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EXISTENCE OF POSITIVE PERIODIC SOLUTIONS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We establish some criteria for existence of positive periodic solutions of nonlinear functional differential equations with parameter. The eigenvalue theory of operators is used.

1. Introduction

In this paper we will study the existence of positive periodic solutions of the first order functional differential equation

$$y'(t) = -a(t)y(t) + \lambda h(t) f\big(y\big(t-\tau(t)\big)\big), \qquad t \in R, \tag{1}$$

where $a = a(t)$ and $\tau = \tau(t)$ are continuous T-periodic functions, $h = h(t)$ is a positive continuous T-periodic function, $T > 0$, $\lambda > 0$ and

$$\int_0^T a(t)\,\mathrm{d}t > 0\,.$$

Recently Xi-Lan Lin and Wan-Tong Li [6] investigated the existence of positive periodic solutions of Eq. (1) by using the eigenvalue theory of operators under the conditions

$$0 < f_\infty < \infty, \quad f_\infty = \infty, \quad 0 < f_0 < \infty \quad \text{or} \quad f_0 = \infty\,,$$

where

$$f_\infty = \lim_{u\to\infty} \frac{f(u)}{u} \qquad \text{and} \qquad f_0 = \lim_{u\to 0^+} \frac{f(u)}{u}\,.$$

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They assume that the function $f\colon [0,\infty) \to [0,\infty)$ is continuous with the property $f(0) = 0$.

In our article we do not suppose that the function f satisfies the condition $f(0) = 0$. We will establish several sufficient conditions for the existence of positive T-periodic solutions of Eq. (1) by using the eigenvalue theory of operators. Some authors, e.g., Cheng and Zhang [1] investigated the existence of positive solutions of (1) by using Krasnosel'skii fixed point theorem.

The differential equation

$$y'(t) = sf\big(y(t)\big) - ny(t), \qquad s, n \in (0,\infty)\,,$$

emerged in the theory of a neoclassical growth economic model [2]. It is fundamental dynamic equation of the model, where $y(t)$ is a capital/labour ratio. In addition the cognate differential equations have recently found a variety of applications in several fields of natural sciences. In mathematical biology and physiology such equations with retarded argument describe systems with delayed response, for example in respiratory control circuits or neural interactions, etc. It is due to influence of the past history of the processes on their evolution. Such equations were also suggested to model processes in radiophysics and optics.

Now we impose condition on the function f,

$$f\colon [0,\infty) \to [0,\infty) \quad \text{is continuous.} \tag{2}$$

We also assume that

$$\int_0^T h(s)\,\mathrm{d}s \,= B > 0\,.$$

We call a continuously differentiable and T-periodic function a periodic solution of (1) associated with λ^* if it satisfies (1) when $\lambda = \lambda^*$.

2. Auxiliary lemmas

From Eq. (1) we get

$$\begin{aligned} y'(t)\exp\left(\int_{t_0}^t a(s)\,\mathrm{d}s\right) = {}& -a(t)y(t)\exp\left(\int_{t_0}^t a(s)\,\mathrm{d}s\right) \\ & + \lambda h(t) f\big(y\big(t-\tau(t)\big)\big)\exp\left(\int_{t_0}^t a(s)\,\mathrm{d}s\right), \qquad t_0 \in R\,. \end{aligned}$$

Then we have

$$\left[y(t)\exp\left(\int_{t_0}^{t} a(s)\,\mathrm{d}s\right)\right]' = \lambda h(t) f\big(y\big(t-\tau(t)\big)\big)\exp\left(\int_{t_0}^{t} a(s)\,\mathrm{d}s\right).$$

Integrating from t to $t+T$ we obtain

$$y(t) = \lambda \int_{t}^{t+T} G(t,s)h(s)f\big(y\big(s-\tau(s)\big)\big)\,\mathrm{d}s\,, \tag{3}$$

where

$$G(t,s) = \frac{\exp\left(\int_t^s a(u)\,\mathrm{d}u\right)}{\exp\left(\int_0^T a(u)\,\mathrm{d}u\right)-1}, \qquad s\in[t,t+T], \quad t\in R\,.$$

