Tatra Mt. Math. Publ. **35** (2007), 71–89

TATRA MOUNTAINS
Mathematical Publications

TWO-DIMENSIONAL ANALOGS OF FIRST-RETURN CONTINUITY

Michael J. Evans — Paul D. Humke

ABSTRACT. It is known that a function $f : [-1,1] \to \mathbb{R}$ belongs to Baire class one and has the Darboux property if and only if it is first-return continuous. The goal of this work is to exhibit several two-dimensional analogs of first-return continuity which force a function $f : [-1,1] \times [-1,1] \to \mathbb{R}$ to be of Baire class one and have something analogous to the Darboux property. To this end we shall explore three candidates for such an analog: radially first-return approachable functions, sectorially first-return approachable functions, and radially sectorially first-return approachable functions. All three types of functions belong to Baire class one and preserve the connectedness of some subclasses of the collection of connected sets. We also investigate the set of points of continuity of such functions.

1. Introduction

We begin by describing our goal in fairly broad terms, using the language of "first-return analysis." In the second section we shall provide the definitions necessary to make these comments intelligible to readers not familiar with the terminology. Readers interested in learning more about the general field of "first-return analysis" are referred to the survey article [6].

In [4] it was shown that a function $f\colon \mathbb{I} \equiv [-1,1] \to \mathbb{R}$ belongs to Baire class one if and only if it is first-return recoverable. That this is also true for a function $f\colon \mathbb{I}^2 \to \mathbb{R}$ follows from [2]. Also, in [5] it was shown that a function $f\colon \mathbb{I} \to \mathbb{R}$ is Baire one and has the Darboux property if and only if f is first-return continuous. Our goal here is to exhibit several two-dimensional analogs of first-return continuity which force a function to be of Baire class one and have something analogous to the Darboux property. We shall explore three candidates for such an analog in $\mathbb{I}^2$: radially first-return approachable functions, sectorially first-return approachable functions, and radially sectorially first-return approachable functions. All of these classes were introduced by James D. Shelton in

2000 Mathematics Subject Classification: 26A21, 26B05.
Keywords: first-return, Baire one, continuity, approachability, recoverability, Darboux.

The authors dedicate this paper to the life and work of Professor Janina Ewert.

his undergraduate honors thesis [8]. (Also, see the article [9], which is based on that thesis.) We shall show that these three types of functions belong to Baire class one and preserve the connectedness of some subclasses of the collection of connected sets. We also investigate the set of points of continuity of such functions.

2. Background information and definitions

We shall be considering real-valued functions defined on the n-dimensional cube, $\mathbb{I}^n \equiv [-1,1] \times [-1,1] \times \cdots \times [-1,1]$ in Euclidean n-space, $\mathbb{R}^n$. Of course, a continuous function $f\colon \mathbb{I}^n \to \mathbb{R}$ is uniquely determined by its values on any countable dense set in $\mathbb{I}^n$. One of the early triumphs in the field of investigation that has come to be known as first-return analysis was establishing the fact that a Baire class one function is also completely determined by its values on a countable dense set, and that the algorithm for computing the values everywhere based on knowing the values at the countable set, can be extremely simple. Of course, unlike the situation for continuous functions, not just any countable dense set will work for a Baire class one function. Before explaining further we need to specify some notation and terminology.

Underlying most of our subsequent definitions is the notion of what we shall call a *trajectory*. A *trajectory* is any sequence $\overline{x} = \{x_n\}_{n=1}^{\infty}$ of distinct points in $\mathbb{I}^n$, whose range is dense in $\mathbb{I}^n$. Any countable dense subset S of $\mathbb{I}^n$ is called a support set and, of course, any enumeration of S becomes a trajectory. We use the symbol $B(x,\rho) \equiv \{y \in \mathbb{I}^n \colon ||x-y|| < \rho\}$ to denote the open ball of radius ρ centered at x. Furthermore, we let $r\big(B(x,\rho)\big)$ be defined as the first element of the trajectory $\overline{x}$ that falls inside the ball $B(x,\rho)$.

