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OSCILLATION PROPERTIES OF TWO-DIMENSIONAL NEUTRAL DIFFERENTIAL SYSTEMS

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ABSTRACT. Asymptotic properties of solutions of two dimensional system of differential equations of neutral type are studied.

1. Introduction

We consider two dimensional systems of neutral differential equations of the form

$$\begin{aligned} \big(x_1(t) - p_1(t)x_1\left(\tau_1(t)\right)\big)' &= a_1(t)f_1\big(x_2(g_2(t))\big) \\ \big(x_2(t) - p_2(t)x_2\left(\tau_2(t)\right)\big)' &= -\,a_2(t)f_2\big(x_1\left(g_1(t)\right)\big) \end{aligned} \tag{1}$$

under the following conditions:

(A) $p_i,\ a_i,\ \tau_i,\ g_i \in C\big([t_0,\infty), R_+\big),\ i = 1,2\,,\ t_0 \ge 0$,

(B) $0 < p_i(t) \le \lambda_i < 1,\ t \ge t_0,\ i = 1,2\,,$

(C) $\tau_i(t) < t$ on $[t_0,\infty),\ \lim\limits_{t\to\infty} \tau_i(t) = \infty$
and $\tau_i(t)$ is a strictly increasing function, $i = 1,2\,,$

(D) $\lim\limits_{t\to\infty} g_i(t) = \infty,\ i = 1,2\,,$

(E) $f_i \in C(R,R),\ f_i(u)u > 0$ for $u \ne 0,\ i = 1,2\,,$
f_2 is a nondecreasing function and there exists a positive constant L such that $|f_1(u)| \ge L|u|$ for $u \in R$.

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The problem of oscillation of neutral functional differential equations has received considerable attention in the last few years (see, for example [1] and the references cited therein). However, very little has been published on the systems of neutral differential equations; see, for example Marušiak [3], Špániková, Šamajová [5]–[7].

Our aim in the present paper is to establish sufficient conditions under which all proper solutions of systems (1) are oscillatory or their components tend to zero for $t \to \infty$. By a *proper solution* of (1) we mean a continuous vector function $\boldsymbol{x} = (x_1, x_2)$ on $[t_x, \infty)$ such that $x_1(t) - p_1(t)x_1(\tau_1(t))$, $x_2(t) - p_2(t)x_2(\tau_2(t))$ are continuously differentiable, $\boldsymbol{x}$ satisfies system (1) for all sufficiently large $t \geq t_x$ and $\sup\{|x_1(t)| + |x_2(t)| : t \geq T\} > 0$ for any $T \geq t_x$. Such a solution $\boldsymbol{x}$ of (1) is called *nonoscillatory* if there exists a $T_0 \geq t_x$ such that its every component is different from zero for all $t \geq T_0$, and it is called *oscillatory* otherwise.

The obtained results extend some other ones which have been given in the paper [4].

2. Properties of nonoscillatory solutions

We begin by analyzing the asymptotic behavior of possible nonoscillatory solutions of system (1). The following notation will be used extensively:

$$
\begin{aligned}
&\tau_j^0(t) = t, && && j = 1,2\,,\\
&\tau_j^i(t) = \tau_j\bigl(\tau_j^{i-1}(t)\bigr), && i = 1,2,\dots, && j = 1,2\,,\\
&\tau_j^{-i}(t) = \tau_j^{-1}\bigl(\tau_j^{-(i-1)}(t)\bigr), && i = 1,2,\dots, && j = 1,2\,,
\end{aligned}
$$

where $\tau_j^{-1}(t)$ denotes the inverse function of $\tau_j(t)$, $j = 1,2$,

$$P_{j0}(t) \equiv 1, \quad P_{ji}(t) = \prod_{k=0}^{i-1} p_j\bigl(\tau_j^k(t)\bigr), \quad i = 1,2,\dots, \quad j = 1,2\,.$$

Let $\boldsymbol{x} = (x_1, x_2)$ be a nonoscillatory solution of (1). From the system (1) it follows that the functions

$$u_j(t) = x_j(t) - p_j(t)x_j\bigl(\tau_j(t)\bigr), \quad j = 1,2\,, \tag{2}$$

are eventually monotone, so that $u_j(t)$, $j = 1,2$ have to be of the constant sign. Therefore

$$x_j(t)u_j(t) > 0, \quad j \in \{1,2\} \tag{3}$$

or

$$x_j(t)u_j(t) < 0, \quad j \in \{1,2\} \tag{4}$$

for all sufficiently large t. Denote by N^+ (respectively N^-) the set of components of all nonoscillatory solutions (x_1, x_2) of system (1) such that (3) (respectively (4)) is satisfied.

