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ON A GENERALIZED BOUNDARY VALUE PROBLEM FOR DIFFERENTIAL SYSTEMS

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ABSTRACT. We prove the existence of a solution of a boundary value problem for differential system with bounded nonlinearity. We prove that linear part defines a Fredholm operator. We use a degree theory method assuming Landesman-Lazer type of condition for nonlinear problem.

We deal with the generalized boundary value problem for the differential system

$$x' = A(t)x + F(t,x), \tag{1}$$

$$l_i(x) = 0, \qquad i = 1, \dots, m, \tag{2}$$

where $A(t)$ is a matrix of continuous functions $a_{ij}(t) \in C\big([a,b]\big)$, $F(t,x)\colon [a,b] \times R^n \to R^n$ is a continuous function and $l_i\colon C\big([a,b]; R^n\big) \to R$ are linearly independent continuous linear functionals. We suppose that $n \geq m$.

The aim of this paper is twofold. In the first part we generalize a result of [5] and prove that the linear operator defined by the linear part of the problem (1), (2) is a Fredholm operator. In the second part we give an existence result for a nonlinear system with bounded nonlinearity based on the degree theory [1] and Landesman-Lazer type of conditions [3].

For existence results and methods based on Landesman-Lazer condition for a certain type of boundary value problems for a second order nonlinear differential equation see [2], [4], [7].

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1. A linear problem

We prove that the linear problem

$$x' = A(t)x\,, \tag{3}$$

$$l_i(x) = 0\,, \qquad i = 1,\dots,n\,, \tag{4}$$

defines always a Fredholm operator of index zero. For the definition and properties of Fredholm operators see [11], [12]. Further results about Fredholm operators and boundary value problems can be found in [9], [10].

Our result is based on the Nikol'skij's Theorem.

Theorem 1.1 ([11] Nikol'skij's Theorem, [12] Theorem 5.C)**.** *Let X, Z be real Banach spaces.*

A linear bounded operator $L\colon X \to Z$ is Fredholm of index zero if and only if $L = C + T$, where C is a linear homeomorphism of X onto Z and $T\colon X \to Z$ is a linear completely continuous operator.

We put $X = \bigl\{x \in C^1([a,b];R^n),\quad l_i(x) = 0,\ i = 1,\dots,n\bigr\}$ and $Z = C\bigl([a,b];R^n\bigr)$. Then the following theorem is true.

Theorem 1.2. *The operator $L\colon X \to Z$, $Lx(t) = x'(t) - A(t)x(t)$ is a Fredholm operator of index zero.*

P r o o f. We shall find a suitable homeomorphism C and a completely continuous operator T such that $L = C + T$.

Assume that there are given linearly independent vector functions

$$\phi_i(t) \in C^1\bigl([a,b];R^n\bigr), \qquad i = 1,\dots,n\,,$$
$$\phi_i(t) = \bigl(\phi_{i1}(t),\dots,\phi_{in}(t)\bigr)\,.$$

For $i = 1,\dots,n$ we denote by M_i the matrix

$$M_i = \begin{pmatrix} x_i' & \phi_{1i}' & \phi_{2i}' & \dots & \phi_{ni}' \\ x_1 & \phi_{11} & \phi_{21} & \dots & \phi_{n1} \\ \dots & & & & \\ x_n & \phi_{1n} & \phi_{2n} & \dots & \phi_{nn} \end{pmatrix}.$$

Each vector function ϕ_i, $i = 1,\dots,n$ is a solution of the system of differential equations

$$\det M_i = 0, \qquad i = 1,\dots,n\,.$$

Let us denote by $M_{i[j,k]}$ a submatrix of M_i obtained by omitting the jth row and the kth column of M_i.

As $M_{i[1,1]}$ is the same for each i, then in the case when $\det M_{i[1,1]} \neq 0$, our system is of the form

$$x_i' = \sum_{j=1}^{n} (-1)^{(j+1)} \frac{\det M_{i[j+1,1]}}{\det M_{i[1,1]}} x_j \, .$$

Now we choose by the induction vector functions $\phi_i(t)$ such that $\det M_{i[1,1]} \neq 0$, and each nontrivial linear combination

$$\phi = \sum_{i=1}^{n} c_i \phi_i$$

doesn't satisfy the set of the generalized boundary conditions (4).