Definition 2.1. Let $x \in \mathbb{I}^n$ and let $\overline{x} = \{x_n\}$ be a fixed trajectory. The *first-return route to* x, $\mathcal{R}_x = \{w_k\}_{k=1}^{\infty}$, is defined recursively via

$$w_1 = x_1\,,$$

$$w_{k+1} = \begin{cases} r\big(B(x, ||x - w_k||)\big) & \text{if } x \neq w_k\,, \\ w_k & \text{if } x = w_k\,. \end{cases}$$

We say that a function $f\colon \mathbb{I}^n \to \mathbb{R}$ is *first-return recoverable with respect to* $\overline{x}$ at x provided that

$$\lim_{k\to\infty} f(w_k) = f(x)\,,$$

and if this happens for each $x \in I$, we say that f is *first return recoverable with respect to* $\{x_n\}$. Finally, we say that f is *first-return recoverable* if it is first-return recoverable with respect to some trajectory.

The fundamental result concerning first-return recoverable functions is the following.

THEOREM 2.1. *A function $f\colon \mathbb{I}^n \to \mathbb{R}$ belongs to Baire class one if and only if f is first-return recoverable.*

This was first proved in the one-variable case by Darji, Evans, and O'Malley in [4] and extended to a situation that includes the several variable case by Darji and Evans in [2].

A somewhat more restrictive notion is that of first-return approachability:

DEFINITION 2.2. For each $x \in \mathbb{I}^n$ the *first-return approach to x based on* $\overline{x} = \{x_n\}$, $\mathcal{A}_x = \{u_k\}$, is defined recursively via

$$u_1 = r\big(\mathbb{I}^n \setminus \{x\}\big), \quad \text{and} \quad u_{k+1} = r\big(B(x, ||x - u_k||) \setminus \{x\}\big)\,.$$

We say that a function $f\colon \mathbb{I}^n \to \mathbb{R}$ is *first-return approachable at x with respect to the trajectory $\overline{x}$* provided

$$\lim_{\substack{u \to x \\ u \in \mathcal{A}_x}} f(u) = f(x)\,.$$

We say that f is *first-return approachable with respect to $\overline{x}$* provided it is first-return approachable with respect to $\overline{x}$ at each $x \in \mathbb{I}^n$. Likewise, f is said to be *first-return approachable* provided there exists a trajectory with respect to which f is first-return approachable.

If $\mathcal{O}$ is an open set having x as a limit point, we define the first-return approach to x based on $\{x_n\}$ relative to $\mathcal{O}$ as the sequence $\mathcal{A}_{x,\mathcal{O}} = \{y_k\}$ where

$$y_1 = r\big(\mathcal{O} \setminus \{x\}\big) \text{ and}$$
$$y_{k+1} = r\big(\mathcal{O} \cap B(x, ||x - y_k||) \setminus \{x\}\big)$$

and say that a function $f\colon \mathbb{I}^n \to \mathbb{R}$ is first return approachable at x with respect to $\overline{x}$, and relative to the specified open set $\mathcal{O}$,

$$\lim_{\substack{y \to x \\ y \in \mathcal{A}_{x,\mathcal{O}}}} f(y) = f(x)\,.$$

The following characterization was presented by Darji, Evans and Humke in [3].

THEOREM 2.2. *The function $f\colon \mathbb{I} \to \mathbb{R}$ is Baire one with no isolated points on its graph if and only if f is first-return approachable.*

Next, we say that a function $f\colon \mathbb{I} \to \mathbb{R}$ is *first-return continuous* if there is a trajectory $\overline{x}$ such that at each $x \in \mathbb{I}$, f is approachable with respect to $\overline{x}$ relative to both of the intervals $(-1, x)$ and $(x, 1)$; i.e., f is first-return approachable from both the left and right at x. (Naturally, this condition is interpreted as one-sided at the two points -1 and 1.) In [5] Darji, Evans and O'Malley showed the following.