If $x_j \in N^+$, $j \in \{1,2\}$, then for every $T \geq T_0$ and every integer $k \geq 0$ there exists a $T_k \geq T$ such that

$$\tau_j^i(t) \geq T, \quad i = 0, 1, \dots, k+1, \quad j \in \{1,2\}$$

and

$$|x_j(t)| \geq \sum_{i=0}^{k} P_{ji}(t) \big| u_j\big(\tau_j^i(t)\big) \big| , \quad j \in \{1,2\} \tag{5}$$

for $t \geq T_k$.

Similarly, if $x_j \in N^-$, $j \in \{1,2\}$, then for every sufficiently large T and every integer $m \geq 1$ there exists a $T_m \geq T$ such that

$$\tau_j^{-i}(t) \geq T, \quad i = 1, \dots, m\,, \quad j \in \{1,2\}$$

and

$$|x_j(t)| \geq \sum_{i=1}^{m} \frac{\big| u_j\big(\tau_j^{-i}(t)\big) \big|}{Pji\big(\tau_j^{-i}(t)\big)}\,, \quad j \in \{1,2\} \tag{6}$$

for $t \geq T_m$.

The simple known lemma below indicates that additional restrictions upon p_j and τ_j, $j \in \{1,2\}$ may force some properties of the nonoscillatory solutions.

Lemma 2.1. *Suppose that* (A)–(C) *hold. If* $x_j \in N^-$, *then* $\lim\limits_{t\to\infty} x_j(t) = 0$, $\lim\limits_{t\to\infty} u_j(t) = 0$, $j \in \{1,2\}$.

The proof can be found in [2].

Lemma 2.2. *Suppose that* (A)–(C) *hold and* $x_j \in N^+$, $j \in \{1,2\}$. *If* $\lim\limits_{t\to\infty} u_j(t) = 0$ *then* $\lim\limits_{t\to\infty} x_j(t) = 0$, $j \in \{1,2\}$.

Proof. Without loss of generality we suppose that $x_1 \in N^+$ and $x_1(t) > 0$, $u_1(t) > 0$ for $t \geq T_0$.

First note that x_1 is bounded. To prove it assume on the contrary that x_1 is unbounded. Hence there exists a sequence $\{t_m\}$ such that $\lim\limits_{m\to\infty} t_m = \infty$ and $\lim\limits_{m\to\infty} x_1(t_m) = \infty$, where $x_1(t_m) = \max\big\{x_1(s) : T_0 \leq s \leq t_m\big\}$.
As $\tau_1(t) < t$ we have

$$\begin{aligned} x_1\big(\tau_1(t_m)\big) &\leq \max\big\{x_1(s) : T_0 \leq s \leq \tau_1(t_m)\big\} \\ &\leq \max\big\{x_1(s) : T_0 \leq s \leq t_m\big\} = x_1(t_m)\,. \end{aligned}$$

Therefore for all large m

$$\begin{aligned} u_1(t_m) &= x_1(t_m) - p_1(t_m)x_1\big(\tau_1(t_m)\big) \ge x_1(t_m) - \lambda_1 x_1(t_m) \\ &\ge x_1(t_m)(1-\lambda_1)\,. \end{aligned}$$

Thus $u_1(t_m) \to \infty$ as $m \to \infty$. A contradiction, therefore x_1 is bounded.