1. Put $\hat{\phi}_1$ such that $\hat{\phi}_{11} = 1$.

If $l_1(\hat{\phi}_1) \neq 0$, we take $\phi_1 = \hat{\phi}_1$ as the first basis function. If $l_1(\hat{\phi}_1) = 0$, we choose arbitrary $\bar{\phi}_1$ such that $l_1(\bar{\phi}_1) \neq 0$. Being from $C^1\big([a,b]\big)$, $\bar{\phi}_{11}$ achieves a minimum on $[a,b]$, $|\min \bar{\phi}_{11}(t)| = m$. Now

$$\phi_1(t) = \bar{\phi}_1(t) + (m+1)\hat{\phi}_1(t)$$

is such that $\phi_{11}(t) > 0$ for $t \in [a,b]$. Obviously $l_1(\phi_1) = l_1(\bar{\phi}_1) \neq 0$.

2. Assume that functions $\phi_1, \dots, \phi_{k-1}$, $\phi_i(t) \in C^1\big([a,b]; R^n\big)$ satisfy

$$\begin{aligned} \det \varPhi_{(k-1)} &> 0, \qquad && \text{for} \quad t \in [a,b], \\ l_i(\phi_i) \neq 0, \quad l_j(\phi_i) &= 0 && \text{for} \quad i = 1, \dots, k-1, \qquad j < i \, , \end{aligned} \tag{5}$$

where

$$\varPhi_{(k-1)} = \begin{pmatrix} \phi_{11} & \dots & \phi_{(k-1)1} \\ \dots & & \\ \phi_{1(k-1)} & \dots & \phi_{(k-1)(k-1)} \end{pmatrix}.$$

We find a function $\phi_k \in C^1\big([a,b]; R^n\big)$ such that (5) holds for k instead of $k-1$. At first we find a function $\hat{\phi}_k \in C^1\big([a,b]; R^n\big)$ such that $\det \hat{\varPhi}_k = 1$. This is possible choosing $\hat{\phi}_{ki}$, $i = 1, \dots, k-1$ arbitrary and setting

$$\hat{\phi}_{kk} = \frac{1 - \sum\limits_{i=1}^{k-1} \hat{\phi}_{ki} \det \hat{\varPhi}_{k[i,k]}}{\det \varPhi_{k-1}} \, .$$

Denote

$$\bar{\phi}_k(t) = \hat{\phi}_k(t) + c_1\phi_1(t) + \dots + c_{k-1}\phi_{k-1}(t) \, .$$

Now beginning by c_1, we choose the coefficients $\{c_i\}_{i=1}^{k-1}$ such that $l_j(\bar{\phi}_k) = 0$ for $j = 1, \dots, k-1$. The choice is possible while $l_j(\phi_i) = 0$ for $j < i$.

If $l_k(\bar{\phi}_k) \neq 0$, we set $\phi_k = \bar{\phi}_k$. If $l_k(\bar{\phi}_k) = 0$, we choose a function $\tilde{\phi}_k$ arbitrary but such that $l_j(\tilde{\phi}_k) = 0$ for $j = 1, \dots, k-1$ and $l_k(\tilde{\phi}_k) \neq 0$, and we calculate the determinant

$$\tilde{\Phi}_k = \mathrm{d}(t).$$

Denote $m = |\min \mathrm{d}(t)|$. Now the vector function $\phi_k = \tilde{\phi}_k + (m+1)\bar{\phi}_k$ satisfies (5). That means, the boundary value problem

$$x'(t) = B(t)x(t)\,, \tag{6}$$

$$l_i(x) = 0, \qquad i = 1, \dots, n\,, \tag{4}$$

where

$$b_{ij}(t) = (-1)^{(j+1)} \frac{\det M_{i[j+1,1]}}{\det M_{i[1,1]}},$$

admits only the trivial solution.

Then the operator $C\colon X \to Z$ given by

$$Cx(t) = x'(t) - B(t)x(t)\,,$$

i.e., the operator generated by the problem (6), (4) is continuous, invertible and onto Z. Then

$$A = C + T$$

where $T\colon X \to Z$, $Tx(t) = \big(B(t) - A(t)\big)x(t)$. Obviously $T\colon C^1\big([a,b]; R^n\big) \to C\big([a,b]; R^n\big)$ is a completely continuous operator.

□

We use a generalization of Nikoľskij's theorem to characterize a situation when the number of the boundary conditions is different from the number of the equations.

Theorem 1.3 ([6]). *Let X, Z be real Banach spaces.*

A linear bounded operator $L\colon X \to Z$ is Fredholm of a positive index k if and only if $L = CP + T$ where $P\colon X \to X_b$ is a continuous projection onto X_b, $X = X_a \oplus X_b$, $\dim X_a = k$, C is a linear homeomorphism of X_b onto Z and $T\colon X \to Z$ is a linear completely continuous operator.

