Tatra Mt. Math. Publ. **35** (2007), 47–50

Tatra Mountains
Mathematical Publications

ON REARRANGEMENTS OF NON-ABSOLUTELY CONVERGENT SERIES

Pavel Kostyrko

ABSTRACT. Let (E,ρ) be a space of the set E of all permutations of positive integers furnished with the Fréchet metric ρ. In the paper a class of non-absolutely convergent series $\sum_{n=1}^{\infty} c(n)$ is given such that the set $\{x=(x_n):\ \sum_{n=1}^{\infty} c(x_n) \text{ converges}\} \subset E$ is an $(x^{v^{-1}})$-σ-porous set for each $\sum_{n=1}^{\infty} c(n)$ of the above class.

In the paper [A] the author deals with the space E of all permutations of the set $\{1,2,\dots\}$ of all positive integers endowed with the Fréchet metric

$$\rho(x,y)=\sum_{n=1}^{\infty} 2^{-n}|x_n-y_n|\big(1+|x_n-y_n|\big)^{-1},$$

$x=(x_n)\in E$, $y=(y_n)\in E$. In [A] it is proved that the metric space (E,ρ) is of the second category at each of its points, and the following statement, too.

Theorem A. *Let $\sum_{n=1}^{\infty} c(n)$ be a non-absolutely convergent series. Then for each $x=(x_n)\in E$*

$$\liminf_{N\to\infty}\sum_{n=1}^{N} c(x_n)=-\infty \quad \textit{and} \quad \limsup_{N\to\infty}\sum_{n=1}^{N} c(x_n)=+\infty$$

hold, with the exception of points of the set of the first category.

Put $A=\left\{x\in E\colon \sum_{n=1}^{\infty} c(x_n) \text{ converges}\right\}$ for a non-absolutely convergent series $\sum_{n=1}^{\infty} c(n)$. Then we have

2000 Mathematics Subject Classification: 40A05.
Keywords: non absolutely convergent series, (f)-σ-porous set.
Supported by the grant VEGA 1/3018/06.

Corollary. *The set $A \subset E$ is a set of the first category in E.*

Professor T. Šalát raised a question whether in Corollary the notion "a set of the first category" can be replaced by "a σ-porous set". Our aim is to prove the statement of Corollary in a sharper form for a class of non-absolutely convergent series. We shall use the notion of (f)-porosity and (f)-σ-porosity of sets in a metric space introduced by L. Zajíček in [Z] in the following way: Let f be an increasing and continuous function on $(0,\eta)$ for some $\eta > 0$. Let (P,ρ) be a metric space, $M \subset P$, $x \in P$, $R > 0$ and $B(x,r) = \{y \in P\colon \rho(x,y) < r\}$, $r > 0$. Put $\gamma(x,R,M) = \sup\{r > 0\colon$ for some $z \in P$, $B(z,r) \subset B(x,R)$ and $B(z,r) \cap M = \emptyset\}$. Then the point x is said to be a point of (f)-porosity of M if $p^f(x,M) = \limsup\limits_{R\to 0+} \frac{1}{R} f\bigl(\gamma(x,R,M)\bigr) > 0$. A set M is said to be (f)-porous if any point $x \in M$ is a point of (f)-porosity of M. We say that M is an (f)-σ-porous set if it is the union of a sequence of (f)-porous sets. In [Z] it is shown, that for the functions $f(x) = x^p$ and $g(x) = x^q$, $0 < p < q < 1$, the (f)-σ-porosity and (g)-σ-porosity of a set coincide.

Let

$$\sum_{n=1}^{\infty} c(n) \tag{1}$$

be a non-absolutely convergent series with non-zero elements. Let $\sum\limits_{k=1}^{\infty} c'(k)$ be the series of positive elements of (1) ordered in such a way that $c'(k+1) \leqslant c'(k)$. Analogously, let $\sum\limits_{k=1}^{\infty} c''(k)$ be the series of absolute values of negative elements of (1) ordered in such a way that $c''(k+1) \leqslant c''(k)$. Obviously $c'(k) \to 0$, $c''(k) \to 0$ as $k \to \infty$ and $\sum\limits_{k=1}^{\infty} c'(k) = \sum\limits_{k=1}^{\infty} c''(k) = +\infty$.

Definition 1. Let

$$\sum_{k=1}^{\infty} d(k) \tag{2}$$

be a divergent series, $d(k) > 0$, $d(k+1) \leqslant d(k)$, $d(k) \to 0$ as $k \to \infty$. The series (2) is said to be v-divergent, v-positive integer, $v > 1$, if there is $\varepsilon > 0$ and n_0 such that for each $n \geqslant n_0$ $\sum\limits_{k=n+1}^{vn} d(k) > \varepsilon$, i.e.,

$$\exists_{\varepsilon>0} \exists_{n_0} \forall_{n \geqslant n_0} \sum_{k=n+1}^{vn} d(k) > \varepsilon\,. \tag{3}$$

Definition 2. The series (1) is said to be v-divergent if its positive part $\sum_{k=1}^{\infty} c'(k)$, or negative part $\sum_{k=1}^{\infty} c''(k)$ is v-divergent.

Example. The non-absolutely convergent series $\sum_{n=1}^{\infty} (-1)^n n^{-1}$ is 2-divergent. The series $\sum_{k=1}^{\infty} (2k)^{-1}$ is its positive part and for each $n \geqslant 2$, $\sum_{k=n+1}^{2n} (2k)^{-1} = 2^{-1}\big((n+1)^{-1} + \dots + (2n)^{-1}\big) > 2^{-1}n(2n)^{-1} = 4^{-1}$.