Our goal is to show that $x_1(t) \to 0$ as $t \to \infty$. Assume on the contrary that

$$\limsup_{t\to\infty} x_1(t) = \lim_{n\to\infty} x_1(t_n^*) = c > 0\,.$$

Let $\varepsilon > 0$ satisfy $\varepsilon < \frac{c(1-\lambda_1)}{\lambda_1}$. Then $x_1\big(\tau_1(t)\big) \le c + \varepsilon$ and from (2) we have

$$u_1(t) \ge x_1(t) - p_1(t)(c+\varepsilon) \ge x_1(t) - \lambda_1(c+\varepsilon)\,, \qquad \text{for } t \text{ large enough.}$$

Putting $t = t_n^*$ and letting $n \to \infty$ we get

$$0 \ge c - \lambda_1 c - \lambda_1 \varepsilon > c - \lambda_1 c - c(1-\lambda_1) = 0\,.$$

A contradiction. $\square$

3. Main results

We are interested in the system (1) when f_2 satisfies the following sublinear conditions:

(F) $0 < \int\limits_0^{\varepsilon} \frac{\mathrm{d}u}{f_2(u)} < \infty\,, \quad 0 < \int\limits_0^{-\varepsilon} \frac{\mathrm{d}u}{f_2(u)} < \infty \qquad$ for all $\quad \varepsilon > 0$,

(G) there exists a constant $0 < K$ such that

$$uvf_2(uv) \ge Kuvf_2(u)f_2(v) \qquad \text{for} \quad u, v \in R\,.$$

Theorem 3.1. *Suppose that* (A)–(G) *are satisfied and:*

(i) *g_2 is a nondecreasing function, $g_2\big(g_1(t)\big) \le t$, $\tau_2^{-1}\big(g_2(t)\big) \le t$,*

(ii) *$\tau_1\big(g_1(t)\big) \le t$.*

If there exist non-negative integer numbers n, m such that

$$\int^{\infty} a_1(s)P_{2n}\big(g_2(t)\big)\,\mathrm{d}s = \infty\,, \tag{7}$$

$$\int^{\infty} a_2(t) f_2\left(\sum_{i=0}^{m} P_{1i}\big(g_1(t)\big) \int\limits_T^{\tau_1^i(g_1(t))} a_1(s)P_{2n}\big(g_2(s)\big)\,\mathrm{d}s\right)\mathrm{d}t = \infty\,, \tag{8}$$

then every proper solution of the system (1) *is oscillatory or $\lim\limits_{t\to\infty} x_i(t) = 0$, $i = 1, 2$.*

Proof. Assume the converse and suppose that (1) has a nonoscillatory solution $\boldsymbol{x} = (x_1, x_2)$ with positive first component $x_1(t) > 0$ on $[T_0; \infty)$. It follows from the system (1) that u_2 is a nonincreasing function and hence there exists such $T_1 \ge T_0$ that there are two possibilities for u_2:

1. $u_2(t) < 0$ for $t \ge T_1$,
2. $u_2(t) > 0$ for $t \ge T_1$.

Now we assume that 1. holds and consider the following two subcases:

a) Let $x_2 \in N^+$. Then there exist a constant $c < 0$, $T \ge T_1$, and an integer $n \ge 0$ such that $u_2\bigl(\tau_2^i(g_2(t))\bigr) < c$ for $t \ge T$, $i = 0, \dots, n$ and from (5)

$$x_2\bigl(g_2(t)\bigr) \le c \sum_{i=0}^{n} P_{2i}\bigl(g_2(t)\bigr) \le c\, P_{2n}\bigl(g_2(t)\bigr)\,, \qquad t \ge T\,.$$

Using (E) and the first equation of (1) we get

$$u_1(t) - u_1(T) \le Lc \int_T^t a_1(s) P_{2n}\bigl(g_2(s)\bigr)\,\mathrm{d}s\,, \qquad t \ge T\,. \tag{9}$$

Letting $t \to \infty$ we have $\lim\limits_{t\to\infty} u_1(t) = -\infty$ because of (7). This means $x_1 \in N^-$ which contradicts the Lemma 2.1.

b) Let $x_2 \in N^-$. The function u_2 is nonincreasing and so there exists $\lim\limits_{t\to\infty} u_2(t) = c < 0$ which contradicts Lemma 2.1.