Now let us consider the problem

$$x' = A(t)x\,, \tag{3}$$

$$l_i(x) = 0, \qquad i = 1, \dots, m\,, \tag{2}$$

with $m < n$. We define $X = \big\{x \in C^1([a,b]; R^n),\, l_i(x) = 0,\ i = 1, \dots, m\big\}$ and $Z = C([a,b]; R^n)$.

Theorem 1.4. *The operator $L\colon X \to Z$, $Lx(t) = x'(t) - A(t)x(t)$ is a Fredholm mapping of the index $k = n - m$.*

Proof. Let $\tilde{l}_i\colon C\big([a,b];R^n\big) \to R$ for $i = 1,\dots,k$ be continuous linear functionals such that $l_i,\ i = 1,\dots,m,\ \tilde{l}_j,\ j = 1,\dots,k$ are linearly independent and denote by $\phi_j \in X$, $j = 1,\dots,k$ such functions that $\tilde{l}_i(\phi_j) = 0$ for $i < j$ and $\tilde{l}_j(\phi_j) \neq 0$.

Set $X_b = \big\{x \in X,\quad \tilde{l}_i(x) = 0 \quad i = 1,\dots,k\big\}$ and $X_a = \operatorname{span}\{\phi_1,\dots,\phi_k\}$. Obviously $X = X_a \oplus X_b$. Let P be a projection onto X_b and $P' = I - P$.

Theorems 1.1 and 1.2 imply $L_1\colon X_b \to Z$, $L_1x = x' - A(t)x$ is a Fredholm operator of index zero and $L_1 = C + T_1$, where C is a homeomorphism $C\colon X_b \to Z$ and a T_1 is a completely continuous operator $T_1\colon X_b \to Z$.

Now for the operator $L\colon X \to Z$, $Lx = x' - A(t)x$ there is

$$L = L(P + P') = LP + LP' = L_1P + LP' = CP + (T_1P + LP') = CP + T\,,$$

where

$$T\colon X \to Z, \qquad T = T_1P + LP'$$

is a completely continuous operator. □

2. Nonlinear boundary value problem

In this part we consider a nonlinear boundary value problem (1), (4) with the bounded nonlinearity F.

Theorem 1.2 implies that the operator $L\colon X \to Z$, $Lx = x' - A(t)x$, $X = \big\{x \in C^1([0,T];R^n),\ l_i(x) = 0,\ i = 1,\dots,n\big\}$, $Z = C\big([0,T];R^n\big)$ is a Fredholm operator of index zero.

We suppose that the null space $Ker(L)$ is a one dimensional subspace of X, $Ker(L) = \operatorname{span}\big\{\phi_1(t)\big\}$. Then R(L) is of codimension one and $Z = R(L) \oplus Z_b$.

Further we suppose that there is a dual operator $L^D\colon Y \to W$, $L^Dy = y' - B(t)y$ with $Y = \big\{y \in C^1([0,T];R^n),\ \tilde{l}_i(y) = 0,\quad i = 1,\dots,n\big\}$, $W = Z$, where $\{X, W\}$ and $\{Y, Z\}$ form dual pairs of Banach spaces. The duality is given by a pair of bilinear operators $D\colon Z \times Y \to R$ and $\bar{D}\colon W \times X \to R$.

Then $D(Lx, y) = \bar{D}(L^Dy, x)$, for each $x \in X$, $y \in Y$.

The operator $L^D\colon Y \to W$ defines a dual boundary value problem

$$y' = B(t)y,$$
$$\tilde{l}_i(y) = 0, \qquad i = 1,\dots,n.$$

A notion of a dual boundary value problem is a generalization of the adjoint problem. A dual problem is not uniquely defined. See [7], [12]. More about the adjoint problem for a generalized boundary value problem can be found in [8].

Theorem 1.2 implies L^D is Fredholm of index 0. Moreover $Ker(L^D)$ is one dimensional and $Ker(L^D) = \operatorname{span}\big\{\phi_2(t)\big\}$.

Now $D(L\phi_2, y) = \bar{D}(L^D y, \phi_2) = 0$ and $\operatorname{codim} R(L) = 1$ imply $R(L) = \big\{z \in Z,\ D(z, \phi_2) = 0\big\}$. We choose a complement Z_b of $R(L)$ by $Z_b = \operatorname{span}\{\phi_2\}$.

