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THE CROSSING NUMBERS OF PRODUCTS OF CYCLES WITH 6-VERTEX TREES

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ABSTRACT. There are known exact values of the crossing numbers of Cartesian products of cycles with graphs of order four and with several graphs of order five. In this paper the crossing numbers of the Cartesian products of cycles with all 6-vertex trees, except of the star are given. For the star on six vertices an upper bound is given.

1. Introduction

Let G be a simple graph with vertex set V and edge set E. The *crossing number* $cr(G)$ of a graph G is the minimum number of edge crossings in any drawing of G in the plane. It is implicit that the edges in a drawing are Jordan arcs (hence, nonselfintersecting), and it is easy to see that a drawing with the minimum number of crossings (an optimal drawing) must be a *good* drawing; that is, each two edges have at most one point in common, which is either a common end-vertex or a crossing. Let D be a good drawing of the graph G. We denote the number of crossings in D by $cr_D(G)$. Let G_i and G_j be edge disjoint subgraphs of G. We denote by $cr_D(G_i, G_j)$ the number of crossings between edges of G_i and edges of G_j, and by $cr_D(G_i)$ the number of crossings among edges of G_i in D.

Computing the crossing number of a given graph is in general an elusive problem, and the crossing numbers of few families of graphs are known. Most of them are Cartesian products of special graphs. (For a definition of Cartesian product, see [4].)

Let C_n be the cycle with n vertices. Harary *et al.* [7] conjectured that the crossing number of $C_m \times C_n$ is $(m-2)n$, for all m, n satisfying $3 \leqslant m \leqslant n$. This has been proved only for m, n satisfying $n \geqslant m$, $m \leqslant 7$, [1], [2], [3], [4], [5], [11], [12],[13]. It has been recently proved by L. Y. Glebsky and

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G. Salazar [6] that the crossing number of $C_m \times C_n$ equals its long-conjectured value at least for $n \geqslant m(m+1)$. The crossing numbers of the Cartesian products of cycles and all graphs of order four are determined in [4], [8]. The table in [10] shows the summary of known crossing numbers for Cartesian products of cycles and connected graphs of order five. Thus it seems natural to inquire about the crossing numbers of the products of 6-vertex graphs with cycles. It is the purpose of this paper to give the crossing numbers of Cartesian products of all trees of order six (see Figure 1) with cycles, except of the star. For the star on six vertices an upper bound is given.

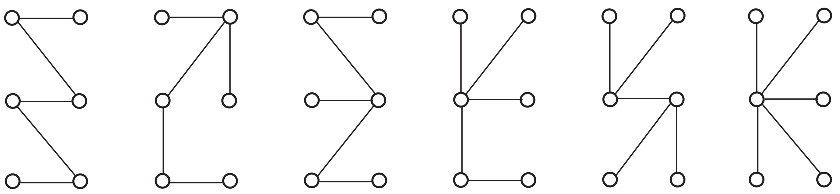


FIGURE 1.

2. The crossing number of $G_j \times C_n$ for $j = 1, 2, 3$ and 4

One can easily verify that the graph $G_1 \times C_n$ is planar, and hence $cr(G_1 \times C_n) = 0$. Figure 2(a) shows the drawing of the graph $G_2 \times C_n$ in which the edges of every subgraph isomorphic to G_2 are crossed exactly once. As the graph $G_2 \times C_n$ contains n copies of G_2, the crossing number of $G_2 \times C_n$ is not greater than n. The graph $G_2 \times C_n$ contains the graph $K_{1,3} \times C_n$ as a subgraph and, since $cr(K_{1,3} \times C_n) = n$ (see [8]), $cr(G_2 \times C_n) \geqslant n$. Therefore, the crossing number of the graph $G_2 \times C_n$ is n. Using the drawing of one copy of the graph G_3 in Figure 2(b), the same analysis shows that $cr(G_3 \times C_n) = n$. In [9] it is proved that $cr(K_{1,4} \times C_n) = 2n$. The fact that $K_{1,4} \times C_n$ is a subgraph of the graph $G_4 \times C_n$, together with the drawing of one copy of G_4 in Figure 2(c) imply that $cr(G_4 \times C_n) = 2n$. For the graph $G_6 = K_{1,5}$, we have the upper bound $4n$ for the crossing number of $G_6 \times C_n$ obtained from the drawing of one copy of G_6 in Figure 2(d). Thus, we can summarize these results in the forthcoming Theorem 1.

