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RANDOMLY $2C_n$ GRAPHS

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ABSTRACT. A graph G is said to be a randomly H graph if and only if any subgraph of G without isolated vertices, which is isomorphic to a subgraph of H, can be extended to a subgraph H_1 of G such that H_1 is isomorphic to H. In this paper the problem of randomly H graphs, where $H = 2C_n$, is discussed.

1. Introduction

In 1951 Ore [14] studied arbitrarily traceable graphs which were later referred to as randomly eulerian graphs. This concept was later extended by Chartrand and White [5], and Erickson [8]. In 1968 Chartrand and Kronk [2] introduced and characterized the concept of randomly hamiltonian graphs. Analogous questions were studied in [4], [6], [7].

In 1986 Chartrand, Oellermann, and Ruiz [3] generalized these concepts and introduced the term 'randomly H graph' as follows: Let G be a graph containing a subgraph H without isolated vertices. Then G is called a randomly H graph if whenever F is a subgraph of G without isolated vertices that is isomorphic to a subgraph of H, then F can be extended to a subgraph H_1 of G such that H_1 is isomorphic to H.

Note that a subgraph H_1 does not have to maintain the partial isomorphism between F and the subgraph of H.

In [1] and [3] authors pointed out the problems concerning isolated vertices in the definition of randomly H graphs. That is why we consider that both H and F are free of isolated vertices.

The graph G shown in Figure 1 is not randomly P_4 since the subgraph F of G cannot be extended to a subgraph of G isomorphic to P_4, while the graph $K_{3,3}$ is randomly P_4 as well as randomly C_4.

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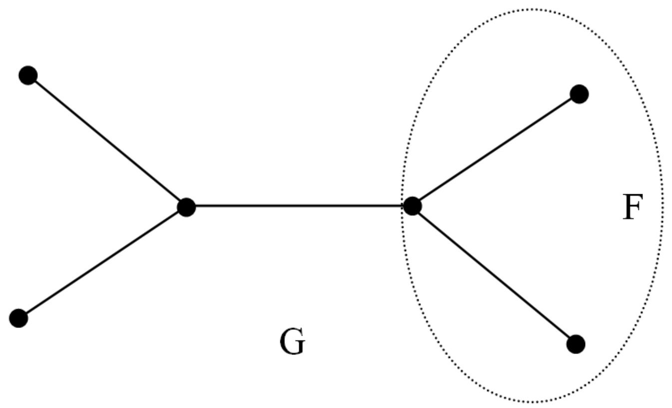

FIGURE 1.

2. Preliminaries

Let H be a class of graphs. The problem is to characterize those classes of graphs G that are randomly H graphs. The problem was completely solved for complete bipartite graphs and complete n-partite graphs. Only particular results have been achieved for r-regular graphs so far.

Every nonempty graph is randomly K_2, while every graph G without isolated vertices is a randomly G graph. K_n is randomly H for every $H \subseteq K_n$. The graph $K_{3,3}$ is randomly H for every subgraph H of $K_{3,3}$ (see [3, Theorem 1]).

In [11] the characterization of randomly $K_{r,s}$ graphs was given, but in terms of H-closed graphs. In [1] Alavi, Lick, and Tian studied randomly complete n-partite graphs and characterized them.

The problem of characterization of randomly H graphs, where H is r-regular graph on p vertices, was given by Tomasta and Tomová (see [16]). Characterization of such graphs in general seems to be difficult. However, there exist several results for some special values of r and p. The list of the most important results follows.

Theorem A. (See Sumner [15]). *Let H be 1-regular graph on $2p$ vertices. A graph G on $2p$ vertices is randomly H (perfectly matchable) if and only if*

1. $G = K_{2p}$, *or*
2. $G = K_{p,p}$, *or*
3. $G = H$.

Theorem B. (See Chartrand, Oellermann, and Ruiz [3]). *A graph G is randomly C_3 if and only if each component of G is a complete graph of order at least 3.*

THEOREM C. (See Chartrand, Oellermann and Ruiz [3]). *A graph G is randomly C_4 if and only if*

1. $G = K_p$, *where* $p \geqslant 4$, *or*
2. $G = K_{r,s}$, *where* $2 \leqslant r \leqslant s$.