We conclude that Eq. (1) has T-periodic solutions if and only if (3) has T-periodic solutions [1], [6]. Further we have

$$0 < N \equiv \min_{t\in[0,T]} G(t,s) \le G(t,s) \le \max_{t\in[0,T]} G(t,s) \equiv M, \qquad t \le s \le t+T,$$

and

$$1 \ge \frac{G(t,s)}{\max_{t\in[0,T]} G(t,s)} \ge \frac{\min_{t\in[0,T]} G(t,s)}{\max_{t\in[0,T]} G(t,s)} = \frac{N}{M} = k > 0, \quad s\in[t,t+T].$$

Let X be the set of all real T-periodic continuous functions endowed with the usual linear structure and the norm

$$||y|| = \sup_{t\in[0,T]} |y|.$$

Then X is a Banach space. Define the cone of X by

$$P = \{y\in X : y \ge k||y||\}$$

and an operator $A\colon X\to X$ by

$$(Ay)(t) = \int_t^{t+T} G(t,s)h(s)f\big(y\big(s-\tau(s)\big)\big)\,\mathrm{d}s, \qquad t\in R\,. \tag{4}$$

Then for any $y\in P$ we obtain

$$(Ay)(t) \le M\int_t^{t+T} h(s)f\big(y\big(s-\tau(s)\big)\big)\,\mathrm{d}s$$

and

$$(Ay)(t) \geq N \int_{t}^{t+T} h(s) f\big(y\big(s-\tau(s)\big)\big)\,\mathrm{d}s \geq k||Ay||\,.$$

Now we conclude our consideration above in the next lemma.

Lemma 2.1. ([1], [6]). *$Ay \in P$ for any $y \in P$, that is $AP \subset P$.*

In order to prove our results the following lemmas are also needed.

Lemma 2.2. ([3], [6]). *Suppose D is an open subset of the infinite-dimensional real Banach space E, $\theta \in D$ and P is a cone of E. If the operator $\Gamma\colon P \cap \bar{D} \to P$ is completely continuous with $\Gamma\theta = \theta$ and*

$$\inf_{x \in P \cap \partial D} ||\Gamma x|| > 0,$$

then Γ has a proper element on $P \cap \partial D$ associated with a positive eigenvalue. That is, there exist $x_0 \in P \cap \partial D$ and $\mu_0 > 0$ such that $\Gamma x_0 = \mu_0 x_0$.

Lemma 2.3. ([4]) *(Schauder's fixed point theorem). Let Ω be a closed, convex and nonempty subset of a Banach space X. Let $A : \Omega \to \Omega$ be a continuous mapping such that $A\Omega$ is a relatively compact subset of X. Then A has at least one fixed point in Ω. That is, there exists an $x \in \Omega$ such that $Sx = x$.*

3. Existence of positive periodic solutions

In this section we shall establish sufficient conditions for the existence of positive periodic solutions of Eq. (1).

Theorem 3.1. *Suppose that (2) holds, $0 < f_\infty < l_2$ and $Ml_2B \leq 1$. Then there exists a positive number R_0 such that for any $r > R_0$, (1) has a positive T-periodic solution $y_r^*(t)$ associated with $\lambda^* \in [\lambda_1, \lambda_2]$ and $||y_r^*(t)|| = r$, where λ_1 and λ_2 are two positive finite numbers.*

Proof. With regard to (3) and (4), Equation (1) has a positive T-periodic solution $y_r^*(t)$ associated with $\lambda^* > 0$ if and only if the operator A has a proper element y_r^* associated with the eigenvalue $1/\lambda^* > 0$. Since $0 < f_\infty < l_2$ there exists $l_1 > 0$ and large enough $\eta > 0$ such that

$$l_1 u < f(u) < l_2 u \qquad \text{for any} \quad u \geq \eta.$$

Let $R_0 = k^{-1}\eta$ and $\Omega_r = \{y \in X : ||y|| < r\}$, where $r > R_0$. Then Ω_r is a bounded open subset of the Banach space X.