THEOREM 2.3. *A function $f\colon \mathbb{I} \to \mathbb{R}$ belongs to Baire class one and has the Darboux property if and only if f is first-return continuous.*

First-return continuity at a point x technically entails approachability relative to every open connected set having x as a limit point, but really this boils down to being first-return approachable at x relative to the two open connected sets $(-1, x)$ and $(x, 1)$. Thus, it seems logical to ask what type of connected open sets one might use in $\mathbb{I}^2$ if, again, one wishes the function to be of Baire class one and preserve the connectivity of some significant subclass of the collection of connected sets. Following Shelton, [8] and [9], we consider several reasonable possibilities:

DEFINITION 2.3. Let $x, y \in \mathbb{I}^2$. An *arc from y to x* is a continuous one-to-one function $g\colon [0,1] \to \mathbb{I}^2$ with $g(0) = y$ and $g(1) = x$. An *arc at x* is a continuous one-to-one function $g\colon [0,1] \to \mathbb{I}^2$ with $g(1) = x$. When the context makes the usage clear, we shall not distinguish between an arc and its range. If g is a linear arc from y to x, we denote it $[y, x]$. If g is an arc at x, then any open set containing $g \setminus \{x\}$ is called an envelope for g. If $x \in \mathbb{I}^2$ and $y \in \partial(\mathbb{I}^2)$, then $[y, x]$ is called the *ray from y to x*. If this ray makes an angle θ with the horizontal, then we also denote the ray by $R(x, \theta)$.

As one candidate for a two-dimensional analog of the one-dimensional notion of first-return continuity we could consider the following:

DEFINITION 2.4. Let $f\colon \mathbb{I}^2 \to \mathbb{R}$ and let $\overline{x}$ be a trajectory. We say that f is *arcwise first-return approachable with respect to $\overline{x}$ at x* if for each arc g at x there is an envelope G_g of g such that for every subenvelope $G \subseteq G_g$ of g we have that f is first-return approachable at x with respect to $\overline{x}$ relative to G. If a function is arcwise first-return approachable at $x \in \mathbb{I}^2$ with respect to a trajectory $\overline{x}$ for all $x \in \mathbb{I}^2$ we say it is *arcwise first-return approachable.*

If we weaken the requirement from every arc at x to only rays at x we have the notion of radial first-return approachability:

DEFINITION 2.5. A function $f\colon \mathbb{I}^2 \to \mathbb{R}$ will be said to be *radially first-return approachable at $x \in \mathbb{I}^2$ with respect to a trajectory $\overline{x}$* if for every ray g at x, there is an envelope G_g of g such that f is first-return approachable relative to every envelope G of g for which $G \subseteq G_g$. If a function is radially first-return approachable at $x \in \mathbb{I}^2$ with respect to a trajectory $\overline{x}$ for all $x \in \mathbb{I}^2$ we say it is *radially first-return approachable.*

Another natural candidate is to consider first-return approachability with respect to every sector (Stolz angle) at x:

Definition 2.6. A function $f\colon \mathbb{I}^2 \to \mathbb{R}$ will be said to be *sectorially first-return approachable at $x \in \mathbb{I}^2$ with respect to a trajectory $\overline{x}$* if it is first-return approachable relative to every sector at x. If a function is sectorially first-return approachable at $x \in \mathbb{I}^2$ with respect to a trajectory $\overline{x}$ for all $x \in \mathbb{I}^n$ we say it is *sectionally first-return approachable.*

Lastly, we could combine the previous two notions to obtain the concept of radially sectorially first-return approachable:

Definition 2.7. A function $f\colon \mathbb{I}^2 \to \mathbb{R}$ will be said to be *radially-sectorially first-return approachable at $x \in \mathbb{I}^2$ with respect to a trajectory $\overline{x}$* if for every ray g at x, there is a sector A at x which is an envelope for g such that f is first-return approachable relative to every sector $B \subseteq A$ at x which is also an envelope for g. If a function is radially-sectorially first-return approachable at $x \in \mathbb{I}^n$ with respect to a trajectory $\overline{x}$ for all $x \in \mathbb{I}^2$ we say it is *radially-sectorially first-return approachable.*

In [8] and [9] Shelton demonstrated the relationships among these concepts and summarized these in the following chart. This is a "pointwise" chart; for example, the arrow from "arcwise" to "radially" signifies that if $f\colon \mathbb{I}^2 \to \mathbb{R}$ is arcwise first-return approachable at x with respect to a fixed trajectory $\overline{x}$, then f is also radially first-return approachable at x with respect to the same trajectory $\overline{x}$. Furthermore, Shelton provides examples to show that no non-trivial arrows can be added to this chart; that is, he, for example, provides a function which is everywhere radially first-return approachable with respect to a specific trajectory $\overline{x}$ yet fails to be arcwise first return approachable via $\overline{x}$ at at least one point.