THEOREM 1. *For $n \geqslant 3$, $cr(G_1 \times C_n) = 0$, $cr(G_2 \times C_n) = cr(G_3 \times C_n) = n$, $cr(G_4 \times C_n) = 2n$ and $cr(G_6 \times C_n) \leqslant 4n$.*

The exact value of the crossing number for the Cartesian product of the remaining tree of order six with cycle is derived in the next section.

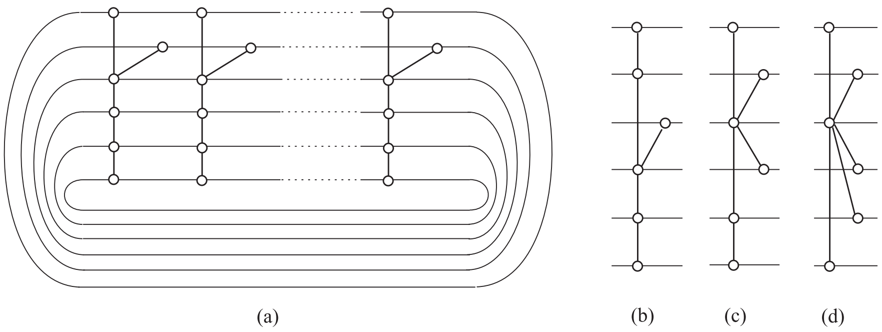


FIGURE 2.

3. The crossing number of $G_5 \times C_n$

We assume $n \geqslant 3$ and find it convenient to consider the graph $G_5 \times C_n$ in the following way: it has $6n$ vertices and edges that are the edges in the n copies G_5^i, $i = 0, 1, \dots, n-1$, and in the six cycles of length n. For $i = 0, 1, \dots, n-1$, let a_i, b_i, e_i and f_i be the vertices of G^i of degree one, c_i and d_i the vertices of degree three (see Figure 3). Thus, for $x \in \{a, b, c, d, e, f\}$, the n-cycle C_n^x is induced by the vertices $x_0, x_1, \dots, x_{n-1}$. For $i = 0, 1, \dots, n-1$, let M^i denote the subgraph of $G_5 \times C_n$ containing the vertices of G_5^i and G_5^{i+1} and six edges joining G_5^i to G_5^{i+1}, i taken modulo n. Let T^x, $x = a, b$ ($x = e, f$), be the subgraph of the graph $G_5 \times C_n$ consisting of the cycle C_n^x together with the vertices of C_n^c (C_n^d) and of the edges joining C_n^x with C_n^c (C_n^d). Denote by I the subgraph of $G_5 \times C_n$ consisting of the vertices of C_n^c and C_n^d and of the edges $\{c_i, d_i\}$ for all $i = 0, 1, \dots, n-1$, and let $H = C_n^c \cup I \cup C_n^d$. It is not difficult to see that

$$G_5 \times C_n = T^a \cup T^b \cup C_n^c \cup I \cup C_n^d \cup T^e \cup T^f = T^a \cup T^b \cup H \cup T^e \cup T^f .$$

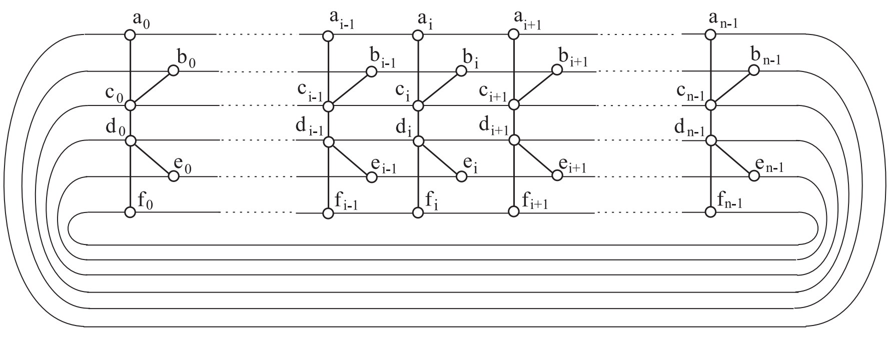


FIGURE 3.

Let P_n be the path with $n+1$ vertices and let S_n denote the star $K_{1,n}$. The graph $P_1 \times C_n$, $n \geqslant 3$, consists of two vertex-disjoint cycles C_n and n edges joining vertices of these n-cycles.