Further we suppose that there is a continuous extension of D onto $L_\infty \times Y$ and that $D(\phi_2, \phi_2) > 0$.

Theorem 2.1. *Suppose that*

(i) *F is continuous and $|F| \le M$ for each $x \in R^n$, $t \in I = [0, T]$,*

(ii) *there exist limits $F_1(t),\ F_2(t) \in L_\infty(I, R^n)$*

$$F_1(t) = \lim_{k\to\infty} F\big(t, k\phi_1(t)\big), \qquad F_2(t) = \lim_{k\to-\infty} F\big(t, k\phi_1(t)\big)\,,$$

(iii)

$$D\big(F_1(t), \phi_2(t)\big) < 0 < D\big(F_2(t), \phi_2(t)\big)\,.$$

Then there exists a solution of the problem (1), (4).

P r o o f. The operator form of the problem (1), (4) is

$$Lx = Nx$$

where $Lx = x' - Ax$, $Nx = F(t, x)$,

The Ljapunov-Schmidt reduction gives the equivalent problem

$$\begin{aligned} 0 &= QNx, \\ x_b &= L^{-1}(I - Q)Nx \end{aligned}$$

or

$$(x_a, x_b) = S(x_a, x_b)\,,$$

where $S\colon \big(Ker(L), X_b\big) \to \big(Ker(L), X_b\big)$,

$$S(x_a, x_b) = \big(x_a + JQNx, L^{-1}(I - Q)Nx\big)$$

is a completely continuous operator, $Q\colon Z \to Z_b$ is a continuous projection onto $Z_b = \operatorname{span}\big\{\phi_2(t)\big\}$ and L^{-1} is the inverse operator to $L|_{X_b}$ defined on $R(L)$ which is continuous as an operator onto X_b and completely continuous as an operator to Z, J is a natural isomorphism $J\colon Z_b \to Ker(L)$ defined by $J(k\phi_2) = k\phi_1$.

We define a homotopy

$$H(\lambda, x) = (I - \lambda S)x\,,$$

which is completely continuous as an operator $H\colon [0,1]\times Z \to Z$.

We use the Leray-Schauder degree and prove that there exists an open bounded set Ω such that

$$\deg\big(H(\lambda,x),\Omega,0\big)$$

is constant and nonzero for each $\lambda \in [0,1]$.

The homotopy invariance of the degree is satisfied provided that there is no solution of the equation

$$H(\lambda,x)=0 \tag{7}$$

on $\partial\Omega$.

Suppose $x=x_a+x_b$ is a solution of (7). Then

$$x_b=\lambda L^{-1}(I-Q)Nx$$

and boundedness of F implies there is a real constant R_b such that

$$\|x_b\|<R_b\,.$$

Now (7) is equivalent to

$$Lx=\lambda Nx-(1-\lambda)J^{-1}x_a$$

and for each $u\in Z_b$

$$\lambda D(Nx,u)=(1-\lambda)D(J^{-1}x_a,u)\,.$$

Assume now that there are sequences $\lambda_n\in(0,1]$, $x_{an}\in Ker(L)$ such that $\|x_{an}\|\to\infty$ and $H(\lambda_n,x_{an}+x_{bn})=0$, where $\|x_{bn}\|<R_b$. As $x_{an}=k_n\phi_1(t)$ there is $|k_n|\to\infty$. We choose $u_n=\operatorname{sgn}(k_n)\phi_2(t)$. Then

$$D(Nx_n,u_n)=\frac{(1-\lambda)}{\lambda}|k_n|D(\phi_2,\phi_2)>0\,.$$

That means

$$\operatorname{sgn}(k_n)D(Nx_n,\phi_2)\geq 0\,.$$

The last inequality is in a contradiction with the assumptions (ii), (iii).

That means there is a real constant R_a such that

$$\|x_a\|<R_a$$

for each solution $x=x_a+x_b$ of (7) and there is no solution of (7) on $[0,1]\times\partial\Omega$ where

$$\Omega=\big\{x\in C(I;R^2),\ x=x_a+x_b,\ \|x_a\|<R_a,\ \|x_b\|<R_b\big\}\,.$$

Then the Leray-Schauder degree

$$\deg\big(H(1,x),\Omega,0\big)=\deg\big(H(0,x),\Omega,0\big)\neq 0$$

and there exists a solution $x\in\Omega$ of $H(1,x)=0$. $\square$

3. Examples

Let $a(t)\colon R_0^+ \to R^+$ be a continuous function and $T > 0$ be a constant such that $\int_0^T a(s)\,\mathrm{d}s = 2\pi$.