$$\begin{tikzcd}
& \text{Arcwise} \arrow[dl] \arrow[dr] & \\
\text{Radially} & & \text{Sectorially} \arrow[d] \\
& & \text{Radially-Sectorially} \arrow[d] \\
& & \text{Approachable}
\end{tikzcd}$$

If we look to this chart for possible analogs to first-return continuity, we can quickly dismiss the two classes of arcwise first-return approachable functions and first-return approachable functions. Indeed, Shelton [8], [9] has shown that if $f\colon \mathbb{I}^2 \to \mathbb{R}$ is arcwise first-return approachable, then f is actually continuous on $\mathbb{I}^2$. Of course, on the other extreme, first-return approachability is too weak a concept since, for example, the characteristic function of the upper half of $\mathbb{I}^2$ is first-return approachable but does not preserve the connectivity of $\mathbb{I}^2$. Thus we narrow our search to the remaining three classes. In [8], [9] it was shown that none of these three classes of functions have the property that they preserve the connectedness of *every* connected set in $\mathbb{I}^2$, but that is probably too much to expect. Indeed, even derivatives (gradients) do not have that property. However, as we shall see, all three classes present some claim to being a natural analog of first-return continuity.

3. Radially first-return approachable functions are Baire one, "Darboux"

Here we shall show that radially first-return approachable functions belong to Baire class one and map polygonally connected sets to connected sets. (Shelton [8], [9] established a preliminary version of this result by showing that if a function is *both* first-return approachable and radially first-return approachable with respect to the same trajectory $\overline{x}$, then it will map polygonally connected sets to connected sets.)

Recall that a first-return recoverable function $f\colon \mathbb{I} \to \mathbb{R}$ is of Baire class one. However, as shown in [8], [9], it is possible for a function to be radially first-return approachable on $\mathbb{I}^2$ via a trajectory $\overline{x}$, yet fail to be first return recoverable via $\overline{x}$. Thus, we must look for another means to show that such a function belongs to Baire class one. We begin with the following lemma, which has a couple of interesting applications.

Lemma 3.1. *Let $f\colon \mathbb{I}^2 \to \mathbb{R}$, $x_0 \in \mathbb{I}^2$, and $\theta \in [0, 2\pi)$. Suppose $\overline{x}$ is an enumeration of a support set D and that $x_o \notin \overline{[R(x_o,\theta) \setminus \{x_o\}] \cap D}$. Suppose further that there is an envelope G for $R(x_0,\theta)$ such that f is first-return approachable via $\overline{x}$ at x_0 relative to every subenvelope of G for $R(x_0,\theta)$. Then for every $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$B(x_0,\delta) \cap \overline{A_\varepsilon(x_0)} \cap \bigl(R(x_0,\theta) \setminus \{x_o\}\bigr) = \emptyset,$$

where $A_\varepsilon(x_0) = \bigl\{x \in D\colon |f(x) - f(x_0)| \geqslant \varepsilon\bigr\}$.

Proof. Without loss of generality we may assume that $\bigl(R(x_0,\theta) \setminus \{x_0\}\bigr) \cap D = \emptyset$. Let $\varepsilon > 0$ and suppose that no δ satisfies the stated conclusion. Let E be any subenvelope of G for $R(x_0,\theta)$. Then there exists a subsequence $\{x_{n_i}\}$ of $\overline{x} = \{x_n\}$ such that $\{x_{n_i}\} \subset E$ and

(1) $|f(x_{n_i}) - f(x_0)| \geqslant \varepsilon$ for all $i \in \mathbb{N}$.

(2) $\delta_{i+1} < \delta_i/2$ for each $i \in \mathbb{N}$, where $\delta_i = \|x_0 - x_{n_i}\|$.

(3) $L(x_{n_i}, z_i) \subset E$ for each $i \in \mathbb{N}$. Here $z_i \in R(x_0,\theta)$ with $\|x_0 - z_i\| = \delta_i$. As $[x_{n_i}, z_i]$ divides $\overline{B(x_0,\delta_i)}$ into two closed regions, we let $L(x_{n_i}, z_i)$ denote the region not containing x_0.