Lemma 1. *For $n \geqslant 4$, there is no good drawing of the graph $P_1 \times C_n$ with one crossing.*

Proof. Assume there is a good drawing of $P_1 \times C_n$ with exactly one crossing. As in a good drawing no two edges incident with the same vertex cross each other, for $n \geqslant 4$ one can easily verify that in any good drawing of $P_1 \times C_n$ the edges that cross each other must lie in two different edge-disjoint cycles. Since two edge-disjoint cycles cannot cross only once, we have a contradiction. □

In the proofs of the paper, we will often use the term "region" also in nonplanar drawings. In this case, crossings are considered to be vertices of the "map".

Theorem 2. $cr(G_5 \times C_3) = 2$.

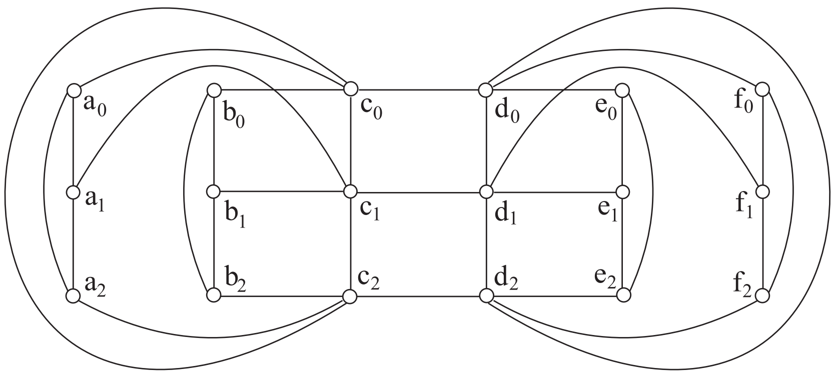
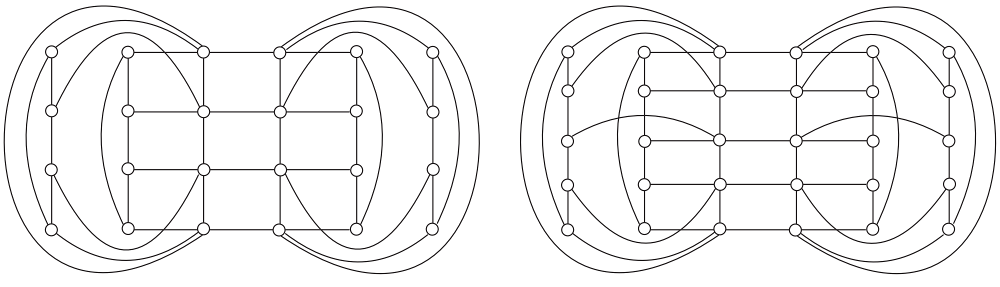


Figure 4.

Proof. It follows from Figure 4 that $cr(G_5 \times C_3) \leqslant 2$. Assume there is a good drawing of $G_5 \times C_3$ with fewer than two crossings and let D be such a drawing. The subgraphs $T^a \cup T^b \cup H$ and $T^e \cup T^f \cup H$ of the graph $G_5 \times C_3$ are both isomorphic to $S_3 \times C_3$ and $cr(S_3 \times C_3) = 1$ (see [8]). So, if $cr_D(H) = 0$, then $cr_D(G_5 \times C_3) \geqslant 2cr(S_3 \times C_3) = 2$. This contradiction implies that $cr_D(H) = 1$, $cr_D(T^a \cup T^b) = 0$ and $cr_D(T^a \cup T^b, H) = 0$. Hence, as $T^a \cup T^b$ is the subdivision of the 3-connected planar graph $P_1 \times C_3$, the planar subdrawing of $T^a \cup T^b$ induced from D divides the plane into two triangular and three hexagonal regions in such a way that there is no region with all three vertices c_0, c_1, and c_2 on its boundary. So, an edge of $I \cup C_3^d$ crosses an edge of $T^a \cup T^b$ in D, which contradicts $cr_D(T^a \cup T^b, H) = 0$. This completes the proof. □

Theorem 3. $cr(G_5 \times C_4) = 4$.