We consider a linear boundary value problem

$$x_1' = a(t)x_2, \qquad x_2' = -a(t)x_1\,, \tag{8}$$

$$x_1(0) = x_1(T), \qquad x_2(0) = -x_2(T)\,. \tag{9}$$

Theorem 1.2 implies that the operator $L\colon X \to Z$, $Lx = x' - Ax$, $A = \left(\begin{smallmatrix} 0 & a(t) \\ -a(t) & 0 \end{smallmatrix}\right)$, $X = \bigl\{x \in C^1([0,T];R^2),\ x_1(0) = x_1(T),\ x_2(0) = -x_2(T)\bigr\}$, $Z = C\bigl([0,T];R^2\bigr)$ is Fredholm of index 0.

Set $\phi_1(t) = \left(\cos\left(\int_0^t a(s)\,\mathrm{d}s\right), -\sin\left(\int_0^t a(s)\,\mathrm{d}s\right)\right)$. Then $Ker(L) = \operatorname{span}\bigl\{\phi_1(t)\bigr\}$, $\dim Ker(L) = \operatorname{codim} R(L) = 1$ and $R(L) = \left\{z \in Z,\ \int_0^T \bigl(z(t), \phi_1(t)\bigr)\,\mathrm{d}t = 0\right\}$. So there are complementary subspaces X_b, $Z_b = \operatorname{span}\bigl\{\phi_1(t)\bigr\}$ such that $X = Ker(L) \oplus X_b$ and $Z = R(L) \oplus Z_b$.

We set $X = Y$, $Z = W$, and define dualities $D = \bar{D}$ by

$$D(x,y) = \int_0^T \bigl(x(t), y(t)\bigr)\,\mathrm{d}t = 0\,.$$

Then the dual operator L^D is equal to $-L$ and a dual boundary value problem to (8), (9) is the same problem (8), (9). Now we consider an antiperiodic nonlinear boundary value problem

$$\begin{aligned} x_1' &= a(t)x_2 + f_1(t,x_1,x_2) + h_1(t)\,,\\ x_2' &= -a(t)x_1 + f_2(t,x_1,x_2) + h_2(t)\,, \end{aligned} \tag{10}$$

(9) and we give an existence result based on a Landesman-Lazer type of Condition [3].

Let $t_0 = 0 < t_1 < t_2 < t_3 < t_4 = T$ be such that $\int_{t_i}^{t_{i+1}} a(s)\,\mathrm{d}s = \frac{\pi}{2}$.

Theorem 2.1 implies the following result.

Theorem 3.1. *Suppose that*

(i) *f_i are continuous functions and $|f_i(t,x_1,x_2)| \le M$ for each $x_1, x_2 \in R$, $t \in I = [0,T]$,*

(ii) *for*

$$F(t, x_1, x_2) = \big(f_1(t, x_1, x_2), f_2(t, x_1, x_2)\big)$$

there is

$$F^{\pm}_{\pm}(t) = \lim_{\substack{x_1 \to \pm\infty \\ x_2 \to \pm\infty}} F(t, x_1, x_2)$$

(iii)

$$\begin{aligned}
&\int_0^{t_1} F^{-}_{+}(t) \cdot \phi_1(t)\,\mathrm{d}t + \int_{t_1}^{t_2} F^{+}_{+}(t) \cdot \phi_1(t)\,\mathrm{d}t \\
&\quad + \int_{t_2}^{t_3} F^{+}_{-}(t) \cdot \phi_1(t)\,\mathrm{d}t + \int_{t_3}^{T} F^{-}_{-}(t) \cdot \phi_1(t)\,\mathrm{d}t \\
&> -\int_0^{T} h(t) \cdot \phi_1(t)\,\mathrm{d}t \\
&> \int_0^{t_1} F^{+}_{-}(t) \cdot \phi_1(t)\,\mathrm{d}t + \int_{t_1}^{t_2} F^{-}_{-}(t) \cdot \phi_1(t)\,\mathrm{d}t \\
&\quad + \int_{t_2}^{t_3} F^{-}_{+}(t) \cdot \phi_1(t)\,\mathrm{d}t + \int_{t_3}^{T} F^{+}_{+}(t) \cdot \phi_1(t)\,\mathrm{d}t\,.
\end{aligned}$$

Then there exists a solution of the problem (10), (9).

