let $\phi \in S$. A function $p = p(x)$, $p \in C_0(J)$, is called *automorphic over* ϕ if

$$p \circ \phi(x) = p(x) \qquad \text{for} \qquad x \in J\,. \tag{8}$$

By U_ϕ we denote the set of all automorphic functions p over ϕ such that $p(x) \neq 0$ for all x.

Note 5. The product of two automorphic functions over ϕ is an automorphic function over ϕ.

Assume hereafter $\phi \in S$, $X \in S$ is a canonical function, $\phi(x) > X(x), x \in J = (-\infty, \infty)$. Let $x_\mu = X^* \phi^\mu(x_0)$, where $\mu \in Z$, $x_0 \in J$ denotes an arbitrary point of J and X^* denotes the inverse to X.

Definition 6. Let us define

$$j_\mu = \langle x_\mu, x_{\mu+1}), \qquad \mu \in \mathbb{Z}\,.$$

The set of points $x_\mu = X^*\phi^\mu(x_0)$, $\mu \in \mathbb{Z}$, will be denoted by $O_\phi(x_0)$ and called an *orbit* of the point x_0. The set of points x_μ, $\mu \in \mathbb{N}_0$, will be denoted by $O_\phi^+(x_0)$, the set of intervals j_μ, $\mu \in \mathbb{Z}$, by $O_\phi(j_0)$, where $j_0 = \langle x_0, x_1)$ and the set of intervals j_μ, $\mu \in \mathbb{N}_0$, by $O_\phi^+(j_0)$.

Difference and functional equations

Equation

$$f \circ \phi^k(x) + P_1(x) f \circ \phi^{k-1}(x) + \cdots + P_k(x) f \circ \phi^0(x) = Q(x)\,, \tag{9}$$

where $f, P_k, Q \in C_0(J)$, $P_k(x) \neq 0$, considered for $x \in O_\phi^+(x_0)$ is called a kth *order difference equation* and for all $x \in j$, $j \in O_\phi^+(j_0)$ a kth *order functional equation.*

A solution of functional equation (9) is a function $f(x) \in C_0\langle x_0, \infty)$ which satisfies equation (9) identically on $\langle x_0, \infty)$.

A solution of difference equation (9) is a sequence, it is a function defined on the set $O_\phi^+(x_0)$.

Note 6. If $f(x)$ is a solution of a functional equation (9) on $O_\phi^+(j_0)$, then $f(x)$ on $O_\phi^+(x_0)$ is a solution of the difference equation (9).

Note 7. Solutions of functional equation (9) can be defined and constructed on the interval $(-\infty, \infty)$, by modifying the construction on the right interval $\langle x_0, \infty)$ to the left interval $(-\infty, x_0\rangle$, to give a two-sided solution; see Figure 1.

Difference operator, sum operator

Definition 7. Let $f \in C_0(J)$, $\phi, X \in S$, $\phi > X$, X a canonical function. A *difference operator of a function* f relative to ϕ is denoted by $\Delta_\phi f(x)$ and defined by the equation

$$\Delta_\phi f(x) = f \circ \phi(x) - f(x)\,.$$

If we denote the right side of the equation by $g(x)$ then the *sum operator of a function* g relative to ϕ is denoted by $\Delta_\phi^{-1} g(x)$ and defined by

$$\Delta_\phi^{-1} g(x) = f(x)$$

if and only if

$$\Delta_\phi f(x) = g(x)\,. \tag{10}$$

Choosing $X = x$ and $\phi(x) = x+1$ reproduces the classical definitions of difference and sum operators. We have $\Delta f(x) = f(x+1) - f(x)$. For instance, let $f = kx$, then $\Delta f(x) = k(x+1) - kx = k$. If we define $g = k$, then $\Delta^{-1} g(x)$ equals $f = kx$.

Lemma 1. *Let $\phi, X \in S$, $\phi > X$ on J, X be a canonical function, $g \in C_0(J)$. Let*

$$\Delta_\phi^{-1} g(x) = f_n(x)$$

on j_n, $n \in \mathbb{N}_0$. Then

$$f_n = \bar{f} \circ \phi^{-n} + \sum_{i=1}^{n} g \circ \phi^{-i}$$

on j_n, where $\bar{f}$ is a given continuous function on j_0.

If $\lim_{x \to x_1} \bar{f}(x) = \bar{f}(x_0) + g(x_0)$ then the function $f = f_n$ on j_n is continuous on $\langle x_0, \infty)$.