THEOREM D. (See Chartrand, Oellermann, and Ruiz [3]). *A graph G is randomly C_n, $n \geqslant 5$, if and only if*

1. $G = K_p$, *where* $p \geqslant n$, *or*
2. $G = C_n$, *or*
3. $G = K_{n/2,n/2}$ *and n is even.*

THEOREM E. (See Tomasta and Tomová [16]). *Let H be connected n-vertex regular graph of degree $r \geqslant 2$ different from K_3 and C_4. Then a p-vertex graph G where $p > n$ is randomly H if and only if $G = K_p$.*

All known results about 2-regular graphs deal only with connected graphs. This paper deals with disconnected randomly 2-regular graphs, where $H = C_n \cup C_n = 2C_n$.

All the terms used in this paper can be found in [9]. Especially, if H is a subgraph of G, we will use $G - H = \langle V(G) - V(H) \rangle$ to denote the induced subgraph of the graph G with the vertex set $V(G) - V(H)$.

3. Results

THEOREM 1. A graph G is randomly $2C_3$ if and only if

(i) $G = K_p$, $p \geqslant 6$, or

(ii) $G = K_{p_1} \cup K_{p_2} \cup \cdots \cup K_{p_n}$, where $n \geqslant 2$, $p_i = 3$ or $p_i \geqslant 6$.

Proof. The sufficiency is obvious, as both $G = K_p$, where $p \geqslant 6$, and $G = K_{p_1} \cup K_{p_2} \cup \cdots \cup K_{p_n}$, where $n \geqslant 2$, $p_i = 3$ or $p_i \geqslant 6$, are randomly $2C_3$ graphs.

For necessity, let G be a randomly $2C_3$ graph. Let G be connected. We will prove that no edge is missing.

Assume the contrary. Let the edge $\{u, v\}$ be missing. Since G is connected, there exists a $u - v$ path. Let us choose the shortest $u - v$ path P. Obviously, the length of P is at least 2. Then no subpath of P of the length 2 cannot be expanded into C_3, a contradiction with the assumption that G is a randomly $2C_3$ graph.

Now, let not G be connected. Then G cannot contain components K_{p_i}, where $p_i = 2, 4, 5$. The case $p_i = 2$ is trivial. If $p_i = 4$ or $p_i = 5$, those components contain a subgraph F isomorphic to $2K_2$. Although F is isomorphic to a subgraph of $2C_3$, F cannot be extended to $2C_3$, a contradiction. If some component of G has exactly three vertices, then the component is isomorphic to K_3, as G is a randomly $2C_3$ graph. If some component of G has more than five vertices, then the component is a complete subgraph of G. Otherwise, we could produce a contradiction similarly to the connected case. □

LEMMA 1. *Let G be a disconnected randomly $2C_n$ graph, where $n > 3$. Then G has two components and each of the components has exactly n vertices.*

P r o o f. a) Let G have more than two components. Let a subgraph F of G consist of three edges which belong to three different components of G. Subgraph F is isomorphic to some subgraph of $2C_n$. However, the subgraph F cannot be extended to $2C_n$, a contradiction.

b) Let G have two components. Obviously none of the components has less than n vertices. We will prove that none of the components has more than n vertices. Let C be a component of G which has at least $n+1$ vertices. Then there exists a subgraph F_1 of C, which consists of a path on $n-1$ vertices and of one edge. Let $\{u,v\}$ be an edge of the other component of G. Let $F = F_1 \cup \{u,v\}$. Obviously, F is a subgraph of G which is isomorphic to a subgraph of $2C_n$. However, F cannot be extended to $2C_n$, a contradiction.

According to a) and b), G has two components and each component has exactly n vertices. □

THEOREM 2. *A graph G is randomly $2C_{2n+1}$, where $n \geqslant 2$, if and only if*

(i) $G = 2C_{2n+1}$, *or*

(ii) $G = 2K_{2n+1}$, *or*

(iii) $G = C_{2n+1} \cup K_{2n+1}$, *or*

(iv) $G = K_p$, $p \geqslant 2(2n+1)$.

P r o o f. It is not difficult to verify that each of the graphs listed in the statement of the theorem has the desired property. For the converse then, we assume that G is a randomly $2C_{2n+1}$ graph.

1. Assume first that G is disconnected. We will prove that G fits part (i), (ii), or (iii) in the statement of the theorem.