We assume now that 2. holds and consider next subcases:

a) Let $x_2 \in N^+$, $x_1 \in N^-$. The function u_2 is positive, nonincreasing and so there exists $\lim\limits_{t\to\infty} u_2(t) = d \ge 0$. We shall show that $d = 0$. Otherwise, $d > 0$ and so there exist a constant $T \ge T_1$ and an integer $n \ge 0$ such that $u_2\bigl(\tau_2^i(g_2(t))\bigr) \ge d$ for $t \ge T$, $i = 0, \dots, n$ and from (5)

$$x_2\bigl(g_2(t)\bigr) \ge d \sum_{i=0}^{n} P_{2i}\bigl(g_2(t)\bigr) \ge dP_{2n}\bigl(g_2(t)\bigr)\,, \qquad t \ge T\,.$$

Using (E) we get from the first equation of (1)

$$u_1(t) - u_1(T) \ge Ld \int_T^t a_1(s) P_{2n}\bigl(g_2(s)\bigr)\,\mathrm{d}s\,. \tag{10}$$

With regard to (7), letting $t \to \infty$ we have $\lim\limits_{t\to\infty} u_1(t) = \infty$ which contradicts the negativity of u_1. Therefore $\lim\limits_{t\to\infty} u_2(t) = 0$ and $\lim\limits_{t\to\infty} x_2(t) = 0$ according to Lemma 2.2 and Lemma 2.1 implies that $\lim\limits_{t\to\infty} x_1(t) = 0$.

b) Let $x_1 \in N^+$, $x_2 \in N^+$. Then we obtain from the system (1) that u_2 is a nonincreasing function. From (5), (C) we see that there exist $T \ge T_1$, an integer $n \ge 0$ such that

$$x_2(t) \ge \sum_{i=0}^{n} P_{2i}(t) u_2\big(\tau_2^i(t)\big) \ge u_2(t) P_{2n}(t)\,,$$

and so using (E)

$$f_1\big(x_2(g_2(t))\big) \ge L u_2\big(g_2(t)\big) P_{2n}\big(g_2(t)\big), \qquad t \ge T\,. \tag{11}$$

Putting (11) into the first equation of (S), integrating on $[T,t]$, $t \ge T$ and using monotonicity of u_2, g_2 we get

$$u_1(t) \ge L u_2\big(g_2(t)\big) \int_T^t a_1(s) P_{2n}\big(g_2(s)\big)\,\mathrm{d}s\,, \qquad t \ge T\,. \tag{12}$$

Taking into account (5), (12) and the properties of τ_1^i, g_2, g_1 there exist $T_2 \ge T$ and an integer $m \ge 0$ such that

$$x_1\big(g_1(t)\big) \ge L u_2(t) \sum_{i=0}^{m} P_{1i}\big(g_1(t)\big) \int_T^{\tau_1^i(g_1(t))} a_1(s) P_{2n}\big(g_2(s)\big)\,\mathrm{d}s, \qquad t \ge T_2\,. \tag{13}$$

Then the second equation of (1) according to (13) implies

$$\frac{u_2'(t)}{f_2\big(u_2(t)\big)} \le -M\, a_2(t) f_2 \left(\sum_{i=0}^{m} P_{1i}\big(g_1(t)\big) \int_T^{\tau_1^i(g_1(t))} a_1(s) P_{2n}\big(g_2(s)\big)\,\mathrm{d}s \right)\,,$$

where $M = K^2 f_2(L)$. An integration of the last inequality from T_2 to $t \ge T_2$ yields

$$\int_{u_2(t)}^{u_2(T_2)} \frac{\mathrm{d}z}{f_2(z)} \ge M \int_{T_2}^{t} a_2(s) f_2 \left(\sum_{i=0}^{m} P_{1i}\big(g_1(s)\big) \int_T^{\tau_1^i(g_1(s))} a_1(u) P_{2n}\big(g_2(u)\big)\,\mathrm{d}u \right) \mathrm{d}s\,. \tag{14}$$

Then (F) together with (14) for $t \to \infty$ contradicts (8).