Fix an $i \in \mathbb{N}$ and set $N_i = \{x_k\colon k \neq n_i,\ k \leqslant n_{i+2}\}$. As $R(x_0,\theta) \cap D = \emptyset$, $[z_{i+1}, z_i] \cap N_i = \emptyset$. Moreover, there exists an arc α_i from z_i to x_{n_i} in $L(x_{n_i}, z_i)$ such that $\alpha_i \cap N_i = \emptyset$. Then $\alpha_i \cup [z_{i+1}, z_i]$ is a closed set which misses N_i and, thus, there is an open set $U_i \subset \mathbb{I}^2 \setminus B(x_0, \delta_{i+2})$ such that

(i) $U_i \cap N_i = \emptyset$.

(ii) $\alpha_i \cup [z_{i+1}, z_i] \subset U_i$.

Let U_0 be an open set in E which contains $R(x_0,\theta) \setminus [z_1, x_0]$ such that U_0 misses $\{x_k\colon k < n_1\}$. Then set $U = \bigcup\limits_{i=0}^{\infty} U_i$. Then U is subenvelope of E for $R(x_0,\theta)$. However, the first-return approach to x_0 relative to U is $\{x_{n_i}\}$, and since $\left|f(x_{n_i} - f(x_0)\right| \geqslant \varepsilon$ for all $i \in \mathbb{N}$, this contradicts the the assumption that f is first-return approachable via $\overline{x}$ at x_0 relative to every subenvelope of G for $R(x_0,\theta)$. □

Theorem 3.1. *If $f\colon \mathbb{I}^2 \to \mathbb{R}$ is radially first-return approachable, then f is of Baire class one.*

Proof. Let $\overline{x}$ be an enumeration of a support set D such that f is radially first-return approachable with respect to $\overline{x}$. Let $P \subseteq \mathbb{I}^2$ be perfect and suppose that $f|_P$ has no point of continuity. Then for each $x \in P$, $\operatorname{osc}(f|_P, x) > 0$. Let $\mathcal{O}_\varepsilon = \left\{x \in P\colon \operatorname{osc}(f|_P, x) \geqslant \varepsilon\right\}$. Since for each $\varepsilon > 0$, $\mathcal{O}_\varepsilon$ is closed, there is an $\varepsilon > 0$ and a portion P^* of P such that $P^* \subseteq \mathcal{O}_\varepsilon$.

Now, for each $x_o \in P^*$ there exists a sequence $\{z_n\} \subset P$ and intervals $I_1(x_o)$ and $I_2(x_o)$ such that $\{z_n\} \to x_o$, $\operatorname{dist}\bigl(I_1(x_o), I_2(x_o)\bigr) > 3\varepsilon/8$, $f(x_o) \in I_1(x_o)$, and $f(z_n) \in I_2(x_o)$ for each $n \in \mathbb{N}$. Hence there exist two fixed intervals $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ with $\operatorname{dist}(I_1, I_2) > 3\varepsilon/8$ such that the set $S \equiv \bigl\{x \in P^*\colon f(x) \in I_1$ and there exists a sequence $\{z_n\} \subset P$ with $f(z_n) \in I_2$ for all n and $\{z_n\} \to x\bigr\}$ is of second category in a portion $P^{**} \subseteq P^*$.

Fix $x_o \in S$ and set $T^*(x_o) = \bigl\{\theta\colon R(x_o,\theta) \cap D = \emptyset\bigr\}$. From Lemma 3.1 it follows that for each $\theta \in T^*(x_o)$ there exists an envelope $E\bigl(R(x_o,\theta)\bigr)$ and a radius $\delta(x_o,\theta)$ such that

$$E\bigl(R(x_o,\theta)\bigr) \cap B\bigl(x_o, \delta(x_0,\theta)\bigr) \cap A_{\frac{\varepsilon}{10}}(x_o) = \emptyset .$$