We now show that $A\colon P\cap\bar{\Omega}_r\to P$ is completely continuous. First we will show that A is continuous. Let $y_k\in P\cap\bar{\Omega}_r$ and $y_k(t)\to y(t)$ as $k\to\infty$. Since $P\cap\bar{\Omega}_r$ is closed $y(t)\in P\cap\bar{\Omega}_r$. For $t\in R$ we get

$$\begin{aligned}
&|(Ay_k)(t)-(Ay)(t)|\\
&=\left|\int_t^{t+T} G(t,s)h(s)\Big[f\big(y_k\big(s-\tau(s)\big)\big)-f\big(y\big(s-\tau(s)\big)\big)\Big]\,\mathrm{d}s\right|\\
&\le \int_t^{t+T} G(t,s)h(s)\big|f\big(y_k\big(s-\tau(s)\big)\big)-f\big(y\big(s-\tau(s)\big)\big)\big|\,\mathrm{d}s\,.
\end{aligned}$$

Since $\big|f\big(y_k(s-\tau(s))\big)-f\big(y(s-\tau(s))\big)\big|\to 0$ as $k\to\infty$, by applying the Lebesgue dominated convergence theorem we conclude that

$$\lim_{k\to\infty}||(Ay_k)(t)-(Ay)(t)||=0\,.$$

This means that A is continuous.

Next we show that $A(P\cap\bar{\Omega}_r)$ is relatively compact. By Arzela's theorem it is sufficiently to show that the family of functions $\{Ay : y\in P\cap\bar{\Omega}_r\}$ is uniformly bounded and equicontinuous on R.

For any $y(t)\in P\cap\bar{\Omega}_r,\ t\in R$ we have

$$\begin{aligned}
(Ay)(t) &\le M\int_t^{t+T} h(s)f\big(y\big(s-\tau(s)\big)\big)\,\mathrm{d}s\\
&\le Ml_2\int_t^{t+T} h(s)y\big(s-\tau(s)\big)\,\mathrm{d}s\\
&\le Ml_2B||y||\le Ml_2Br\,.
\end{aligned}$$

This means the uniform boundedness.

For the equicontinuity we need to show that for any given $\varepsilon>0$ interval $(-\infty,\infty)$ can be decomposed into finite subintervals in such a way that on each subinterval all functions of the family have a change of amplitude less than ε.

For $y(t) \in P \cap \bar{\Omega}_r,\ t_1, t_2 \in (-\infty, \infty)$ we have

$$|(Ay)(t_2) - (Ay)(t_1)| = \left| \int_{t_2}^{t_2+T} G(t_2, s)h(s)f\big(y\big(s - \tau(s)\big)\big)\,\mathrm{d}s - \int_{t_1}^{t_1+T} G(t_1, s)h(s)f\big(y\big(s - \tau(s)\big)\big)\,\mathrm{d}s \right| .$$

From this if follows that for any $\varepsilon > 0$ we can find $\delta > 0$ such that $0 < |t_2 - t_1| < \delta$ implies $|(Ay)(t_2) - (Ay)(t_1)| < \varepsilon$.

Thus $\{Ay : y \in P \cap \bar{\Omega}_r\}$ is uniformly bounded and equicontinuous on $(-\infty, \infty)$ and hence $A(P \cap \bar{\Omega}_r)$ is relatively compact.

Now we shall show that the operator A has a fixed point $\theta \in \Omega_r$, i.e., $A\theta = \theta \in \Omega_r$. So we consider the set

$$\Omega_b = \big\{y(t) \in X : y(t) \geq 0,\ ||y|| \leq b < r,\ b > 0,\ t \in R\big\} .$$

Then $\Omega_b \subset \Omega_r$. For any $y \in \Omega_b$ we get

$$\begin{aligned}
(Ay)(t) &= \int_t^{t+T} G(t, s)h(s)f\big(y\big(s - \tau(s)\big)\big)\,\mathrm{d}s \\
&\leq Ml_2 \int_t^{t+T} h(s)y\big(s - \tau(s)\big)\,\mathrm{d}s \\
&\leq Ml_2 \int_t^{t+T} h(s) \sup_{s \in [t, t+T]} \big|y\big(s - \tau(s)\big)\big|\,\mathrm{d}s \\
&\leq Ml_2||y|| \int_t^{t+T} h(s)\,\mathrm{d}s \leq Ml_2bB .
\end{aligned}$$

Since $Ml_2B \leq 1$, thus $Ml_2bB \leq b$. So we have $||Ay|| \leq b$ and $A\Omega_b \subset \Omega_b$. By a similar way as in the first part of the proof we can prove that A is continuous on Ω_b and that $A\Omega_b$ is a relatively compact subset of X. With regard to Lemma 2.3 there exists a $\theta \in \Omega_b$ such that $A\theta = \theta$.

