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ON TOTAL EDGE IRREGULARITY STRENGTH OF THE GRIDS

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ABSTRACT. A total edge irregular labelling ν of a graph G is a labelling of the vertices and edges of G with labels from the set $\{1, \dots, k\}$ in such a way that for any two different edges e and f their weights $\varphi(f)$ and $\varphi(e)$ are distinct, where the weight of an edge $g = uv$ is the sum of the label of g and the labels of vertices u and v. The minimum k for which the graph G has an edge irregular total k-labelling is called *the total edge irregularity strength* of G. In this note we determine exact values of the total edge irregularity strength of $m \times n$ grids.

1. Introduction

In [3], Chartrand *et al.* proposed the following problem: Assign positive integer labels to the edges of a connected graph of order at least 3 in such a way that the graph becomes irregular, i.e., the weights (label sums) at each vertex are distinct. What is the minimum value of the largest label over all such irregular assignments?
This parameter of a graph G is well known as the *irregularity strength* of the graph G, $s(G)$. Finding the irregularity strength of a graph seems to be hard even for graphs with simple structure, see [4], [5], [6]. For recent results see the papers by Amar and Togni [1], Jacobson and Lehel [6], Nierhoff [7], Frieze *at al.* [5]. Motivated by these papers and by a book of Wallis [9], Bača *et al.* [2] started to investigate the total edge irregularity strength of a graph G, an invariant analogous to the irregularity strength for total labellings.

For a graph $G = (V, E)$ we define a labelling $\nu\colon V \cup E \to \{1, 2, \dots, k\}$ to be a *total edge irregular k-labelling* of the graph G if for every two different edges e and f of G one has $\varphi(e) \neq \varphi(e)$, where the weight of an edge $e = \{u, v\}$ in the labelling ν is $\varphi(e) = \nu(u) + \nu(v) + \nu(e)$. The minimum k for which the graph

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G has an edge irregular total k-labelling is called *the total edge irregularity strength* of G, $tes(G)$. Motivated by the papers [4] of Dinitz, Garnick and Gyárfás and [8] of Togni we investigate the total edge irregularity strength of the $m \times n$ *grid*. By an $m \times n$ grid we understand a graph isomorphic to the Cartesian product $P_m \times P_n$ of two paths P_m and P_n. Let us recall the definition of the Cartesian product of two paths $P_m \times P_n$. For integers a and b let $[a, b]$ be an interval of integers x, $a \leqslant x \leqslant b$. Let $V(P_m) = \{u_1, \dots, u_m\}$, $E(P_m) = \{u_i u_{i+1}, i \in [1, m-1]\}$, $V(P_n) = \{v_1, \dots, v_n\}$, $E(P_n) = \{v_i v_{i+1}, i \in [1, n-1]\}$. Then $V(P_m \times P_n) = \{(u_i, v_j), i \in [1, m], j \in [1, n]\}$ and $E(P_m \times P_n) = \{(u_i, v_j)(u_k, v_l), i, k \in [1, m], j, l \in [1, n], |i-k| + |j-l| = 1\}$. Now it is easy to check that the grid $P_m \times P_n$ has $2mn - m - n$ edges.

In [2] it is proved that the total edge irregularity strength of every graph $G = (V, E)$ is at least $\lceil \frac{|E|+2}{3} \rceil$ and for a path on n vertices $tes(P_n) = \lceil \frac{n+1}{3} \rceil$. This implies that $tes(P_m \times P_n) \geqslant \lceil \frac{2mn-m-n+2}{3} \rceil$. The main goal of this note is to prove equality. More precisely we prove

THEOREM 1. *Let m, n be positive integers. Then*

$$tes(P_m \times P_n) = \left\lceil \frac{2mn - m - n + 2}{3} \right\rceil .$$

2. Proof of the theorem

In the proof we write $\nu(u, v)$ instead of $\nu\big((u, v)\big)$ for the vertex (u, v) of the grid under a labelling ν. Analogously we write $\nu(u_1, v_1; u_2, v_2)$ instead of $\nu\big((u_1, v_1)(u_2, v_2)\big)$. The weight of an edge $(u_1, v_1)(u_2, v_2)$ is denoted by $\varphi(u_1, v_1; u_2, v_2)$.
We distinguish three cases. The first case occurs when $m = 3k$, $k \in \mathbb{N}$ and $n \in \mathbb{N}$. The second case concerns $m = 3k + 2$, $k \in \mathbb{N}$ and $n \in \mathbb{N}$. Because the graphs $P_m \times P_n$ and $P_n \times P_m$ are isomorphic, in the third case it is sufficient to complete the proof for $m = 3k + 1$, $k \in \mathbb{N}$ and $n = 3l + 1$, $l \in \mathbb{N}$.