P r o o f. Equation (10) implies

$$f \circ \phi^1(x) - f \circ \phi^0(x) = g(x)\,,$$

thus

$$f \circ \phi^1(x) = f(x) + g(x)\,. \tag{11}$$

Multiplying the equation from the right by $\phi(x)$ we get

$$\begin{aligned}
f \circ \phi^2(x) &= f \circ \phi^1(x) + g \circ \phi^1(x) = f(x) + g(x) + g \circ \phi^1(x)\,,\\
f \circ \phi^3(x) &= f \circ \phi^1(x) + g \circ \phi^1(x) + g \circ \phi^2(x)\\
&= f(x) + g(x) + g(x) \circ \phi^1(x) + g \circ \phi^2(x)\\
&\;\vdots\\
f \circ \phi^n(x) &= f \circ \phi^1(x) + g \circ \phi^1(x) + g(x) \circ \phi^2(x) + \ldots + g \circ \phi^{n-1}(x)\\
&= f(x) + g(x) + g \circ \phi^1(x) + g(x) \circ \phi^2(x) + \ldots + g \circ \phi^{n-1}(x)\,.
\end{aligned}$$

Because $j_n = \big\langle X^*\phi^n(x_0), X^*\phi^{n+1}(x_0)\big)$, $n = 0, 1, 2, \ldots$, those intervals form a decomposition of the interval $\langle x_0, \infty)$. On the interval j_0 we choose for f an arbitrary continuous function $\bar{f}$ which satisfies

$$\lim_{x \to X^*\phi(x_0)-} \bar{f}(x) = \bar{f}(x_0) + g(x_0)\,.$$

Then the function f on the intervals j_n, $n = 1, 2, \ldots$, is defined by

$$f \circ \phi^n(x) = \bar{f}(x) + g(x) + g \circ \phi(x) + \ldots + g \circ \phi^{n-1}(x), \qquad x \in j_0\,. \tag{12}$$

This function f will be shown to be continuous on the interval $\langle x_0, +\infty)$ and is a solution of equation (10) for the given function g.

Let us show now continuity of the function f on $\langle x_0, +\infty)$.

Function f is defined on j_1 by

$$f \circ \phi^1(x) = \bar{f}(x) + g(x), \quad x \in j_0 .$$

Its value at the point $X^*\phi^1(x_0)$ is $f\big(X^*\phi^1(x_0)\big) = \bar{f}(x_0) + g(x_0)$. Because $\lim\limits_{x \to X^*\phi(x_0)} \bar{f}(x) = \bar{f}(x_0) + g(x_0)$, the function f is continuous at the point $X^*\phi^1(x_0)$. A similar proof is used for points $X^*\phi^n(x_0)$.

Let us prove that the function f given by (12) is the sum operator of the function $g(x)$. We calculate the difference of the function f defined by (12) relative to ϕ.

For $x \in j_0$,

$$\Delta_\phi f(x) = \bar{f}(x) + g(x) - \bar{f}(x) = g(x)$$

holds on j_0,

$$\begin{aligned} \Delta_\phi f \circ \phi(x) &= \bar{f}(x) + g(x) + g \circ \phi(x) - (\bar{f}(x) + g(x)) \\ &= g \circ \phi(x) \end{aligned}$$

holds on j_1, etc., and

$$\begin{aligned} \Delta_\phi f \circ \phi^n(x) = {} & \bar{f}(x) + g(x) + g \circ \phi(x) + \ldots + g \circ \phi^n(x) \\ & - (\bar{f}(x) + g(x) + g \circ \phi(x) + \ldots + g \circ \phi^{n-1}(x)) \\ = {} & g \circ \phi^n(x) \end{aligned}$$

holds on j_n. □

Lemma 2. *If f is a solution of the equation* (10) *for a given function g and if p is an automorphic function over ϕ, then $f(x) + p(x)$ is also a solution of equation* (10).

P r o o f. We have

$$(f + p) \circ \phi = f \circ \phi + p \circ \phi = (f + g) + p = (f + p) + g .$$

Thus the sum operator of a function f in S is determined up to additive automorphic functions over ϕ. □

Now we will search for solutions of the first order linear functional equations in the space S. The general theory implies the *superposition principle*: The general solution of the nonhomogeneous equation is given as a sum of the general solution of the homogeneous equation plus a particular solution of the nonhomogeneous equation; for details see [7].

Note 8. The functional or difference equation $\Delta_\phi f(x) + P(x)f(x) = Q(x)$ may be written as $f \circ \phi = A(x)f(x) + B(x)$, where $A(x) = 1 - P(x)$, $B(x) = Q(x)$.

1. First order linear equations with constant coefficients and a constant right side

We consider the first order linear functional equation

$$f \circ \phi(x) = Af(x) + B\,, \tag{13}$$

where A, B are constants, $A \neq 0$.