That is, if $x \in E\bigl(R(x_o,\theta)\bigr) \cap B\bigl(x_o, \delta(x_0,\theta)\bigr)$, then

$$a_1 - \frac{\varepsilon}{10} < f(x) < b_1 + \frac{\varepsilon}{10} .$$

As $T^*(x_o)$ is co-countable, there exist $\delta(x_o) > 0$ and $\theta_1(x_o) < \theta_2(x_o)$ such that

$$\Theta(x_0) \equiv \{\theta \in T^*(x_o) \colon \delta(x_o, \theta) > \delta(x_o)\}$$

is dense in $(\theta_1(x_o), \theta_2(x_o))$. But S is second category in P and so there exists an open set U and fixed $\theta_1 < \theta_2$, $\delta > 0$ such that $\{x \in S \colon \theta_1(x) < \theta_1 < \theta_2 < \theta_2(x)$ and $\delta < \delta(x)\}$ is dense in $S \cap U \neq \emptyset$. From the definition of S it follows that there exists a $z \in U \cap P$ such that $f(z) \in I_2$. Let $\theta^* \in T^*(z)$. Then according to Lemma 3.1, there is an envelope $E(R(z, \theta^*))$ and a $\delta^* > 0$ such that

$$|f(x) - f(z)| < \frac{\varepsilon}{10} \text{ whenever } x \in E(R(z, \theta^*)) \cap B(z, \delta^*). \tag{1}$$

That is, for every $x \in E(R(z, \theta^*)) \cap B(z, \delta^*)$, $a_2 - \frac{\varepsilon}{10} < f(x) < b_2 + \frac{\varepsilon}{10}$. As S is dense in $P \cap U$, there is a sequence $\{u_i\} \subset S \cap U$ such that $\{u_i\} \to z$. Moreover, there exist $\theta_i \in \Theta(u_i)$ such that $\{\theta_i\} \to \theta^*$. It follows that $|f(x) - f(u_i)| < \frac{\varepsilon}{10}$ for all $x \in R(u_i, \theta_i) \cap B(u_i, \delta)$, and as $u_i \in S$, $f(u_i) \in I_1$. Hence, $a_1 - \frac{\varepsilon}{10} < f(u_i) < b_1 + \frac{\varepsilon}{10}$ for all $x \in R(u_i, \theta_i) \cap B(u_i, \delta)$. However, $\{u_i\} \to z$ and $\{\theta_i\} \to \theta^*$, so eventually

$$R(u_i, \theta_i) \cap B(u_i, \delta) \cap E(R(z, \theta^*)) \cap B(z, \delta^*) \neq \emptyset.$$

This contradicts (1) and completes the proof. □

Another application of Lemma 3.1 is the following characterization.

Lemma 3.2. *Let $f \colon \mathbb{I}^2 \to \mathbb{R}$ and fix $x_o \in \mathbb{I}^2$, $\theta \in [0, 2\pi)$, a support set D, and an enumeration $\overline{x}$ of D. Then f is radially first-return approachable at x_o via $\overline{x}$ in the direction θ if and only if one of the following criteria holds:*

(a) *$x_o \notin \overline{[R(x_o, \theta) \setminus \{x_o\}] \cap D}$ and there exists an envelope G for $R(x_o, \theta)$ such that $\lim\limits_{x \to x_o} f|_{G \cap D}(x) = f(x_o)$.*

(b) *$x_o \in \overline{[R(x_o, \theta) \setminus \{x_o\}] \cap D}$ and f is first-return recoverable at x_o relative to the sequence $\{x_n \in \overline{x} \colon x_n \in R(x_o, \theta)\}$.*

P r o o f. Throughout this proof we shall, without loss of generality, assume that $x_o = (0, 0) \equiv 0$. First, if (a) holds then clearly f is first-return approachable at 0 via $\overline{x}$ relative to every subenvelope $G^* \subseteq G$ for $R(0, \theta)$. Hence, f is radially first-return approachable at 0 via $\overline{x}$ in the direction θ.

Suppose (b) holds. Let $\{x_{n_i}\}$ be the first-return approach to 0 obtained from the sequence $\{x_n \in \overline{x} \colon x_n \in R(0, \theta)\}$. Let G_1 be an open set such that

$$R(0, \theta) \setminus B(0, \|x_{n_1}\|) \subseteq G_1 \subseteq$$

$$\mathbb{I}^2 \setminus \Big[B\big(0, (\|x_{n_1}\| + \|x_{n_2}\|)/2\big) \cup \big(\{x_k : k < n_2\} \setminus R(0, \theta)\big)\Big].$$