Case 1. Let $m = 3k$. We proceed by induction according to t, where $t = \lceil \frac{n-1}{3} \rceil$. Let $\mathcal{A}(t)$ denote that

(1) $tes(P_m \times P_n) = \lceil \frac{2mn-m-n+2}{3} \rceil$ for $n = 3t-1$, $3t$ and $3t+1$

(2) the vertex (u_i, v_{3t+1}) is assigned with the value $\nu(u_i, v_{3t+1}) = \lceil \frac{i}{3} \rceil + t(6k-1)$

(3) the edge $\{(u_i, v_{3t+1})(u_{i+1}, v_{3t+1})\}$ of the $P_n \times P_m$ receives the value $\nu(u_i, v_{3t+1}; u_{i+1}, v_{3t+1}) = \lceil \frac{i+2}{3} \rceil + t(6k-1)$ for every $i \in [1, 3k-1]$

(4) the maximum edge weight in $P_m \times P_n$ has value $3t(6k-1) + 3k + 1$.

To prove $\mathcal{A}(1)$ put $\nu(u_i, v_1) = \lceil \frac{i}{3} \rceil$ for every $i \in [1, 3k]$ and put $\nu(u_i, v_1; u_{i+1}, v_1) = \lceil \frac{i+2}{3} \rceil$ for every $i \in [1, 3k-1]$. Clearly, for $i \neq j$ we have $\varphi(u_i, v_1; u_{i+1}, v_1) \neq \varphi(u_j, v_1; u_{j+1}, v_1)$.
Next, assuming that $\mathcal{A}(t)$ is true, we prove $\mathcal{A}(t+1)$. We distinguish three subcases.

Case 1.1. Let $n = 3t+2 = 3(t+1)-1$. Assume the subgraph $P_m \times P_{3t+1}$ of $P_m \times P_{3t+2}$ to be labelled as mentioned in $\mathcal{A}(t)$. The vertices of the path $Q(3t+2) = \big((u_1, v_{3t+2})(u_2, v_{3t+2}) \dots (u_m, v_{3t+2})\big)$ are assigned $\nu(u_i, v_{3t+2}) = \lceil \frac{i}{3} \rceil + t(6k-1) + 2k$ for $i \in [1, 3k]$ and the edges $\big(u_i, v_{3t+2}; u_{i+1}, v_{3t+2}\big)$ are assigned with the values $\nu\big(u_i, v_{3t+2}; u_{i+1}, v_{3t+2}\big) = \lceil \frac{i+2}{3} \rceil + t(6k-1) + 2k - 1$. The weights of these edges run between $3t(6k-1) + 6k + 2$ and $3t(6k-1) + 9k$. Clearly, for different edges their weights are distinct. To complete the assignment of the graph $P_m \times P_{3t+2}$, we label the edge $\big(u_i, v_{3t+1}\big)\big(u_i, v_{3t+2}\big)$ with $\nu\big(u_i, v_{3t+1}; u_i, v_{3t+2}\big) = t(6k-1) + k - 1 + \lceil \frac{i}{3} \rceil + (i - 1 - 3\lfloor \frac{i-1}{3} \rfloor)$ for every $i \in [1, 3k]$. The weight of this edge is $i + 3\big(\lceil \frac{i}{3} \rceil - \lfloor \frac{i-1}{3} \rfloor\big) + 3t(6k-1) + 3k - 2 = i + 3 + 3t(6k-1) + 3k - 2 = 3t(6k-1) + 3k + i + 1$ and so the weights of these edges form the interval $\big[3k(6k-1) + 3k + 2, 3t(6k-1) + 6k + 1\big]$.

Case 1.2. Let $n = 3(t+1) = 3t+3$ and assume the subgraph $P_m \times P_{3t+2}$ of $P_m \times P_{3t+3}$ to be labelled as mentioned in the Case 1.1. The elements of the path $Q(3t+3)$ are assigned with the values $\nu\big(u_i, v_{3t+3}\big) = \lceil \frac{i}{3} \rceil + t(6k-1) + 4k$ for $i \in [1, 3k]$ and $\nu\big(u_i, v_{3t+3}; u_{i+1}, v_{3t+3}\big) = \lceil \frac{i+2}{3} \rceil + t(6k-1) + 4k - 2$ for $i \in [1, 3k-1]$. The edges between the paths $Q(3t+2)$ and $Q(3t+3)$ are labelled as follows: $\nu\big(u_i, v_{3t+2}; u_i, v_{3t+3}\big) = t(6k-1) + 3k - 2 + \lceil \frac{i}{3} \rceil + \big(i - 1 - 3\lfloor \frac{i-1}{3} \rfloor\big)$ for $i \in [1, 3k]$. It is easy to check again that the labelling ν is irregular and the largest edge weight is $3t(6k-1) + 15k - 1$.