First we will deal with the homogeneous equation

$$F \circ \phi(x) = AF(x)\,. \tag{14}$$

Let $A = 1$. Then (8) implies that the homogeneous equation

$$F \circ \phi(x) = F(x)$$

has a solution $F = a(x)$, where $a(x)$ is an automorphic function over ϕ which satisfies $a(x) \neq 0$ for all x.

Let F be any solution of (14). Because $C(x) = F(x)/a(x)$ is automorphic over ϕ, then $F(x) = C(x)a(x)$ is a general solution formula.

Let $A \neq 1$ and $A > 0$. Then the homogeneous equation

$$f \circ \phi(x) - Af(x) = 0$$

has characteristic equation $\lambda = A$. A solution exists of the form $f = A^{X^*\alpha(x)}$, where $\alpha(x)$ is a solution of the Abel functional equation $\alpha \circ \phi(x) = X(x+1) \circ \alpha(x)$; see Theorem 1 in [5]. Let $f \circ \phi - Af = 0$. Because $C(x) = f(x)A^{-X^*\alpha(x)}$ satisfies $C \circ \phi = C$, then a general solution of the homogeneous equation (14) is

$$f(x) = C(x)A^{X^*\alpha(x)}\,, \tag{15}$$

where $C(x)$ is automorphic over ϕ.

Now we will discuss solutions of the nonhomogeneous equation (13).

Theorem 1. *Assume* $A = 1$ *and* B *is a constant. The general solution of the nonhomogeneous equation* (13) *is a sum of the general solution of the homogeneous equation* (14) *and a particular solution of the nonhomogeneous equation* (13). *That is,*

$$f(x) = C(x)a(x) + \Delta_\phi^{-1} B\,, \tag{16}$$

where $a(x) \in U_\phi$, $C(x)$ *is automorphic over* ϕ *and* $B \in \mathbb{R}$.

Proof. Let $A = 1$. A particular solution f of the equation

$$f \circ \phi(x) = f(x) + B$$

is obtained from the equation

$$\Delta_\phi f(x) = B$$

or

$$f(x) = \Delta_\phi^{-1} B\,,$$

where Δ_ϕ^{-1} denotes the sum operator. □

Theorem 2. *Assume* $A \neq 1, A > 0, B$ *are constants. Then the general solution of the nonhomogeneous equation* (13) *is a sum of the general solution of homogeneous equation* (14) *and a particular solution of the nonhomogeneous equation* (13). *Therefore,*

$$f(x) = C(x)A^{X^*\alpha(x)} + \frac{B}{1-A}\,, \tag{17}$$

where $C(x)$ *is automorphic over* ϕ *and* $\alpha(x)$ *is a solution of the Abel equation* $\alpha \circ \phi(x) = X(x+1) \circ \alpha(x)$ (see [5]) .

Proof. Let $A \neq 1$. We search for a particular solution of the nonhomogeneous equation (13) of the form

$$\psi(x) = a(x)\,,$$

where $a(x) \in U_\phi$.

We substitute $\psi(x) = a(x)$ for f in the given equation and get

$$a \circ \phi(x) = Aa(x) + B\,.$$

Because $a \circ \phi(x) = a(x)$, we have

$$a(x)(1-A) = B \quad \text{or} \quad a(x) = \frac{B}{1-A}\,.$$

Thus

$$\psi(x) = \frac{B}{1-A}.$$

□

2. First order linear equations with nonconstant coefficients

Consider for continuous functions A, B on $\langle x_0, \infty)$, $A(x)$ never zero, the equation

$$f \circ \phi(x) - A(x) f \circ \phi^0(x) = B(x)\,. \tag{18}$$

Theorem 3. *The general solution f of the first order nonhomogeneous equation* (18) *with continuous nonconstant coefficients A, B is in j_{n+1} is a continuous function*

$$\begin{aligned} f(x) = \quad & A \circ \phi^{-1}(x) A \circ \phi^{-2}(x) \dots A \circ \phi^{-n-1}(x) \\ & \left\{ \Delta_\phi^{-1} \left[\tfrac{B(x)}{A \circ \phi^0(x) A \circ \phi^{-1}(x) \dots A \circ \phi^{-n-1}(x)} \right] + k(x) \right\}, \end{aligned} \tag{19}$$

where $k(x)$ is automorphic over ϕ.

P r o o f. Let $x_0 \in J = (-\infty, \infty)$ be an arbitrary point, $x_n = X^* \phi^n(x_0)$, where ϕ^n denotes the nth iteration of a function ϕ in S, $n \in \mathbb{N}_0$. Let $j_n = \langle x_n, x_{n+1})$.