Proof. Figure 5(a) shows that $cr(G_5 \times C_4) \leqslant 4$. Assume there is a good drawing of $G_5 \times C_4$ with at most 3 crossings and let D be such a drawing. Both subgraphs $T^a \cup T^b \cup H$ and $T^e \cup T^f \cup H$ of the graph $G_5 \times C_4$ are isomorphic to $S_3 \times C_4$. As $cr(S_3 \times C_4) = 2$ (see [8]), if $cr_D(H) = 0$, then $cr_D(G_5 \times C_4) \geqslant 2cr(S_3 \times C_4) = 4$, a contradiction. From Lemma 1 we know that $cr_D(H) \geqslant 2$ and $cr_D(T^a \cup T^b) + cr_D(T^a \cup T^b, H) = 0$ or $cr_D(T^e \cup T^f) + cr_D(T^e \cup T^f, H) = 0$. Without loss of generality, let $cr_D(T^a \cup T^b) + cr_D(T^a \cup T^b, H) = 0$. As $T^a \cup T^b$ is the subdivision of the 3-connected planar graph $P_1 \times C_4$, the planar subdrawing of $T^a \cup T^b$ induced from D divides the plane into two quadrangular and four hexagonal regions in such a way that there are at most two of the vertices c_0, c_1, c_2, and c_3 on the boundary of one hexagonal region. So $cr_D(T^a \cup T^b, I \cup C_4^d) \neq 0$, and this contradiction completes the proof. □



(a) (b)

Figure 5.

Theorem 4. $cr(G_5 \times C_5) = 8$.

Proof. The drawing in Figure 5(b) shows that $cr(G_5 \times C_5) \leqslant 8$. Assume, there is a good drawing of $G_5 \times C_5$ with at most 7 crossings and let D be such a drawing. Both subgraphs $T^a \cup T^b \cup H$ and $T^e \cup T^f \cup H$ of the graph $G_5 \times C_5$ are isomorphic to $S_3 \times C_5$. As $cr(S_3 \times C_5) = 4$ (see [8]), if $cr_D(H) = 0$, then $cr_D(G_5 \times C_5) \geqslant 2cr(S_3 \times C_5) = 8$, a contradiction. Lemma 1 implies that $cr_D(H) \geqslant 2$, and $cr_D(T^a \cup T^b) + cr_D(T^a \cup T^b, H) \leqslant 2$ or $cr_D(T^e \cup T^f) + cr_D(T^e \cup T^f, H) \leqslant 2$. Without loss of generality, let $cr_D(T^a \cup T^b) + cr_D(T^a \cup T^b, H) \leqslant 2$.

Assume first that $cr_D(T^a \cup T^b) = 0$. As $T^a \cup T^b$ is the subdivision of the 3-connected planar graph $P_1 \times C_5$, the planar subdrawing of $T^a \cup T^b$ induced from D divides the plane into two pentagonal and five hexagonal regions in such a way that there are at most two of the vertices c_0, c_1, c_2, c_3, and c_4 on the boundary of one hexagonal region, see Figure 6(a). So, if $cr_D(T^a \cup T^b, C_5^d) \neq 0$, then $cr_D(T^a \cup T^b, C_5^d) = 2$ and C_5^d lies in D in at most two regions of the subdrawing induced by $T^a \cup T^b$. In this case $cr_D(T^a \cup T^b, I) \neq 0$, a contradiction. If

$cr_D(T^a \cup T^b, C_5^d) = 0$, then C_5^d lies in only one region of the subdrawing induced by $T^a \cup T^b$ and, in D, the edges joining C_5^d with the vertices c_0, c_1, c_2, c_3, and c_4 cross $T^a \cup T^b$ more than two times, a contradiction with $cr_D(T^a \cup T^b, H) \leqslant 2$. So $cr_D(T^a \cup T^b) \neq 0$, and the condition $cr_D(T^a \cup T^b) + cr_D(T^a \cup T^b, H) \leqslant 2$, implies that $cr_D(T^a \cup T^b, H) \leqslant 1$. Hence $cr_D(T^a, I \cup C_5^d) = 0$ or $cr_D(T^b, I \cup C_5^d) = 0$.

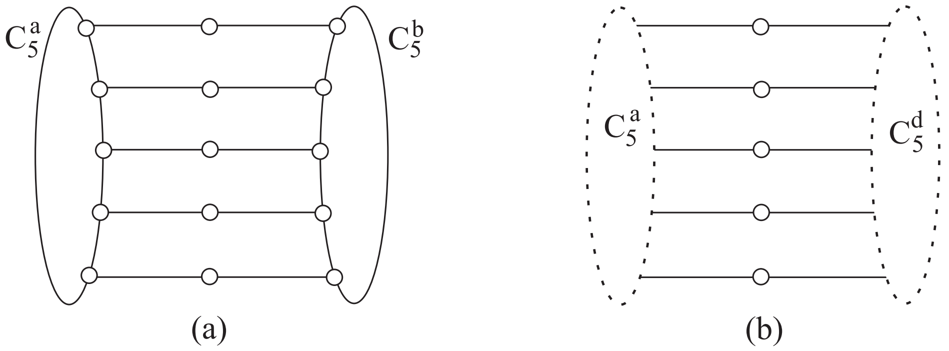


FIGURE 6.