c) Let $x_2 \in N^-$, $x_1 \in N^+$. For this case one gets that u_1, u_2 are nonincreasing functions and there exists $\lim\limits_{t\to\infty} u_1(t) = b \ge 0$. We shall show that $b = 0$. To prove it suppose on the contrary that $b > 0$. Then there exists a constant $T \ge T_1$ such that

$$b \le u_1(t) \le 2b, \qquad t \ge T\,. \tag{15}$$

Combining (6), (i), (B) and monotonicity of u_2 we get $x_2\big(g_2(t)\big) \le -u_2(t)$ and according to (E)

$$f_1\big(x_2(g_2(t))\big) \le Lx_2\big(g_2(t)\big) \le -Lu_2(t)\,, \qquad t \ge T_2 \ge T\,.$$

Using the last inequality, (15) and monotonicity of u_2 the first equation of (1) implies

$$-u_1(t) \le -b \le u_1(t) - u_1(T_1) \le -Lu_2(t)\int_{T_2}^{t} a_1(s)\,\mathrm{d}s \tag{16}$$

and so

$$u_1(t) \ge Lu_2(t)\int_{T_2}^{t} a_1(s)\,\mathrm{d}s, \qquad t \ge T_2\,.$$

Putting the last inequality in (5) (j=1) we have

$$\begin{aligned} x_1\big(g_1(t)\big) \quad &\ge \quad L\sum_{i=0}^{m} P_{1i}\big(g_1(t)\big)u_2\big(\tau_1^i(g_1(t))\big)\int_{T_2}^{\tau_1^i(g_1(t))} a_1(s)\,\mathrm{d}s \\ &\ge \quad Lu_2(t)\sum_{i=0}^{m} P_{1i}\big(g_1(t)\big)\int_{T_2}^{\tau_1^i(g_1(t))} a_1(s)\,\mathrm{d}s, \qquad t \ge T_2 \ge T_1\,. \end{aligned}$$

Combining the last inequality, (G), the second equation of (1) we get

$$-\frac{u_2'(t)}{f_2\big(u_2(t)\big)} \ge K^2 f_2(L)a_2(t)f_2\left(\sum_{i=0}^{m} P_{1i}\big(g_1(t)\big)\int_{T_1}^{\tau_1^i(g_1(t))} a_1(s)\,\mathrm{d}s\right)\,.$$

Integrating the obtained inequality from T_2 to $t \ge T_2$, letting $t \to \infty$ we get in view of (F)

$$\infty > \int_{T_2}^{\infty} a_2(s)f_2\left(\sum_{i=0}^{m} P_{1i}\big(g_1(s)\big)\int_{T_1}^{\tau_1^i(g_1(s))} a_1(u)\,du\right)\,\mathrm{d}s.$$

This contradicts (8). Therefore $\lim\limits_{t\to\infty} u_1(t) = 0$ and taking into account Lemma 2.2 $\lim\limits_{t\to\infty} x_1(t) = 0$. Because of $x_2 \in N^-$, using Lemma 2.1 we have $\lim\limits_{t\to\infty} x_2(t) = 0$.

d) Let $x_2 \in N^-$, $x_1 \in N^-$. Then $\lim_{t\to\infty} x_1(t) = 0$, $\lim_{t\to\infty} x_2(t) = 0$ according to Lemma 2.1. □

We are interested now in the system (1) if $f_2(u)$ satisfies the following superlinear condition

(H) $0 < \int\limits_{\varepsilon}^{\infty} \frac{\mathrm{d}u}{f_2(u)} < \infty, \quad 0 < \int\limits_{-\varepsilon}^{\infty} \frac{\mathrm{d}u}{f_2(u)} < \infty \qquad$ for all $\quad \varepsilon > 0$.

THEOREM 3.2. *Suppose that* (A)–(E), (H) *are satisfied and:*

(i) $g_1'(t) > 0$, $g_1(t) < t$, $\tau_2^{-1}\big(g_2(t)\big) < t$,

(ii) $f_2(u)$ *is a nondecreasing, differentiable function.*

If there exists an integer $n \geq 0$*, such that* (7) *holds and*

$$\int\limits_T^{\infty} a_2(u) \int\limits_T^{g_1(u)} a_1(s) P_{2n}\big(g_2(s)\big)\,\mathrm{d}s\,\mathrm{d}u = \infty, \tag{17}$$

then every proper solution of the system (1) is oscillatory or $\lim_{t\to\infty} x_i(t) = 0$*,* $i = 1, 2$.