Let us consider a nonlinear boundary value problem (10), with the periodic boundary condition

$$x_1(0) = x_1(T)\,. \tag{11}$$

Set $\phi_2(t) = \left(\sin\left(\int_0^t a(s)\,\mathrm{d}s\right), \cos\left(\int_0^t a(s)\,\mathrm{d}s\right)\right)$.

Theorem 1.4 implies that the operator $L\colon X \to Z$, is a Fredholm operator of index 1, where $X = \{x \in C^1([0,T]; R^2),\ x_1(0) = x_1(T),\}$, $Z = C\big([0,T]; R^2\big)$, $Ker(L) = \operatorname{span}\{\phi_1(t), \phi_2(t)\}$, $\dim Ker(L) = 2$, $\operatorname{codim} R(L) = 1$, and $R(L) = \left\{y \in Z,\ \int_0^T (y, \phi_1(t))\,\mathrm{d}t\ = 0\right\}$.

We use the decomposition

$$X = X_0 \oplus \operatorname{span} \phi_2(t), \qquad X_0 = \{x \in X,\ x_2(0) = -x_2(T)\}\,.$$

Setting $x = x_0 + c\phi_2(t)$, $x_0 \in X_0$ and using Theorem 3.1 in the problem

$$Lx_0 + N\big(x_0 + c\phi_2(t)\big) = h$$

for c a fixed real constant we prove the following existence result.

Theorem 3.2. *Suppose that assumptions* (i), (ii), (iii) *of* Theorem 3.1 *are satisfied.*

Then for each fixed $c \in R$ *there exists* $x_0 \in X_0$ *such that* $x(t) = x_0(t) + c\phi_2(t)$ *is a solution of the problem* (10), (11).

Theorem 3.2 implies that for each $c \in R$ there exists a solution of the problem (10),

$$x_1(0) = x_1(T), \qquad x_2(0) + x_2(T) = c\,.$$

Now we consider a nonlinear boundary value problem (10),

$$x_2(0) = -x_2(T). \tag{12}$$

with the antiperiodic boundary condition.

Theorem 3.3 below asserts that there are infinitely many solutions of (10), (12) again, but the situation is somewhat different.

The operator $L\colon X \to Z$, $X = \big\{x \in C^1\big([0,T]; R^2\big),\ x_2(0) = -x_2(T),\big\}$, $Z = C\big([0,T]; R^2\big)$ is a Fredholm operator of index 1, but the dimension of $Ker(L) = \operatorname{span}\big\{\ \phi_1(t)\big\}$ is $\dim Ker(L) = 1$ and L is surjective, $\operatorname{codim} R(L) = 0$.

We choose a vector function $\phi_3(t) = \big(\int_0^t a(s)\,\mathrm{d}s\,, 0\big) \in X$. Obviously $\phi_3(t)$ does not satisfy (11) and integration by parts leads to $\int_0^T (L(\phi_3)\cdot\phi_1) = 2\pi$.

We use the decomposition

$$X = X_0 \oplus \operatorname{span}\big\{\phi_3(t)\big\}$$

where $X_0 = \{x \in X,\ x_1(0) = x_1(T)\}$. Now the restriction $L|_{X_0}$ is a Fredholm operator of index zero with $Ker(L) = Ker(L|_{X_0})$ and $R(L|_{X_0}) = \Big\{y \in Z,\ \int_0^T (y, \phi_1(t))\,\mathrm{d}t\ = 0\Big\}$.

We solve the problem

$$L\big(x_0 + c\phi_3(t)\big) = N\big(x_0 + c\phi_3(t)\big) + h\,,$$

reducing to

$$L(x_0) = N\big(x_0 + c\phi_3(t)\big) + h - c\left(a(t), a(t)\int_0^t a(s)\,\mathrm{d}s\right).$$

Setting $h_c(t) = h(t) - c\big(a(t), a(t)\int_0^t a(s)\,ds\big)$ and assuming (iv) in Theorem 3.3 below there are $c_1, c_2 \in R$ such that for each $c \in (c_1, c_2)$ assumption (iii) of Theorem 3.1 is satisfied for $h_c(t)$. Then the following theorem is true.