Without loss of generality, let $cr_D(T^a, I \cup C_5^d) = 0$. Consider now the subdrawing D' of the drawing D induced by $T^a \cup I \cup C_5^d$. As $cr_D(T^a, I \cup C_5^d) = 0$, D' divides the plane in such a way that on the boundary of a region there are at most two vertices of C_5^c and no two regions with a common boundary contain more than three vertices of C_5^c on their boundaries. Figure 6(b) shows the drawing of D' in which possible crossings among edges of T^a are inside the left disc bounded by the dotted cycle and possible crossings among edges of $I \cup C_6^d$ are inside the right disc bounded by the dotted cycle. We can suppose that if, in D, an edge of T^b passes through one of these two discs, then it crosses the edges of $T^a \cup I \cup C_5^d$ at least twice. Then the similar analysis as in the previous paragraph shows that $cr_D(T^a \cup I \cup C_5^d, T^b) \geqslant 3$, and this contradiction with $cr_D(T^a \cup T^b) + cr_D(T^a \cup T^b, H) \leqslant 2$ completes the proof. □

THEOREM 5. $cr(G_5 \times C_6) = 12$

Proof. In the drawing of the graph $G_5 \times C_n$ in Figure 3 one can easily see that $cr(G_5 \times C_6) \leqslant 12$. Assume there is a good drawing D of $G_5 \times C_6$ with at most 11 crossings. The graph $G_5 \times C_6$ contains $T^a \cup T^b \cup H$ and $T^e \cup T^f \cup H$ as subgraphs and both are isomorphic to $S_3 \times C_6$. $T^a \cup T^b$ and $T^e \cup T^f$ are edge--disjoint graphs and $cr(S_3 \times C_6) = 6$ (see [8]). So, if $cr_D(H) = 0$, then $cr_D(G_5 \times C_6) \geqslant 2cr(S_3 \times C_6) = 2.6 = 12$, a contradiction. Therefore $cr_D(H) \neq 0$ and, by Lemma 1, $cr_D(H) \geqslant 2$. This implies that $cr_D(T^a \cup T^b) + cr_D(T^a \cup T^b, H) \leqslant 4$ or $cr_D(T^e \cup T^f) + cr_D(T^e \cup T^f, H) \leqslant 4$. Without loss of generality assume that $cr_D(T^a \cup T^b) + cr_D(T^a \cup T^b, H) \leqslant 4$.

Consider now the subgraph $T^a \cup T^b$ of the graph $G_5 \times C_6$ and let D' be its subdrawing induced by D. Suppose $cr_D(T^a \cup T^b) = 0$. As $T^a \cup T^b$ is the subdivision of the 3-connected planar graph $P_1 \times C_6$, the planar subdrawing D' divides the plane in a similar way as in Figure 6(a) into two hexagonal regions with no vertices of $c_0, c_1, \ldots, c_5$ (the vertices of C_6^c in the original graph) on their boundaries and into six hexagonal regions with exactly two of the vertices of C_6^c on the boundary of one region. So, if in the view of the subdrawing D' the cycle C_6^d lies in D outside the regions having the vertices of C_6^c on their boundaries, then the edges of I joining in D the vertices of C_6^d with the vertices of C_6^c cross the edges of D' at least six times. If the vertices of C_6^d lie in D in only one region of D' having the vertices of C_6^c on its boundary, then four vertices of C_6^c are outside this region. In the case when $cr_D(T^a \cup T^b, C_6^d) = 0$, at least one of the vertices of C_6^c is separated from C_6^d by at least two vertex--disjoint cycles of D' and $cr_D(T^a \cup T^b, I) \geqslant 5$, otherwise $cr_D(T^a \cup T^b, C_6^d) \geqslant 2$ and $cr_D(T^a \cup T^b, I) \geqslant 4$. If the vertices of C_6^d lie in D in more than one region of D' having the vertices of C_6^c on its boundary, then $cr_D(T^a \cup T^b, C_6^d) \geqslant 2$. In this case, if $cr_D(T^a \cup T^b, C_6^d) \leqslant 3$, at least three vertices of C_6^c lie outside the regions of D' containing the vertices of C_6^d and $cr_D(T^a \cup T^b, I) \geqslant 3$. If C_6^d crosses the edges in D' only four times, it lies at most in three regions of D' with the vertices of C_6^c on their boundaries and in this case $cr_D(T^a \cup T^b, I) \geqslant 1$. Thus, in all possible cases $cr_D(T^a \cup T^b, I \cup C_6^d) \geqslant 5$. This contradiction with $cr_D(T^a \cup T^b) + cr_D(T^a \cup T^b, H) \leqslant 4$ imply that $cr_D(T^a \cup T^b) \neq 0$ and $cr_D(T^a \cup T^b, H) \leqslant 3$