P r o o f. Let $\boldsymbol{x} = (x_1, x_2)$ be a nonoscillatory solution of the system (1) and $x_1(t) > 0$ for $t \geq T_0$. We can proceed exactly as in the proof of Theorem 3.1 to see that u_2 is a nonincreasing function and there are two possibilities for u_2:

1. $u_2(t) < 0 \qquad$ for $\quad t \geq T_1 \geq T_0$,
2. $u_2(t) > 0 \qquad$ for $\quad t \geq T_1 \geq T_0$.

Exactly as in the proof of Theorem 3.1 the cases 1a), 1b) lead to the contrary. The cases 2a), 2d) give solutions with properties $\lim_{t\to\infty} x_i(t) = 0$, $i = 1, 2$. We need investigate only the cases 2b), 2c).

We consider now the case 2b), e.g., $x_2 \in N^+$, $x_1 \in N^+$ and define a function

$$F(t) = \frac{u_2(t) A\big(g_1(t)\big)}{f_2\big(u_1(g_1(t))\big)} \geq 0\,, \qquad t \geq T_1\,, \tag{18}$$

where $A(t) = \int\limits_{T_0}^{t} a_1(s) P_{2n}\big(g_2(s)\big)\mathrm{d}s$. Then

$$\begin{aligned} F'(t) = {} & \frac{u_2'(t) A\big(g_1(t)\big)}{f_2\big(u_1(g_1(t))\big)} + \frac{u_2(t) a_1\big(g_1(t)\big) P_{2n}\big(g_2(g_1(t))\big) g_1'(t)}{f_2\big(u_1(g_1(t))\big)} \\ & - \frac{u_2(t) A\big(g_1(t)\big) f_2'\big(u_1(g_1(t))\big) u_1'\big(g_1(t)\big) g_1'(t)}{f_2^2\big(u_1(g_1(t))\big)}\,. \end{aligned} \tag{19}$$

Taking into account $x_1 \in N^+$, (5) and the monotonicity of the function f_2 the second equation of system (1) gives

$$u_2'(t) \le -a_2(t) f_2\big(u_1(g_1(t))\big)\,.$$

Combining the last inequality, the positivity of the functions g_1', f_2', u_1', u_2 we get from (19)

$$F'(t) \le -a_2(t) A\big(g_1(t)\big) + \frac{u_2(t) a_1\big(g_1(t)\big) P_{2n}\big(g_2(g_1(t))\big) g_1'(t)}{f_2\big(u_1(g_1(t))\big)}\,. \tag{20}$$

In the same way as in the proof of Theorem 1 we obtain the inequality (11). According to this inequality, the monotonicity of u_2, (E) we have

$$\begin{aligned} a_1\big(g_1(t)\big) f_1\big(x_2(g_2(g_1(t)))\big) &\ge L a_1\big(g_1(t)\big) u_2\big(g_2(g_1(t))\big) P_{2n}\big(g_2(g_1(t))\big) \\ &\ge L a_1\big(g_1(t)\big) u_2(t) P_{2n}\big(g_2(g_1(t))\big)\,, \end{aligned} \tag{21}$$

for $t \ge T_2 \ge T_1$. Combining (20), (21), the first equation of (1), we obtain

$$F'(t) \le -a_2(t) A\big(g_1(t)\big) + \frac{u_1'\big(g_1(t)\big) g_1'(t)}{L f_2\big(u_1(g_1(t))\big)}\,, \qquad t \ge T_1\,,$$

and integrating from T_2 to $t \ge T_2$

$$0 \le F(t) \le F(T_2) - \int_{T_2}^{t} a_2(s) A\big(g_1(s)\big)\,\mathrm{d}s + \frac{1}{L} \int_{z_1}^{z} \frac{\mathrm{d}u}{f_2(u)}\,, \tag{22}$$

where $z_1 = u_1\big(g_1(T_2)\big)$, $z = u_1\big(g_1(t)\big)$. Then (H) together with (22) for $t \to \infty$ contradict (17).