Case 1.3. Let $n = 3(t+1) + 1 = 3t+4$ and assume the subgraph $P_m \times P_{3t+3}$ of $P_m \times P_{3t+4}$ to be labelled as in Case 1.2. We continue with the following assignment: $\nu\big(u_i, v_{3(t+1)+1}\big) = \lceil \frac{i}{3} \rceil + t(6k-1) + 6k - 1 = \lceil \frac{i}{3} \rceil + (t+1)(6k-1)$ for $i \in [1, 3k]$. $\nu\big(u_i, v_{3(t+1)+1}; u_{i+1}, v_{3(t+1)+1}\big) = \lceil \frac{i+2}{3} \rceil + t(6k-1) + 6k - 1$ for $i \in [1, 3k-1]$. The weight of this edge is $\lceil \frac{i}{3} \rceil + \lceil \frac{i+1}{3} \rceil + \lceil \frac{i+2}{3} \rceil + 3(t+1)(6k-1)$ for $i \in [1, 3k-1]$. $\nu\big(u_i, v_{3t+3}; u_i, v_{3(t+1)+1}\big) = t(6k-1) + 5k - 2 + \lceil \frac{i}{3} \rceil + \big(i - 1 - 3\lfloor \frac{i-1}{3} \rfloor\big)$ for $i \in [1, 3k]$. The weight of this edge is $3t(6k-1) + 15k + i - 1$ for $i \in [1, 3k]$. The largest weight among edge weights has value $3t(6k-1) + 21k - 2 = 3(t+1)(6k-1) + 3k + 1$.
In all above cases the total edge irregular $d-$labelling is constructed with $d = \lceil \frac{2mn-m-n+2}{3} \rceil$, which is the smallest possible value for which such a labelling can exist for.

Case 2. Let $m = 3k-1$. Similarly as in the Case 1 let $\mathcal{B}(t)$ denote that

(1) $tes(P_m \times P_t) = \lceil \frac{2mt-m-t+2}{3} \rceil$ for $t \in [1,n]$.

(2) $\nu(u_i, v_t) = \lceil \frac{i}{3} \rceil + (t-1)(2k-1)$ for $i \in [1, 3k-1]$.

(3) $\nu(u_i, v_t; u_{i+1}, v_t) = \lceil \frac{i+2}{3} \rceil + (t-1)(2k-1)$ for every $i \in [1, 3k-2]$

(4) The maximum edge weight in $P_m \times P_n$ is $\varphi(e) = 3(t-1)(2k-1) + 3k$.

It is easy to see that $\mathcal{B}(1)$ is true, see [2]. Assuming that $\mathcal{B}(t)$ is true we prove the truth of $\mathcal{B}(t+1)$. To see this, let us accept the labelling of the subgraph $P_m \times P_t$ of $P_m \times P_{t+1}$ and let us label the rest of the vertices and edges of $P_m \times P_{t+1}$ as follows:
$\nu(u_i, v_{3t+2}) = \lceil \frac{i}{3} \rceil + t(2k-1) - 2k + 1 + 2k - 1$ for $i \in [1, 3k-1]$
$\nu(u_i, v_{t+1}; u_{i+1}, v_{t+1}) = \lceil \frac{i+2}{3} \rceil + t(2k-1) - 2k + 1 + 2k - 1$ for $i \in [1, 3k-2]$
$\nu(u_i, v_t; u_i, v_{t+1}) = \lceil \frac{i}{3} \rceil + \left(i - 1 - 3\lfloor \frac{i-1}{3} \rfloor\right) + t(2k-1) - 2k + 1 + k - 1$ for $i \in [1, 3k-1]$.
It is a routine matter to check that the largest weight among all edge weights has value $3t(2k-1) + 3k$ and the largest used label is $t(2k-1) + k$. Hence $\mathcal{B}(t+1)$ is proved.

Case 3. Let $m = 3k+1$. As the graphs $P_m \times P_n$ and $P_n \times P_m$ are isomorphic, it is sufficient to consider the case when $n = 3l + 1$. Let $\mathcal{C}(t)$ denote that the graph $P_{3k+1} \times P_{3t+1}$ has a labelling ν having the following properties:

(1) $tes(P_m \times P_n) = \lceil \frac{2mn-m-n+2}{3} \rceil$ for $m = 3k+1$ and $n = 3t+1$.

(2) $\nu(u_i, v_{3t+1}) = \lceil \frac{i}{3} \rceil + t(6k+1)$ for $i \in [1, 3k+1]$.

(3) $\nu(u_i, v_{3t+1}; u_{i+1}, v_{3t+1}) = \lceil \frac{i+2}{3} \rceil + t(6k+1)$ for every $i \in [1, 3k]$.

(4) The maximum edge weight in $P_m \times P_n$ has value $3t(6k+1) + 3k + 2$.