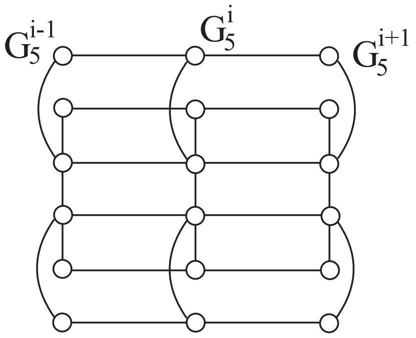
Assume now that $cr_D(T^a, I \cup C_6^d) = 0$. Then, regardless of whether or not the edges of T^a cross each other or the edges of $I \cup C_6^d$ cross each other, the subdrawing D'' of D induced by $T^a \cup I \cup C_6^d$ divides the plane in such a way that on the boundary of a region there are at most two vertices of C_6^c and no two regions with common boundary contain more than three vertices of C_6^c on their boundaries. Figure 6(b) of $T^a \cup I \cup C_5^d$ can be redrawn for showing the drawing D'' in which possible crossings among edges of T^a are inside the left disc bounded by the dotted cycle and possible crossings among edges of $I \cup C_6^d$ are inside the right disc bounded by the dotted cycle. We can suppose again that if, in D, an edge of T^b passes through one of these two discs, then it crosses the edges of $T^a \cup I \cup C_6^d$ at least twice. Then the similar analysis as in the previous paragraph shows that $cr_D(T^a \cup I \cup C_6^d, T^b) \geqslant 5$, and this is in contradiction with $cr_D(T^a \cup T^b) + cr_D(T^a \cup T^b, H) \leqslant 4$. Hence $cr_D(T^a, I \cup C_6^d) \geqslant 1$. The same analysis for the subgraph T^b gives $cr_D(T^b, I \cup C_6^d) \geqslant 1$. This, together with $cr_D(T^a \cup T^b) \neq 0$ and $cr_D(T^a \cup T^b, H) \leqslant 3$, implies that $cr_D(T^a \cup T^b, C_6^c) \leqslant 1$ and $1 \leqslant cr_D(T^a \cup T^b) \leqslant 2$.

Assume first that $cr_D(T^a, T^b) = 0$. Regardless of whether or not the edges of T^a cross each other or the edges of T^b cross each other, the subdrawing of $T^a \cup T^b$ induced by D divides the plane in the same way as the subdrawing

D'' of $T^a \cup I \cup C_6^d$ in the previous paragraph. So, using the same analysis one can find that $cr_D(T^a \cup T^b, I \cup C_6^d) \geqslant 5$, a contradiction with $cr_D(T^a \cup T^b) + cr_D(T^a \cup T^b, H) \leqslant 4$. Therefore $cr_D(T^a, T^b) \geqslant 1$ and, as $cr_D(T^a \cup T^b) \leqslant 2$, the edges of T^a or the edges of T^b do not cross each other. Without loss of generality suppose that $cr_D(T^a) = 0$. If $cr_D(T^a \cup T^b, C_6^c) = 0$ the subdrawing of $T^a \cup C_6^c$ induces the map without crossings in such a way that all edges of T^a lie in one of two regions bounded by C_6^c. So, it follows from $cr_D(T^b, C_6^c) = 0$ and $cr_D(T^a, T^b) \neq 0$ that, in D, the cycles C_6^b and C_6^a lie in the same region in the view of the subdrawing of C_6^c. The similar analysis as above shows that it is not possible with less than three crossings between the edges of T^a and T^b, a contradiction. If $cr_D(T^a \cup T^b, C_6^c) = 1$ then $cr_D(T^a \cup T^b) = 1$ and the fact $cr_D(T^a, T^b) \neq 0$ implies that $cr_D(T^a) = cr_D(T^b) = 0$ and at least one of T^a and T^b does not cross C_6^c. Assume that $cr_D(T^a, C_6^c) = 0$. In this case the subdrawing of $T^a \cup C_6^c$ is planar, and therefore $cr_D(C_6^c) = 0$. Hence, using $cr_D(T^b) = 0$ and $cr_D(T^b, C_6^c) = 1$, we have that $cr_D(T^b \cup C_6^c) = 1$ and this contradiction with Lemma 1 completes the proof. □