We shall investigate the last case e.g. $x_2 \in N^-$, $x_1 \in N^+$. Then $u_1(t) > 0$ is a nonincreasing function and there exists $\lim\limits_{t\to\infty} u_1(t) = b \ge 0$. We shall show that $b = 0$. To prove it assume on the contrary that $b > 0$. As in the proof of Theorem 3.1 we obtain the inequality (16) from which

$$b \ge L u_2\big(g_1(t)\big) \int_{T_2}^{g_1(t)} a_1(s)\,\mathrm{d}s \ge L u_2(t) A\big(g_1(t)\big)\,. \tag{23}$$

Taking into account (6), $x_2 \in N^-$ we have $x_2(t) \le -u_2\big(\tau_2^{-1}(t)\big)$ and in view of (i), and monotonicity of u_2

$$x_2\big(g_2(g_1(t))\big) \le -u_2\big(\tau_2^{-1}(g_2(g_1(t)))\big) \le -u_2(t)\,.$$

The first equation of (1) and (E) imply

$$u_1'\big(g_1(t)\big) \le -L a_1\big(g_1(t)\big) u_2(t)$$

and in regard to (B)

$$a_1\big(g_1(t)\big)P_{2n}\big(g_2(g_1(t))\big)u_2(t) \le a_1\big(g_1(t)\big)u_2(t) \le -\frac{1}{L}u_1'\big(g_1(t)\big)\,, \tag{24}$$

for $t \ge T$, T-large enough. Taking into account the monotonicity of f_2, $x_1 \in N^+$, (5)

$$f_2\big(x_1(g_1(t))\big) \ge f_2\big(u_1(g_1(t))\big)\,. \tag{25}$$

Now using the function F defined in (18) and (19), (23)–(25) the next inequality holds

$$F'(t) \le -a_2(t)A\big(g_1(t)\big) - \frac{u_1'\big(g_1(t)\big)g_1'(t)}{Lf_2\big(u_1(g_1(t))\big)} - \frac{bf_2'\big(u_1(g_1(t))\big)u_1'\big(g_1(t)\big)g_1'(t)}{Lf_2^2\big(u_1(g_1(t))\big)}\,.$$

Then integrating from T to $t \ge T$ we have

$$\begin{aligned}0 \le F(t) \le F(T) - \int_T^t a_2(s)A\big(g_1(s)\big)\,\mathrm{d}s - \frac{1}{L}\int_{z_1}^{z} \frac{\mathrm{d}u}{f_2(u)}\,\mathrm{d}u \\ + \frac{b}{L}\left(\frac{1}{f_2(z)} - \frac{1}{f_2(z_1)}\right),\end{aligned}$$

where $z_1 = u_1\big(g_1(T)\big)$, $z = u_1\big(g_1(t)\big)$. The last two expressions are finite according to $b \le u_1(t) \le 2b$, $t \ge T$. Then the last inequality for $t \to \infty$ contradicts to (17). Therefore $\lim\limits_{t\to\infty} u_1(t) = 0$ and $\lim\limits_{t\to\infty} x_1(t) = 0$ because of Lemma 2.2. Lemma 2.1 implies that $\lim\limits_{t\to\infty} x_2(t) = 0$ too. □

Remark 3.3. Theorem 3.1, Theorem 3.2 extend results of V.N. Shevelo, N.V. Varech, A.G. Gritsai in paper [4]. Since for $p_i(t) \equiv 0$, $i = 1,2$ the system (1) reduces to

$$\begin{aligned} x_1'(t) &= a_1(t)f_1\big(x_2(g_2(t))\big), \\ x_2'(t) &= -a_2(t)f_2\big(x_1(g_1(t))\big)\,. \end{aligned}$$

Our assumptions (F), (G) imply their conditions

$$0 < \int_0^{\varepsilon} \frac{\mathrm{d}u}{f_2\big(f_1(u)\big)} < \infty\,, \quad 0 < \int_0^{-\varepsilon} \frac{\mathrm{d}u}{f_2\big(f_1(u)\big)} < \infty \qquad \text{for all} \quad \varepsilon > 0\,.$$

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