Lemma 1 implies that G has two components and each component has exactly $2n+1$ vertices. Since G is randomly $2C_{2n+1}$, each component is randomly C_{2n+1}. According to Theorem D either $G = 2C_{2n+1}$, or $G = 2K_{2n+1}$, or $G = C_{2n+1} \cup K_{2n+1}$.

2. We henceforth assume that G is connected. We will prove that $G = K_p$, $p \geqslant 2(2n+1)$. Let H be a subgraph of G which is isomorphic to C_{2n+1}.

a) Let $G' = G - H$. Let us notice that G' is a randomly C_{2n+1} graph. We will prove that G' is a complete graph.

If $|V(G')| > 2n+1$, then, by Theorem E, $G' = K_{p'}$, $p' > 2n+1$.

Let $|V(G')| = 2n+1$. By Theorem D either $G' = C_{2n+1}$ or $G' = K_{2n+1}$. We will prove that $G' \neq C_{2n+1}$. Assume the contrary. Let G' be isomorphic to C_{2n+1}. Then $V(G') = \{v_1, v_2, \dots, v_{2n+1}\}$ and $E(G') = \{\{v_i, v_{i+1}\};\ i = 1, 2, \dots 2n\} \cup \{\{v_{2n+1}, v_1\}\}$. Since G is connected, there exists an edge $\{u, v\}$, where $u \in V(H)$, $v \in V(G')$. Without loss of generality we may assume that $v = v_1$. Let us construct the path $u, v_1, v_2, \dots, v_{2n-1}, v_{2n}$. This path can be extended to C_{2n+1} only by adding the edge $\{v_{2n}, u\}$. Now let us construct the path $v_{2n+1}, v_{2n}, u, v_1, v_2, \dots, v_{2n-3}, v_{2n-2}$. This path can be extended to C_{2n+1} only by adding $\{v_{2n-2}, v_{2n+1}\}$. So G' is not isomorphic to C_{2n+1}, a contradiction. Then, by Theorem D, $G' = K_{p'}$, $p' = 2n+1$.

b) Now we show that $G'' = \langle V(H) \rangle$ is complete, too. Assume the contrary. Let $G'' = C_{2n+1}$. To produce a contradiction, it is sufficient to use the same method as we used in the previous paragraph. Then, by Theorem D, $G'' = K_{p'}$, $p' = 2n+1$.

Now it is necessary to prove that for every $u \in V(G')$, $v \in V(G'')$ the graph G contains the edge $\{u, v\}$. It is sufficient to choose $u{-}v$ path on $2n{+}1$ vertices. Since both G' and G'' are complete and G is connected, the path always exists and can be extended to C_{2n+1} only if $\{u, v\}$ edge exists. Since both u and v are arbitrary vertices, G is complete. The formula $p \geqslant 2(2n+1)$ follows from the fact that G is randomly $2C_{2n+1}$. □

LEMMA 2. *Let G be a randomly $2C_4$ graph, $K_{r,s} \subset G$, where $4 \leqslant r \leqslant s$. Let $V(G) = V(K_{r,s})$. Then G is a complete graph.*

P r o o f. Let $G \supset K_{r,s}$ and $V(G) = V(K_{r,s}) = \{u_1, \dots, u_r\} \cup \{v_1, \dots, v_s\}$. Let $\{u_i, u_j\} \in E(G)$, $\{u_i, u_j\} \notin E(K_{r,s})$. Let v_k and v_l be arbitrary vertices which belong to the different partite set than u_i and u_j. Let us construct the path v_k, u_i, u_j, v_l. Because G is randomly $2C_4$, the path can be extended to C_4 only by adding the edge $\{v_k, v_l\}$. Since v_k and v_l are arbitrary vertices, $\{v_k, v_l\} \in E(G)$ for every k, l. If we use a similar method with the edge $\{v_i, v_j\} \in E(G)$, we prove that G is a complete graph. □

THEOREM 3. *A graph G is randomly $2C_4$ if and only if*

(i) *$G = K_{r,s}$, where $4 \leqslant r \leqslant s$, or*

(ii) *$G = 2C_4$, or*

(iii) *$G = 2K_4$, or*

(iv) *$G = C_4 \cup K_4$, or*

(v) *$G = K_p$, where $p \geqslant 8$.*

P r o o f . It is not difficult to verify that each of the graphs listed in the statement of the theorem has the desired property. For the converse, then, we assume that G is randomly $2C_4$.