Lemma 2. *If D is a good drawing of $G_5 \times C_n$, $n \geqslant 6$, in which every G_5^i has at most one crossing on its edges, then D has at least $2n$ crossings.*

Proof. For $i = 0, 1, \dots, n-1$, let Q^i denote the subgraph of $G_5 \times C_n$ induced by $V(G_5^{i-1}) \cup V(G_5^i) \cup V(G_5^{i+1})$, where i is taken modulo n (see Figure 7(a)). Thus, $Q^i = G_5^{i-1} \cup M^{i-1} \cup G_5^i \cup M^i \cup G_5^{i+1}$. Let us denote by Q_{ab}^i the subgraph of Q^i obtained from Q^i by removing six vertices e_j and f_j for $j = i-1, i, i+1$ and two edges $\{c_{i-1}, c_i\}$ and $\{c_i, c_{i+1}\}$ (Figure 7(b)). Likewise, let Q_{ef}^i be the subgraph of Q^i obtained by removing six vertices a_j and b_j for $j = i-1, i, i+1$ and two edges $\{d_{i-1}, d_i\}$ and $\{d_i, d_{i+1}\}$. It is easy to see that both subgraphs Q_{ab}^i and Q_{ef}^i are subdivisions of the graph $K_{3,3}$.



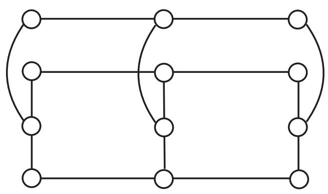


Figure 7.

Let us consider the following types of crossings on the edges of Q^i in a drawing of the graph $G_5 \times C_n$:

(1) a crossing of an edge in $M^{i-1} \cup M^i$ with an edge in G_5^i,

(2) a crossing of an edge in $G_5^{i-1} \cup M^{i-1}$ with an edge in $G_5^{i+1} \cup M^i$,

(3) a crossing of an edge in $G_5^{i-1} \cup G_5^{i+1}$ with an edge in G_5^i,

(4) a crossing of an edge in $G_5^{i-1} \cup M^{i-1}$ with an edge in $G_5^{i+2} \cup M^{i+1}$,

(5) a crossing of an edge in $G_5^{i+1} \cup M^i$ with an edge in $G_5^{i-2} \cup M^{i-2}$.

It is readily seen that every crossing of types (1) or (2) appears in the drawing of the graph $G_5 \times C_n$ only on the edges of the subgraph Q^i. A crossing of type (3) in Q^i between an edge of G_5^{i-1} and an edge of G_5^i appears only in Q^{i-1} as a crossing of type (3) and a crossing in Q^i between an edge of G_5^{i+1} and an edge of G_5^i appears only in Q^{i+1} as a crossing of type (3). A crossing of type (4) in Q^i appears only as a crossing of type (5) in Q^{i+1} and a crossing of type (5) in Q^i appears only in Q^{i-1} as a crossing of type (4).

In a good drawing of $G_5 \times C_n$, we define the *force* $f(Q^i)$ of Q^i in the following way: every crossing of type (1) or (2) contributes the value 1 to $f(Q^i)$ and every crossing of type (3), (4) or (5) contributes the value $\frac{1}{2}$ to $f(Q^i)$ (and $\frac{1}{2}$ to $f(Q^{i-1})$ or to $f(Q^{i+1})$). The *total force* of the drawing is the sum of $f(Q^i)$. It is readily seen that the number of crossings in the drawing is not less then the total force of the drawing. So, the aim of our proof is to show that if every G_5^i has at most one crossing on its edges, then $f(Q^i) \geqslant 2$ for all $i = 0, 1, \dots, n-1$.