Furthermore we get

$$\begin{aligned}(Ay)(t) &= \int_t^{t+T} G(t,s)h(s)f\big(y\big(s-\tau(s)\big)\big)\,\mathrm{d}s \\ &\geq Nl_1 \int_t^{t+T} h(s)y\big(s-\tau(s)\big)\,\mathrm{d}s \\ &\geq Nl_1k||y|| \int_0^T h(s)\,\mathrm{d}s = Nl_1kBr > 0\end{aligned}$$

for any $r > R_0$ and $y \in P \cap \partial\Omega_r$. Thus we have

$$\inf_{y \in P\cap\partial\Omega_r} ||Ay|| \geq Nl_1kBr > 0\,.$$

By Lemma 2.2 for any $r > R_0$ the operator A has a proper element $y_r^* \in P\cap\partial\Omega_r$ associated with the eigenvalue $\mu^* > 0$, i.e., $Ay_r^* = \mu^* y_r^*$ and y_r^* satisfies $||y_r^*|| = r$. If $\lambda^* = 1/\mu^*$, then (1) has a positive T-periodic solution $y_r^*(t)$ associated with λ^*.

Furthermore, for any $r > R_0$ there exists a positive T-periodic solution $y_r^* \in P \cap \Omega_r$ associated with $\lambda^* > 0$ such that

$$y_r^* = \lambda^* \int_t^{t+T} G(t,s)h(s)f\big(y_r^*\big(s-\tau(s)\big)\big)\,\mathrm{d}s$$

with $||y_r^*|| = r$. We obtain

$$\begin{aligned}y_r^*(t) &\leq \lambda^* M \int_t^{t+T} h(s)f\big(y_r^*\big(s-\tau(s)\big)\big)\,\mathrm{d}s \\ &\leq \lambda^* Ml_2r \int_0^T h(s)\,\mathrm{d}s \leq \lambda^* Ml_2Br\,.\end{aligned}$$

So we have

$$||y_r^*|| = r \leq \lambda^* l_2MBr\,,$$

which means that

$$\lambda^* \geq \frac{1}{l_2MB} = \lambda_1\,.$$

On the other hand we get

$$\begin{aligned} y_r^*(t) &\geq \lambda^* N \int_t^{t+T} h(s) f\big(y_r^*\big(s-\tau(s)\big)\big)\,\mathrm{d}s \\ &\geq \lambda^* l_1 N k \|y_r^*\| \int_0^T h(s)\,\mathrm{d}s = \lambda^* l_1 N B k r\,. \end{aligned}$$

Then

$$\|y_r^*\| = r \geq \lambda^* l_1 N B k r\,,$$

which implies that

$$\lambda^* \leq \frac{1}{l_1 N B k} = \lambda_2\,.$$

Hence we have $\lambda^* \in [\lambda_1, \lambda_2]$. The proof is complete. □

Theorem 3.2. *Suppose that* (2) *holds,* $f_\infty = \infty$*, an interval* $[0,b]$, $b > 0$*, is mapped by* f *into itself and* $MB \leq 1$*. Then there exists a positive number* R_0 *such that for any* $r > R_0$, (1) *has a positive* T*-periodic solution* $y_r^*(t)$ *associated with* $\lambda^* \leq \lambda_2$*, where* λ_2 *is a positive finite number and* $\|y_r^*(t)\| = r$.