1. Assume first that G is disconnected. We will prove that G fits part (ii), (iii), or (iv) in the statement of the theorem.

Lemma 1 implies that G has two components and each component has exactly four vertices. Since G is randomly $2C_4$, each of its components has to be randomly C_4. According to Theorem C either $G = 2C_4$, or $G = 2K_4$, or $G = C_4 \cup K_4$.

2. We henceforth assume that G is a connected randomly $2C_4$ graph. We will prove that either $G = K_{r,s}$, where $4 \leqslant r \leqslant s$ or $G = K_p$, where $p \geqslant 8$.

Let H be a subgraph of G isomorphic to C_4. Let $G' = G - H$. Since G is randomly $2C_4$, G' is randomly C_4. According to Theorem C either $G' = K_t$, $t \geqslant 4$ or $G' = K_{a,b}$, $2 \leqslant a \leqslant b$.

a) Let $G' = K_{a,b}$, $2 \leqslant a \leqslant b$. Let $\{w_1, w_2, \dots, w_b\}$ and $\{s_1, s_2, \dots, s_a\}$ be a partition of $V(G')$ into two sets. Let $\{u_1, u_2\}$ and $\{v_1, v_2\}$ be a partition of $V(H)$ into two sets. Since G is connected, it contains an edge which starts in $V(H)$ and ends in $V(G')$. Without loss of generality we may assume that the edge is $\{v_1, w_1\}$. Since the graph G is randomly $2C_4$, every path u_i, v_1, w_1, s_j can be extended to C_4 by adding the edge $\{s_j, u_i\}$ for every $i = 1,2$ and $j = 1, 2, \dots, a$.

Using the same method with the edge $\{u_1, s_1\}$ we prove that the graph G contains all edges $\{v_i, w_j\}$ for every $i = 1,2$ and $j = 1, 2, \dots, b$. So $G \supset K_{a+2,b+2}$ and $V(G) = V(K_{a+2,b+2})$. If $G \neq K_{a+2,b+2}$, by Lemma 2 $G = K_p$, $p \geqslant 8$. Otherwise $G = K_{r,s}$, where $4 \leqslant r \leqslant s$.

b) Let $G' = K_t$, $t \geqslant 4$. Using the same method as in part a) it is not difficult to prove that G is a complete graph, which means that $G = K_p$, $p \geqslant 8$. □

Lemma 3. *Let G be a randomly $2C_{2n}$ graph, $K_{2n,2n} \subset G$, where $n \geqslant 2$. Let $V(G) = V(K_{2n,2n})$. Then G is a complete graph.*

P r o o f . The proof is similar to that of Lemma 2. A major difference is that we have to consider a path on $2n$ vertices instead of a path on 4 vertices. □

Theorem 4. *A graph G is randomly $2C_{2n}$, where $n \geqslant 3$, if and only if*

(i) $G = 2K_{2n}$, *or*

(ii) $G = 2C_{2n}$, *or*

(iii) $G = 2K_{n,n}$, *or*

(iv) $G = C_{2n} \cup K_{n,n}$, *or*

(v) $G = C_{2n} \cup K_{2n}$, *or*

(vi) $G = K_{n,n} \cup K_{2n}$, *or*

(vii) $G = K_{2n,2n}$, *or*

(viii) $G = K_p$, $p \geqslant 4n$.

P r o o f. It is not difficult to verify that each of the graphs listed in the statement of the theorem has the desired property. For the converse, then, we assume that G is randomly $2C_{2n}$.

1. Assume first that G is disconnected. We will prove that G fits one of the parts (i)–(vi) in the statement of the theorem.

Lemma 1 implies that G has two components and each component has exactly $2n$ vertices. Since G is randomly $2C_{2n}$, each of the components is randomly C_{2n}. According to Theorem D, $G = 2K_{2n}$, $G = 2C_{2n}$, $G = 2K_{n,n}$, $G = C_{2n} \cup K_{n,n}$, $G = C_{2n} \cup K_{2n}$, or $G = K_{n,n} \cup K_{2n}$.

2. We henceforth assume that G is connected.

a) Let $|V(G)| > 4n$. We will prove that G is complete. Let H be a subgraph of G which is isomorphic to C_{2n}. Let $G' = G - H$. Since G is randomly $2C_{2n}$, G' is a randomly C_{2n} graph of order $p' > 2n$. Therefore, by Theorem E, G' is complete.