It is easy to see that $\mathcal{C}(1)$ is true, see [2]. To verify the truth of $\mathcal{C}(t+1)$ assuming that $\mathcal{C}(t)$ is true, let us accept the labelling ν of $\mathcal{C}(t)$ for a subgraph $P_{3k+1} \times P_{3t-2}$ of $P_{3k+1} \times P_{3t+1}$. The rest of the vertices and edges are labelled as follows:
$\nu(u_i, v_{3t+2}) = \lceil \frac{i}{3} \rceil + t(6k+1) + 2k + 1$ for $i \in [1, 3k+1]$,
$\nu(u_i, v_{3t+1}; u_i, v_{3t+2}) = t(6k+1) + k - 1 + \lceil \frac{i}{3} \rceil + \left(i - 1 - 3\lfloor \frac{i-1}{3} \rfloor\right)$ for $i \in [1, 3k+1]$.
Hence we have
$\varphi(u_i, v_{3t+1}; u_i, v_{3t+2}) = 3t(6k+1) + 3k + 3 + i - 1$ for $i \in [1, 3k+1]$ and the the maximum weight among these edges is $3t(6k+1) + 6k + 3$.
$\nu(u_i, v_{3t+2}; u_{i+1}, v_{3t+2}) = \lceil \frac{i+2}{3} \rceil + t(6k+1) + 2k - 1$ for $i \in [1, 3k]$,
$\varphi(u_i, v_{3t+2}; u_{i+1}, v_{3t+2}) = 3t(6k+1) + 6k + 1 + \lceil \frac{i}{3} \rceil + \lceil \frac{i+1}{3} \rceil + \lceil \frac{i+2}{3} \rceil$ for $i \in [1, 3k]$, and the maximum weight on these edges is $3t(6k+1) + 9k + 3$.
$\nu(u_i, v_{3t+3}) = \lceil \frac{i}{3} \rceil + t(6k+1) + 4k + 1$ for $i \in [1, 3k+1]$,
$\nu(u_i, v_{3t+2}; u_i, v_{3t+3}) = t(6k+1) + 3k - 1 + \lceil \frac{i}{3} \rceil + \left(i - 1 - 3\lfloor \frac{i-1}{3} \rfloor\right)$ for $i \in [1, 3k+1]$,
$\varphi(u_i, v_{3t+2}; u_i, v_{3t+3}) = 3t(6k+1) + 9k + 4 + i - 1$ for $i \in [1, 3k+1]$ and the

maximum weight is $3t(6k+1)+12k+4$.
$\nu(u_i, v_{3t+3}; u_{i+1}, v_{3t+3}) = \lceil\frac{i+2}{3}\rceil + t(6k+1)+4k$ for $i \in [1,3k]$ and
$\varphi(u_i, v_{3t+2}; u_{i+1}, v_{3t+2}) = 3t(6k+1)+12k+2+\lceil\frac{i}{3}\rceil+\lceil\frac{i+1}{3}\rceil+\lceil\frac{i+2}{3}\rceil$ for $i \in [1,3k]$.
Hence the maximum weight on these edges is $3t(6k+1)+15k+4$.
$\nu(u_i, v_{3(t+1)+1}) = \lceil\frac{i}{3}\rceil+t(6k+1)+6k+1 = \lceil\frac{i}{3}\rceil+(t+1)(6k+1)$ for $i \in [1,3k+1]$,
$\nu(u_i, v_{3t+3}; u_i, v_{3(t+1)+1}) = t(6k+1)+5k+\lceil\frac{i}{3}\rceil+(i-1-3\lfloor\frac{i-1}{3}\rfloor)$ for $i \in [1,3k+1]$, and
$\varphi(u_i, v_{3t+3}; u_i, v_{3(t+1)+1}) = 3t(6k+1)+15k+2+3+i-1$ for $i \in [1,3k+1]$.
Hence the maximum weight among these edges is $3t(6k+1)+18k+5$.
$\nu(u_i, v_{3(t+1)+1}; u_{i+1}, v_{3(t+1)+1}) = \lceil\frac{i+2}{3}\rceil + t(6k+1)+6k+1$ for $i \in [1,3k]$,
$\varphi(u_i, v_{3(t+1)+1}; u_{i+1}, v_{3(t+1)+1}) = 3t(6k+1)+18k+3+\lceil\frac{i}{3}\rceil+\lceil\frac{i+1}{3}\rceil+\lceil\frac{i+2}{3}\rceil$ for $i \in [1,3k]$, and the largest weight on these edges is $3t(6k+1)+21k+5$.
It is a routine matter to verify that for distinct edges e and f there is $\varphi(e) \neq \varphi(f)$. Hence the labelling ν is irregular. The largest weight among edge weights is $3t(6k+1)+21k+5$. Because the largest used label is exactly the possible minimum which a total edge irregular labelling can exist for, we have proved $\mathcal{C}(t+1)$.

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