Proof. Since $f_\infty = \infty$, there exist $l_1 > 0$ and sufficiently large $\eta > 0$ such that $l_1 u < f(u)$ for any $u \geq \eta$. We take $R_0 = k^{-1}\eta$ and $R_0 \geq b$. Let

$$\Omega_r = \big\{y \in X : \|y\| < r\big\},$$

where $r > R_0$. Then Ω_r is a bounded open subset of the Banach space X. We shall show that the operator A has a fixed point $\theta \in \Omega_r$. We consider the closed and convex set

$$\Omega_b = \big\{y(t) \in X : y(t) \geq 0,\ \|y\| \leq b,\ t \in R\big\}\,.$$

Then $\Omega_b \subset \Omega_r$. For any $y \in \Omega_b$ we obtain

$$(Ay)(t) = \int_t^{t+T} G(t,s) h(s) f\big(y\big(s-\tau(s)\big)\big)\,\mathrm{d}s \leq Mb \int_t^{t+T} h(s)\,\mathrm{d}s = MBb\,.$$

Since $MB \leq 1$, it follows that $MBb \leq b$. Hence we have $\|Ay\| \leq b$, i.e., $A\Omega_b \subset \Omega_b$.

We now show that A is continuous on Ω_b. Let $y_k \in \Omega_b$ and $y_k(t) \to y(t)$ as $k \to \infty$. Since Ω_b is closed, $y(t) \in \Omega_b$. For $t \in R$ we get

$$|(Ay_k)(t) - (Ay)(t)| \le \int_t^{t+T} G(t,s)h(s)\big|f\big(y_k\big(s-\tau(s)\big)\big) - f\big(y\big(s-\tau(s)\big)\big)\big|\,\mathrm{d}s\,.$$

Since $\big|f\big(y_k(s-\tau(s))\big) - bl(y(s-\tau(s)))\big| \to 0$ as $k \to \infty$, by applying the Lebesgue dominated convergence theorem we have that $\lim\limits_{k\to\infty} ||(Ay_k)(t) - (Ay)(t)|| = 0$. This means that A is continuous. By a similar way as in the proof of Theorem 1 we can prove that $A\Omega_b$ is a relatively compact subset of X. With regard to Lemma 2.3 there exists a $\theta \in \Omega_b$ such that $A\theta = \theta$.

Proceeding similarly as in the proof of Theorem 3.1 we can show that $A : P \cap \bar{\Omega}_r \to P$ is completely continuous. Furthermore, we have

$$\begin{aligned}(Ay)(t) &\ge Nl_1 \int_t^{t+T} h(s)y\big(s-\tau(s)\big)\,\mathrm{d}s \\ &\ge Nl_1k||y|| \int_0^T h(s)\,\mathrm{d}s = Nl_1kBr\,,\end{aligned}$$

for any $r > R_0$ and $y \in P \cap \partial\Omega_r$. Then we get

$$\inf_{y\in P\cap\partial\Omega_r} ||Ay|| \ge Nl_1kBr > 0\,.$$

By Lemma 2.2 for any $r > R_0$ the operator A has a proper element $y_r^* \in P\cap\partial\Omega_r$ associated with the eigenvalue $\mu^* > 0$. If $\lambda^* = 1/\mu^*$, then (1) has a positive T-periodic solution $y_r^* \in P \cap \partial\Omega_r$ associated with $\lambda^* > 0$ such that

$$y_r^*(t) = \lambda^* \int_t^{t+T} G(t,s)h(s)f\big(y_r^*\big(s-\tau(s)\big)\big)\,\mathrm{d}s$$

and $||y_r^*|| = r$. We have

$$\begin{aligned}y_r^*(t) &\ge \lambda^* N \int_t^{t+T} h(s)f\big(y_r^*\big(s-\tau(s)\big)\big)\,\mathrm{d}s \\ &\ge \lambda^* Nl_1k||y_r^*|| \int_0^T h(s)\,\mathrm{d}s = \lambda^* Nl_1Bkr\,.\end{aligned}$$

Then

$$||y_r^*|| = r \geq \lambda^* N B l_1 k r\,,$$

and we get

$$\lambda^* \leq \frac{1}{NBl_1k} = \lambda_2\,.$$

The proof of Theorem 3.2 is complete. $\square$

Theorem 3.3. *Suppose that* (2) *holds,* $0 < f_0 < d_2$ *and* $Md_2B \leq 1$*. Then there exists a positive number* R_0 *such that for any* $0 < r < R_0$*,* (1) *has a positive* T*-periodic solution* $y_r^*(t)$ *associated with* $\lambda^* \in [\bar\lambda_1, \bar\lambda_2]$*,* $||y_r^*(t)|| = r$ *and* $\bar\lambda_1$*,* $\bar\lambda_2$ *are two positive finite numbers.*