Similarly, if we choose a subgraph of G' isomorphic to C_{2n}, we will obtain that $\langle V(H) \rangle$ is complete.

Now it is sufficient to prove that for every $v \in V(H)$ and for every $w \in V(G')$ there exists the edge $\{v, w\} \in E(G)$. As G is connected and both $\langle V(H) \rangle$ and G' are complete, there exists a $v - w$ path on $2n$ vertices. Let us choose this path. Because G is randomly $2C_{2n}$, the path can be extended to C_{2n} by adding the edge $\{v, w\}$. That is why G is complete.

b) Let $\left|V(G)\right| = 4n$. Let us construct graphs H and G' similarly to case a). Since the order of G' is $2n$ and G is randomly $2C_{2n}$, by Theorem D, both G' and $\langle V(H) \rangle$ are isomorphic to C_{2n}, $K_{n,n}$, or K_{2n}.

b$_1$) We will prove that neither G' nor $\langle V(H) \rangle$ cannot be isomorphic to C_{2n}. Assume the contrary. Let G' be isomorphic to C_{2n}. Then $V(G') = \{v_1, v_2, \ldots, v_{2n}\}$ and $E(G') = \big\{\{v_i, v_{i+1}\};\ i = 1, 2, \ldots, 2n-1\big\} \cup \big\{\{v_{2n}, v_1\}\big\}$. Since G is connected, there exists an edge $\{u, v\}$, where $u \in V(H)$, $v \in V(G')$. Without loss of generality we may assume that $v = v_1$. Let us construct the path $u, v_1, v_2, \ldots, v_{2n-2}, v_{2n-1}$. This path can be extended to C_{2n} by adding an edge $\{v_{2n-1}, u\}$. Now let us construct the path $v_{2n}, v_{2n-1}, u, v_1, v_2, \ldots, v_{2n-4}, v_{2n-3}$. This path can be extended to C_{2n} only by adding $\{v_{2n-3}, v_{2n}\}$. So G' is not isomorphic to C_{2n}, a contradiction. Similarly, we can prove that $\langle V(H) \rangle$ cannot be isomorphic to C_{2n} either.

b$_2$) Let both G' and $\langle V(H) \rangle$ be isomorphic to $K_{n,n}$. Let $\{w_1, w_2, \ldots, w_n\}$ and $\{s_1, s_2, \ldots, s_n\}$ be a partition of $V(G')$ into two sets. Let $\{u_1, u_2, \ldots, u_n\}$

and $\{v_1, v_2, \dots, v_n\}$ be a partition of $V(H)$ into two sets. We will prove that for every $i, j \in \{1, 2, \dots, n\}$ $E(G)$ contains the edge $\{s_i, v_j\}$. Since G is connected, it contains an edge which starts in $V(H)$ and ends in $V(G')$. Without loss of generality we may assume that the edge is $\{v_1, w_1\}$. Let us construct the path $v_i, u_1, w_1, s_1, w_2, s_2, \dots, w_{j-1}, s_{j-1}, w_{j+1}, s_{j+1}, \dots, s_{n-1}, w_n, s_j$. Since G is randomly $2C_{2n}$, the path can be extended to C_{2n} by adding the edge $\{s_j, v_i\}$, so $\{s_j, v_i\} \in E(G)$. Using the similar method we can prove that $\{w_i, u_j\} \in E(G)$ for every $i, j \in \{1, 2, \dots, n\}$. So $G \supset K_{2n,2n}$. If $G \neq K_{2n,2n}$, then by Lemma 3 $G = K_{4n}$.

b_3) If at least one of the graphs G' and $\langle V(H) \rangle$ is isomorphic to K_{2n}, then similarly to b_2 case we can prove that $G \supset K_{2n,2n}$ and $G \neq K_{2n,2n}$. Then, by Lemma 3, $G = K_{4n}$. □

Conclusion

In the paper a complete characterization of randomly H graphs where $H = 2C_n = C_n \cup C_n$ is given. The characterization of randomly H graphs where $H = C_n \cup C_m$ and $n \neq m$ remains open. Similarly, the case of randomly H graphs, where H is 2-regular graph which contain more than two components, remains open, too.

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