Theorem 3.3. *Suppose that the assumptions* (i), (ii) *of* Theorem 3.1 *are satisfied and that*

$$\int_0^{t_1} F_+^-(t)\cdot\phi_1(t)\,\mathrm{d}t + \int_{t_1}^{t_2} F_+^+(t)\cdot\phi_1(t)\,\mathrm{d}t + \int_{t_2}^{t_3} F_-^+(t)\cdot\phi_1(t)\,\mathrm{d}t + \int_{t_3}^{T} F_-^-(t)\cdot\phi_1(t)\,\mathrm{d}t >$$

$$\int_0^{t_1} F_-^+(t)\cdot\phi_1(t)\,\mathrm{d}t + \int_{t_1}^{t_2} F_-^-(t)\cdot\phi_1(t)\,\mathrm{d}t + \int_{t_2}^{t_3} F_+^-(t)\cdot\phi_1(t)\,\mathrm{d}t + \int_{t_3}^{T} F_+^+(t)\cdot\phi_1(t)\,\mathrm{d}t\,.$$

Then for each $h \in Z$ there are infinitely many solutions of the problem (10), (12).

Moreover Theorem 3.3 implies that assuming (i), (ii) there are numbers $k_1 < 0$, $k_2 > 0$ such that for each $k \in (k_1, k_2)$ there is a solution of the problem (10),

$$x_1(T) - x_1(0) = k, \quad x_2(0) = -x_2(T)\,.$$

Finally we consider the problem (10)

$$x_1(T) - x_1(0) = k, \qquad x_2(0) + x_2(T) = c\,. \tag{13}$$

Corollary 3.4. *Suppose that the assumptions* (i), (ii), (iii) *hold.*

Then there are $k_1, k_2 \in R$ such that for each $k \in (k_1, k_2)$ and for each $c \in R$ there exists a solution of the problem (10), (13).

Next we deal with a three point boundary value problem

$$\begin{aligned} x_1' &= a\,x_2 + f_1(t, x_1, x_2) + h_1(t),\\ x_2' &= -a\,x_1 + f_2(t, x_1, x_2) + h_2(t), \end{aligned} \tag{14}$$

$$x_1(0) = x_1(T), \qquad x_2(0) = x_2(t_0). \tag{15}$$

Now $a > 0$ is constant, $T = \frac{2\pi}{a}$ and $0 < t_0 < T$.

A corresponding linear boundary value problem

$$\begin{aligned} x_1' &= a\,x_2, & x_2' &= -a\,x_1,\\ x_1(0) &= x_1(T), & x_2(0) &= x_2(t_0) \end{aligned}$$

defines a linear operator $L\colon X \to Z$, $Lx = x' - Ax$, $A = \left(\begin{smallmatrix} 0 & a\\ -a & 0\end{smallmatrix}\right)$, $X = \big\{x \in C^1\big([0,T]; R^2\big),\ \ x_1(0) = x_1(T),\ x_2(0) = x_2(t_0)\big\}$, $Z = C\big([0,T]; R^2\big)$ which is Fredholm operator of index zero by Theorem 1.2.

It is easy to see that $Ker(L) = \operatorname{span}\{\phi_1(t)\}$, $\phi_1(t) = \big(\sin a(t - \frac{t_0}{2}), \cos a(t - \frac{t_0}{2})\big)$. Then $X = Ker(L) \oplus X_b$ where $\dim Ker(L) = \operatorname{codim} R(L) = 1$.

To determine $R(L)$, we define a (nonunique) dual boundary value problem.

Set $\bar{t}_0 = T - t_0$. Denote $Y = \{y \in C^1\big([0,T]; R^2\big),\ y_1(0) = y_1(T),\ y_2(\bar{t}_0) = y_2(T)\}$, $W = Z = C\big([0,T]; R^2\big)$ and define a pair of bilinear operators $D\colon Z \times \times Y \to R$, $\bar{D}\colon W \times X \to R$

$$D\big(z(t), y(t)\big) = -\int_0^T z(t)y(t)\,\mathrm{d}t + \int_0^{t_0} z(t)y(t+\bar{t}_0)\,\mathrm{d}t + \int_{t_0}^T z(t)y(t-t_0)\,\mathrm{d}t$$

$$\bar{D}\big(w(t), x(t)\big) = -\int_0^T w(t)x(t)\,\mathrm{d}t + \int_{\bar{t}_0}^T w(t)x(t+\bar{t}_0)\,\mathrm{d}t + \int_0^{\bar{t}_0} w(t)x(t-t_0)\,\mathrm{d}t\,.$$

The integration by parts yields $D(Lx, y) = \bar{D}(L^D y, x)$, with a dual operator $L^D\colon Y \to W$, $L^D y = -y' + Ay$. By L^D and the Banach space Y it is defined a dual boundary value problem

$$y_1' = a\,y_2, \quad y_2' = -a\,y_1,$$
$$y_1(0) = y_1(T), \quad y_2(\bar{t}_0) = y_2(T).$$

Theorem 1.2 implies that L^D is Fredholm operator of the index 0 and it is easy to find $Ker(L^D) = \operatorname{span}\{\phi_2(t)\}$, with $\phi_2(t) = \big(\sin a(t + \frac{t_0}{2}), \cos a(t + \frac{t_0}{2})\big)$.