Proof. Since $0 < f_0 < d_2$, there exist $R_0 > 0$ and constant $d_1 > 0$ such that

$$d_1 u < f(u) < d_2 u \qquad \text{for any} \quad 0 < u < R_0.$$

Let

$$U_r = \big\{ y \in X : ||y|| < r \big\}\,,$$

where $0 < r < R_0$. Proceeding now as in the proof of Theorem 3.1 and using the Lemma 2.3 we can show that there exists a $\theta \in U_r$ such that $A\theta = \theta$ and that the operator $A : P \cap \bar U_r \to P$ is completely continuous. Further we have

$$\begin{aligned} (Ay)(t) &= \int\limits_t^{t+T} G(t,s)h(s)f\big(y\big(s-\tau(s)\big)\big)\,\mathrm{d}s \\ &\geq Nd_1 \int\limits_t^{t+T} h(s)y\big(s-\tau(s)\big)\,\mathrm{d}s \;\geq Nd_1k||y|| \int\limits_0^T h(s)\,\mathrm{d}s \\ &= Nd_1kBr > 0 \end{aligned}$$

for any $0 < r < R_0$ and $y \in P \cap \partial U_r$. Then we get

$$\inf_{y\in P\cap\partial U_r} ||Ay|| \geq Nd_1kBr > 0\,.$$

According to Lemma 2.2 for any $0 < r < R_0$ the operator A has a proper element $y_r^* \in P \cap \partial U_r$ associated with the eigenvalue $\mu^* = 1/\lambda^*$ and $||y_r^*|| = r$. Then (1) has a positive T-periodic solution $y_r^*(t)$ associated with $\lambda^* \in [\bar\lambda_1, \bar\lambda_2]$. The proof of the Theorem 3.3 is complete. $\square$

The proof of the next theorem is similar to that of previous theorems, so we only outline it.

Theorem 3.4. *Suppose that (2) holds, $f_0 = \infty$, an interval $[0, b]$, $b > 0$, is mapped by f into itself and $MB \le 1$. Then there exists a positive number R_0 such that for any r, $b < r < R_0$, (1) has a positive T-periodic solution $y_r^*(t)$ associated with $\lambda^* \le \bar{\lambda}_2$, where $\bar{\lambda}_2$ is a positive finite number and $||y_r^*(t)|| = r$.*

Proof. Since $f_0 = \infty$, there exist $R_0 > 0$ and constant $d_1 > 0$ such that

$$d_1 u < f(u) \qquad \text{for any} \quad 0 < u < R_0.$$

Let

$$U_r = \{y \in X : ||y|| < r\},$$

where $0 < r < R_0$.

We also consider a closed and convex set

$$U_b = \{y(t) \in X : y(t) \ge 0,\ ||y|| \le b,\ t \in R\},$$

where $0 < b < r$. Proceeding as in the proof of previous theorems and using Lemma 2.2 we get that for any $b < r < R_0$, (1) has a positive T-periodic solution $y_r^* \in P \cap \partial U_r$ associated with $\lambda^* > 0$ such that

$$y_r^*(t) = \lambda^* \int\limits_t^{t+T} G(t,s) h(s) f\big(y_r^*\big(s - \tau(s)\big)\big)\,\mathrm{d}s$$

and $||y_r^*|| = r$. We obtain

$$\begin{aligned} y_r^*(t) &\ge \lambda^* N \int\limits_t^{t+T} h(s) f\big(y_r^*\big(s-\tau(s)\big)\big)\,\mathrm{d}s \\ &\ge \lambda^* N d_1 \int\limits_t^{t+T} h(s) y_r^*\big(s-\tau(s)\big)\,\mathrm{d}s \\ &\ge \lambda^* N d_1 k ||y_r^*|| \int\limits_0^{T} h(s)\,\mathrm{d}s = \lambda^* N d_1 k B r\,. \end{aligned}$$

Then

$$||y_r^*|| = r \ge \lambda^* N d_1 k B r\,,$$

and we have

$$\lambda^* \le \frac{1}{NBd_1k} = \bar{\lambda}_2\,.$$

The proof is complete. $\square$

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