If the open set G_{i-1} has been selected, we choose an open set G_i such that

$$[x_{n_i}, x_{n_{i-1}}] \subseteq G_i \subseteq$$
$$\mathbb{I}^2 \setminus \left[B\big(0, (\|x_{n_i}\| + \|x_{n_{i+1}}\|)/2\big) \cup \big(\{x_k : k < n_{i+1}\} \setminus R(0,\theta)\big)\right].$$

Let $G = \bigcup_{i=1}^{\infty} G_i$. Then G is an envelope for $R(0,\theta)$ and for each subenvelope $G^* \subseteq G$ for $R(0,\theta)$ we have that the first-return approach to 0 via $\overline{x}$ relative to G^* is $\{x_{n_i}\}$. Hence, f is first-return approachable at 0 via $\overline{x}$ relative to G^* and, thus, f is radially first-return approachable at 0 via $\overline{x}$ in the direction θ.

Turning to the converse, we now suppose that f is radially first-return approachable at 0 in the direction θ via $\overline{x}$. Let G be an envelope for $R(0,\theta)$ such that f is first-return approachable at 0 relative to every subenvelope $G^* \subseteq G$ for $R(0,\theta)$ via $\overline{x}$.

As a first case, suppose that $0 \notin \overline{\big[R(0,\theta) \setminus \{0\}\big] \cap D}$. From Lemma 3.1 we see that for each $i \in \mathbb{N}$ there is a $\delta > 0$ such that

$$\overline{B(0,\delta_i)} \cap \overline{A_{1/i}(0)} \cap \big[R(0,\theta) \setminus \{0\}\big] = \emptyset,$$

where $A_{1/i}(0) = \{x \in D \colon \big|f(x) - f(0)\big| \geqslant 1/i\}$ and where we may assume $\delta_{i+1} < \delta_i$. Thus, for each $i \in \mathbb{N}$ there is an open set G_i such that

$$\big\{z \colon z \in R(0,\theta) \text{ and } \delta_{i+1} \leqslant \|z\| \leqslant \delta_i\big\} \subseteq G_i \subseteq G \setminus \overline{B(0,\delta_{i+2})}$$

and for each $z \in G_i \cap D$, $\big|f(z) - f(0)\big| < 1/i$. Letting $G = \bigcup_{i=1}^{\infty} G_i$ we have $\lim_{x \to 0} f|_{G \cap D}(x) = f(0)$; that is, (a) holds.

Finally, suppose that $0 \in \overline{\big[R(0,\theta) \setminus \{0\}\big] \cap D}$. Let $\{x_{n_i}\}$ be the first-return approach to 0 along $R(0,\theta)$. We shall construct a subenvelope $G^* \subseteq G$ for $R(0,\theta)$ such that the first-return approach to 0 relative to G^* via $\overline{x}$ is also $\{x_{n_i}\}$. From this it will follow that $\lim_{i\to\infty} f(x_{n_i}) = f(0)$ and thus (b.) will hold.

Let G_1 be an open set such that

$$R(0,\theta) \setminus B\big(0, \|x_{n_1}\|\big) \subseteq G_1 \subseteq$$
$$G \setminus \left[B\big(0, (\|x_{n_1}\| + \|x_{n_2}\|)/2\big) \cup \big(\{x_k : k < n_2\} \setminus R(0,\theta)\big)\right].$$

If the open set G_{i-1} has been selected, we choose an open set G_i such that

$$[x_{n_i}, x_{n_{i-1}}] \subseteq G_i \subseteq$$
$$G \setminus \left[B\big(0, (\|x_{n_i}\| + \|x_{n_{i+1}}\|)/2\big) \cup \big(\{x_k : k < n_{i+1}\} \setminus R(0,\theta)\big)\right].$$

Let $G^* = \bigcup_{i=1}^{\infty} G_i$. Then G^* is a subenvelope of G for $R(0,\theta)$ and the first-return approach to 0 via $\overline{x}$ relative to G^* is $\{x_{n_i}\}$, completing the proof. □

When the function is everywhere radially first-return approachable, condition (a) takes on an even more restrictive appearance as seen in the following characterization.