First we will solve an associated homogeneous equation with an unknown function F

$$F \circ \phi(x) - A(x) F \circ \phi^0(x) = 0\,. \tag{20}$$

We look for a nontrivial solution of the equation on the intervals $j_n \in O_\phi^+(j_0)$, which is given by the initial condition

$$F(x) = F_0(x), \quad x \in j_0, \quad \lim_{x \to x_1^-} F_0(x) = A(x_0) F_0(x_0)\,, \tag{21}$$

where $F_0 \in C_0\langle x_0, x_1)$. Thus $F(x) = F_0(x)$ on j_0.

Equation (20) implies

$$\begin{aligned} F \circ \phi(x) \quad & = A(x) F_0(x)\,, \\ F \circ \phi^2(x) \quad & = A \circ \phi(x) F \circ \phi(x) = A \circ \phi(x) A(x) F_0(x)\,, \\ & \dots \\ F \circ \phi^n(x) \quad & = A \circ \phi^{n-1}(x) F \circ \phi^{n-1}(x) \\ & = A \circ \phi^{n-1}(x) A \circ \phi^{n-2}(x) \dots A \circ \phi(x) A(x) F_0(x)\,, \\ F \circ \phi^{n+1}(x) \quad & = A \circ \phi^n(x) F \circ \phi^n(x) \\ & = A \circ \phi^n(x) A \circ \phi^{n-1}(x) \dots A \circ \phi(x) A(x) F_0(x)\,, \end{aligned}$$

for $x \in j_0$, where all factors $A \circ \phi^i(x)$, $i = 0, 1, \dots, n$, have no zeros.

The last equation above implies that the general solution of the homogeneous equation (20) on j_{n+1} is a function

$$F(x) = A \circ \phi^{-1}(x) A \circ \phi^{-2}(x) \cdots A \circ \phi^{-n-1}(x) F_0 \circ \phi^{-n-1}(x)\,. \tag{22}$$

The nonhomogeneous equation (18) will be solved by the method of variation of parameters. We suppose that a solution $f(x)$ of equation (18) has the same form as $F(x)$, but the initial function $F_0(x)$ is replaced by a suitable function $C(x)$, thus we define

$$\begin{aligned} f(x) &= f_n(x), \qquad x \in j_{n+1}, \\ f_n(x) &= A\circ\phi^{-1}(x)A\circ\phi^{-2}(x)\cdots A\circ\phi^{-n-1}(x)C(x)\,. \end{aligned} \tag{23}$$

Let $x \in j_{n+1}$. Substitute (23) for f into (18). Because $f\circ\phi(x)$ is computed from (23) using f_{n+1} (not f_n), we get

$$A\circ\phi^0(x)A\circ\phi^{-1}(x)\cdots A\circ\phi^{-n-1}(x)C\circ\phi(x) \\ - A\circ\phi^0(x)A\circ\phi^{-1}(x)\cdots A\circ\phi^{-n-1}(x)C(x) = B(x)\,.$$

Factoring gives

$$A\circ\phi^0(x)A\circ\phi^{-1}(x)\cdots A\circ\phi^{-n-1}(x)\big[C\circ\phi(x) - C(x)\big] = B(x)\,.$$

Using difference notation Δ_ϕ implies

$$A\circ\phi^0(x)A\circ\phi^{-1}(x)\cdots A\circ\phi^{-n-1}(x)\Delta_\phi C(x) = B(x)\,.$$

Then, for $x \in j_{n+1}$ and $n \in \mathbb{N}$,

$$\Delta_\phi C(x) = \frac{B(x)}{A\circ\phi^0(x)A\circ\phi^{-1}(x)\cdots A\circ\phi^{-n-1}(x)}\,.$$

Thus

$$C(x) = \Delta_\phi^{-1}\left[\frac{B(x)}{A\circ\phi^0(x)A\circ\phi^{-1}(x)\dots A\circ\phi^{-n}(x)A\circ\phi^{-n-1}(x)}\right] + k(x)\,,$$

where $k(x)$ is automorphic over ϕ. This verifies (19).

We verify continuity of f. Equation (20) implies

$$F(x_1) = A(x_0)F(x_0)\,.$$

According to (21), the solution F is continuous at x_1. Then

$$\lim_{x\to x_2} F(x) = \lim_{x\to x_1} F\circ\phi(x) = \lim_{x\to x_1} A(x)F(x) = A(x_1)F(x_1)\,,$$

because F is continuous in x_1. We show that the assumption of continuity at $x_1, x_2, \dots, x_n$ yields the continuity also at x_{n+1}. We have

$$\lim_{x\to x_{n+1}} F(x) = \lim_{x\to x_n} F\circ\phi(x) = \lim_{x\to x_n} A(x)F(x) = A(x_n)F(x_n)\,,$$

because the functions A and F are continuous at x_n according to the assumption. Therefore f is continuous on $\langle x_0, \infty)$. $\square$

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