Now $D(L\phi_2, y) = \bar{D}(L^D y, \phi_2) = 0$ and $\operatorname{codim} R(L) = 1$ imply $R(L) = \{z \in Z,\ D(z, \phi_2) = 0\}$.

After the integration we obtain

$$D(z, \phi_2) = \int_0^T 2\sin\frac{at_0}{2}\big(z(t)\cdot\phi_3(t)\big)\,\mathrm{d}t\,,$$

where $\phi_3(t) = (-\cos at, \sin at)$.

Let $t_k = \frac{kT}{4} - \frac{t_0}{2}$, $k = 1, \dots, 5$, and $\tilde{F}$ be a T–periodic function in the variable t such that $\tilde{F}(t, x_1, x_2) = F(t, x_1, x_2)$ for $t \in [0, T]$.

Then Theorem 2.1 implies the following result.

Theorem 3.5. *Assume* (i), (ii) *of* Theorem 3.1 *and*

$$\begin{aligned}
&\int_{t_1}^{t_2} \tilde{F}_+^-(t) \cdot \phi_3(t)\,\mathrm{d}t + \int_{t_2}^{t_3} \tilde{F}_+^+(t) \cdot \phi_3(t)\,\mathrm{d}t \\
&\quad + \int_{t_3}^{t_4} \tilde{F}_-^+(t) \cdot \phi_3(t)\,\mathrm{d}t + \int_{t_4}^{t_5} \tilde{F}_-^-(t) \cdot \phi_3(t)\,\mathrm{d}t \\
&> -\int_0^T h(t) \cdot \phi_3(t)\,\mathrm{d}t \\
&> \int_{t_1}^{t_2} \tilde{F}_-^+(t) \cdot \phi_3(t)\,\mathrm{d}t + \int_{t_2}^{t_3} \tilde{F}_-^-(t) \cdot \phi_3(t)\,\mathrm{d}t \\
&\quad + \int_{t_3}^{t_4} \tilde{F}_+^-(t) \cdot \phi_3(t)\,\mathrm{d}t + \int_{t_4}^{t_5} \tilde{F}_+^+(t) \cdot \phi_3(t)\,\mathrm{d}t\,.
\end{aligned}$$

Then there exists a solution of the problem (14)*,* (15)*.*

REFERENCES

[1] GAINES, R. E.—MAWHIN, J.: *Coincidence Degree, and Nonlinear Differential Equations*. Lecture Notes in Math. 568, Springer, Berlin, 1977.

[2] GROSSINHO, M. R.: *Some existence results for nonselfadjoint problems at resonance*, Contemp. Math. **72** (1988), 107–119.

[3] LANDESMAN, E. M.—LAZER, A. C.: *Nonlinear perturbations of a linear elliptic boundary value problem*, J. Math. Mech. **19** (1970), 609–623.

[4] PRZERADZKI, B.: *Three methods for the study of semilinear equations at resonance*, Colloq. Math. **66** (1993), 109–129.

[5] RUDOLF, B.: *The generalized boundary value problem is a Fredholm mapping of index zero*, Arch. Math. (Brno) **31** (1995), 55–58.

[6] RUDOLF, B.: *Dual pairs and the generalized boundary value problem.* In: 2nd Workshop Functional analysis and its applications, 1999, Nemecká, pp. 59–61.

[7] RUDOLF, B.: *On the generalized boundary value problem*, Arch. Math. **36** (2000), 125–137.

[8] SCHWABIK, Š.—TVRDÝ, M.—VEJVODA, O.: *Differential and Integral Equations. Boundary Value Problems and Adjoints*, Academia, Praha, 1979.

[9] ŠEDA, V.: *Fredholm mappings and the generalized boundary value problem*, Differential and Integral Equations **8** (1995), 19–40.

[10] ŠEDA, V.: *Generalized boundary value problems with linear growth*, Math. Bohem. **123** (1998), 385–404.
[11] TRENOGIN, V. A.: *Functional Analysis*, Nauka, Moscow, 1980. (In Russian)
[12] ZEIDLER, E.: *Applied Functional Analysis. Applications to Mathematical Physics, Vol. 1.*, Appl. Math. Sci., Vol. 108, Springer-Verlag, Berlin, 1995.

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