Theorem. *Let $\sum_{n=1}^{\infty} c(n)$ be a non-absolutely convergent, v-divergent series with non-zero elements. Then*

$$A = \left\{ x = (x_n)\colon \sum_{n=1}^{\infty} c(x_n) \text{ converges} \right\} \subset E$$

is an $\left(x^{v^{-1}}\right)$-σ-porous set.

Proof. Further, we shall use the following properties of Fréchet metric:

(i) $\rho(x,y) < 2^{-(N+1)}$ implies $x_n = y_n$ for $n = 1, 2, \dots, N$;

(ii) if $x_n = y_n$ for $n = 1, 2, \dots, N$, then $\rho(x,y) < 2^{-N}$.

For $x = (x_k) \in E$ put $s(x_k, n) = \sum_{k=1}^{n} c(x_k)$. It follows from Cauchy-Bolzano condition that $A = \left\{x \in E\colon \forall_{\varepsilon>0} \exists_q \forall_{n \geqslant q} \forall_{m>n\geqslant q} |s(x_k,m) - s(x_k,n)| < \varepsilon\right\} = \bigcap_{p=1}^{\infty} \bigcup_{q=1}^{\infty} A_{pq}$, where $A_{pq} = \left\{ x \in E\colon \forall_{n\geqslant q} \forall_{m>n\geqslant q} \left| \sum_{k=n+1}^{m} c(x_k) \right| \leqslant p^{-1} \right\}$.

Note that each of the sets A_{pq} is closed since the complement $E \setminus A_{pq}$ is open. We show that if $y \in A_{pq}$, then $p^f(y, A_{pq}) > 0$ for $f(x) = x^{v^{-1}}$.

In the next text we can suppose that the positive part $\sum_{k=1}^{\infty} c'(k)$ of the series $\sum_{n=1}^{\infty} c(n)$ is v-divergent, i.e., it fulfils (3) for some $\varepsilon > 0$. Choose such small $\delta > 0$ that $\frac{\delta}{2} \leqslant 2^{-q}$. There exists a unique n for which $2^{-n} < \frac{\delta}{2} \leqslant 2^{-(n-1)}$ and so $q < n$. We construct $z = (z_k) \in E \setminus A_{pq} = \left\{ x \in E\colon \exists_{n\geqslant q} \exists_{m>n\geqslant q} \left| \sum_{k=n+1}^{m} c(x_k) \right| > p^{-1} \right\}$. Put $z_k = y_k$ for $1 \leqslant k \leqslant n$. For $k = n+1,\ n+2, \dots, vn$ we choose positive integers z_k with the aim to get the greatest possible elements of $\sum_{k=1}^{\infty} c'(k)$. For $k > vn$ we can choose z_k in any admissible way. Then we have:

(a) $\rho(y,z) < \frac{\delta}{2}$;

$$\text{(b)} \quad \left|\sum_{k=n+1}^{vn} c(z_k)\right| = \sum_{k=n+1}^{vn} c(z_k) \geqslant \sum_{k=n+1}^{vn} c'(k) > \varepsilon .$$

Suppose that for each $p \geqslant p_0$ we have $p^{-1} \leqslant \varepsilon$. We show $B\big(z, 2^{-(vn+1)}\big) \cap A_{pq} = \emptyset$. If $x \in B\big(z, 2^{-(vn+1)}\big)$, then $\rho(x,z) < 2^{-(vn+1)}$ and $x_k = z_k$ for $k = 1, 2, \ldots, vn$. Consequently $\left|\sum_{k=n+1}^{vn} c(x_k)\right| = \left|\sum_{k=n+1}^{vn} c(z_k)\right| > \varepsilon$. Obviously $B\big(z, 2^{-(vn+1)}\big) \subset B(y,\delta)$. Hence $\gamma(y,\delta,A_{pq}) \geqslant 2^{-(vn+1)}$. Put $f(x) = x^{v^{-1}}$. Since $\frac{1}{\delta} \geqslant 2^{n-2}$ we have $\frac{1}{\delta} f\big(\gamma(y,\delta,A_{pq})\big) \geqslant 2^{-2-v^{-1}}$ and $p^f(y,A_{pq}) \geqslant 2^{-2-v^{-1}} > 0$. The set A_{pq} is (f)-porous and $A(p) = \bigcup_{q=1}^{\infty} A_{pq}$ is an (f)-σ-porous set for each $p \geqslant p_0$. Obviously $A(r) \subset A(p)$ holds for $p < r$. Consequently $A = \bigcap_{p=1}^{\infty} \bigcup_{q=1}^{\infty} A_{pq} = \bigcap_{p=p_0}^{\infty} \bigcup_{q=1}^{\infty} A_{pq}$ is an (f)-σ-porous set. □

Problems. It seems that the result formulated in Theorem can be improved in the following directions:

a) To prove the result for a wider class of non-absolutely convergent series;

b) To prove the result for a wider set of sequences, e.g., for the set of sequences which is, according to Theorem A, a set of the first category;

c) Instead of $\big(x^{v^{-1}}\big)$-σ-porosity to use the notion of (x)-σ-porosity.

REFERENCES

[A] AGNEW, R. P.: *On rearrangements of series*, Bull. Amer. Math. Soc. **46** (1940), 797–799.

[Z] ZAJÍČEK, L.: *Sets of σ-porosity and sets of σ-porosity (q)*, Čas. pěst. mat. **101** (1976), 350–359.

Received October 4, 2005

Faculty of Mathematics
Physics and Informatics
Comenius University
Mlynská dolina
SK–842 48 Bratislava
SLOVAKIA
E-mail: pavel.kostyrko@fmph.uniba.sk