Consider the good drawing D assumed in Lemma 2 and let D_{ab}^i be the subdrawing of the subgraph Q_{ab}^i induced from D. Any drawing of $K_{3,3}$ contains a pair of edges that cross each other and do not meet in a vertex. As Q_{ab}^i can be obtained by an elementary subdivision of six edges of $K_{3,3}$, in any drawing D_{ab}^i of Q_{ab}^i there is a *forced* crossing which appears between an edge in $G_5^{i-1} \cup M^{i-1}$ and an edge in $G_5^{i+1} \cup M^i$, or between an edge in $M^{i-1} \cup M^i$ and an edge in G_5^i, or between an edge in $G_5^{i-1} \cup G_5^{i+1}$ and an edge in G_5^i. Whereas every of the first two considered types of crossings contributes the value 1 to $f(Q^i)$, the contribution of the crossing between G_5^i and $G_5^{i-1} \cup G_5^{i+1}$ to $f(Q^i)$ is only $\frac{1}{2}$. So, the minimal contribution of the subdrawing D_{ab}^i to $f(Q^i)$ is $\frac{1}{2}$, but, in this case, an edge of G_5^i is crossed by an edge of G_5^{i-1} or by an edge of G_5^{i+1}. The same consideration can be repeated for the subdrawing D_{ef}^i of the subgraph Q_{ef}^i. Only three edges $\{c_{i-1}, d_{i-1}\}, \{c_i, d_i\}$, and $\{c_{i+1}, d_{i+1}\}$ appear in both subgraphs Q_{ab}^i and Q_{ef}^i. A possible crossing between two of these three edges cannot be the forced crossing neither in D_{ab}^i nor in D_{ef}^i. Thus, in the subdrawing of Q^i induced from the drawing D there are at least two forced crossings. Since there is at most one crossing on the edges of G_5^i, at least one of these two forced crossings contributes 1 to $f(Q^i)$. If both forced crossings among the edges of Q^i contribute 1 to $f(Q^i)$, then $f(Q^i) \geqslant 2$ and we are done.

Assume that one of the forced crossings among the edges of Q^i, say in D^i_{ab}, contributes only $\frac{1}{2}$ to the force of Q^i. In the rest of the proof we show that in D there is another crossing which contributes $\frac{1}{2}$ to $f(Q^i)$, and hence $f(Q^i) \geqslant 2$. Without loss of generality suppose that in D^i_{ab} an edge of G^i_5 is crossed by an edge of G^{i+1}_5. One can easily see that neither the crossing between the edges $\{a_i, c_i\}$ and $\{a_{i+1}, c_{i+1}\}$ nor the crossing between the edges $\{b_i, c_i\}$ and $\{b_{i+1}, c_{i+1}\}$ is forced in D^i_{ab}. Thus, the forced crossing in D^i_{ab} is one of the following: $\{a_i, c_i\}$ crosses $\{b_{i+1}, c_{i+1}\}$, $\{b_i, c_i\}$ crosses $\{a_{i+1}, c_{i+1}\}$, $\{d_i, c_i\}$ crosses $\{a_{i+1}, c_{i+1}\}$ or $\{b_{i+1}, c_{i+1}\}$, and $\{d_{i+1}, c_{i+1}\}$ crosses $\{a_i, c_i\}$ or $\{b_i, c_i\}$. Up to the symmetry it is enough to consider only three cases: the edge $\{a_i, c_i\}$ is crossed by $\{b_{i+1}, c_{i+1}\}$, the edge $\{c_i, d_i\}$ is crossed by $\{b_{i+1}, c_{i+1}\}$, and the edge $\{c_{i+1}, d_{i+1}\}$ is crossed by $\{a_i, c_i\}$. Since in D there is no other crossing either on the edges of G^i_5 or on the edges of G^{i+1}_5, one can find in Figure 3 that in all three cases two vertex-disjoint cycles $a_i a_{i-1} c_{i-1} d_{i-1} e_{i-1} e_i d_i c_i a_i$ and $b_{i+1} b_{i+2} c_{i+2} d_{i+2} e_{i+2} e_{i+1} d_{i+1} c_{i+1} b_{i+1}$ cross each other in D at least two times in such a way that the path $b_{i+1} b_{i+2} c_{i+2} d_{i+2} e_{i+2} e_{i+1}$ crosses the path $a_i a_{i-1} c_{i-1} d_{i-1} e_{i-1} e_i$. This crossing of type (4) contributes $\frac{1}{2}$ to $f(Q^i)$, and hence $f(Q^i) \geqslant 2$. Since i runs through $0, 1, \dots, n-1$, the total force of the drawing is at least $2n$ and therefore, there are at least $2n$ crossings in the drawing. This completes the proof. □

THEOREM 6. *For* $n \geqslant 6, cr(G_5 \times C_n) = 2n$.

Proof. The drawing in Figure 3 shows that $cr(G_5 \times C_n) \leqslant 2n$ for $n \geqslant 6$. We prove the reverse inequality by the induction on n. By Theorem 5, $cr(G_5 \times C_6) = 12$, so the result is true for $n = 6$. Assume it is true for $n = k$, $k \geqslant 6$, and suppose that there is a good drawing of $G_5 \times C_{k+1}$ with fewer than $2(k+1)$ crossings. By Lemma 2, some G^i_5 must then be crossed at least twice. By the removal of all edges of this G^i_5, we obtain a subdivision of $G_5 \times C_k$ with fewer than $2k$ crossings. This contradicts the induction hypothesis. □

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