THEOREM 3.2. *Let $f\colon \mathbb{I}^2 \to \mathbb{R}$, and let D be a support set with enumeration $\overline{x}$. Then f is radially first-return approachable via $\overline{x}$ if and only if for each $z \in \mathbb{I}^2$ and each $\theta \in [0, 2\pi)$ one of the following holds:*

(A) $z \notin \overline{[R(z,\theta) \setminus \{z\}] \cap D}$ *and there exists an envelope G for $R(z,\theta)$ such that* $\lim\limits_{x \to z} f|_G(x) = f(z)$.

(B) $z \in \overline{[R(z,\theta) \setminus \{z\}]}$ *and f is first-return recoverable at z relative to the sequence* $\{x_n \in \overline{x} \colon x_n \in R(z,\theta)\}$.

Proof. The "if" direction follows immediately from Lemma 3.2 and the "only if" direction follows from the same lemma with the recognition that the assumption of radial first-return approachability assures that $f|_D$ is dense in f. □

We then immediately obtain the following:

COROLLARY 3.1. *Let $f\colon \mathbb{I}^2 \to \mathbb{R}$ be radially first-return approachable with respect to a trajectory $\overline{x}$. Let ℓ be a line segment in $\mathbb{I}^2$ with endpoints z_0 and z_1. If $f(z) \geqslant \alpha$ for all $z \in \ell \setminus \{z_0, z_1\}$, then $f(z_0) \geqslant \alpha$ and $f(z_1) \geqslant \alpha$.*

LEMMA 3.3. *Let $f\colon \mathbb{I}^2 \to \mathbb{R}$ be radially first-return approachable. Let $y_0, y_1 \in \mathbb{I}^2$ and let $g\colon [0,1] \to \mathbb{I}^2$ be a linear arc from y_0 to y_1. Then the function $F \equiv f \circ g\colon [0,1] \to \mathbb{R}$ is Baire one and Darboux.*

Proof. First, note that since Theorem 3.1 assures that f is Baire one and since g is continuous, it is immediate that F is Baire one.

Recall that Zahorski [11] showed that a real-valued Baire one function of one real variable has the Darboux property if and only if the connected components of the inverse image of every closed ray in $\mathbb{R}$ are closed. Now, let $\alpha \in \mathbb{R}$ and let J be a component interval of $F^{-1}\big([\alpha,\infty)\big)$. Let $a < b$ be the endpoints of J. Then by Corollary 3.1 $F(a) \geqslant \alpha$ and $F(b) \geqslant \alpha$. Thus, J is closed. A symmetrical argument holds for $F^{-1}\big((-\infty,\alpha]\big)$. Thus, F has the Darboux property, completing our proof. □

THEOREM 3.3. *Let $f\colon \mathbb{I}^2 \to \mathbb{R}$ be radially first-return approachable with respect to a trajectory $\overline{x}$. Then, if $H \subseteq \mathbb{I}^2$ is polygonally connected, $f(H)$ is connected.*

Proof. Suppose $f(H)$ is disconnected. This implies there exist two disjoint open sets A and B such that $A \cap f(H) \neq \emptyset$, $B \cap f(H) \neq \emptyset$, and $f(H) \subseteq A \cup B$. Consider a point $a \in A \cap f(H)$ and a point $b \in B \cap f(H)$. Now, consider a polygonal arc g from $f^{-1}(a)$ to $f^{-1}(b)$. Since (A, B) disconnects $f(H)$, it also disconnects $f(g)$, and so there is a linear subarc g^* of g such that (A, B) disconnects $f \circ g^*$. However, by Lemma 3.3 we know that $f \circ g^*$ is Darboux, yielding a contradiction. □

4. Points of continuity of a radially first-return approachable function

Before considering the other two candidates for analogs to first-return continuity, we wish to examine the set of points of continuity of a radially first-return approachable function. Of course, since a radially first-return approachable function is of Baire class one, its set of points of discontinuity must be of first category. Here we investigate what further can be said about this set. We begin by showing that the set of points of discontinuity must be significantly smaller than first category; namely, it is sparse. Furthermore, we show that it cannot contain a second category subset of a line segment, although it can contain a full linear measure subset of a line segment.

Recall that W. H. Young [10] defined an *ensemble ridé* to be a set in $\mathbb{R}^2$ lying on a countable collection of curves represented by equations of the form

$$x = a + bt + cf(t), \quad y = a' + b't + c'f(t)$$

where a, b, c, a', b', and c' are constants and $f(t)$ is a monotonic function. H. Blumberg [1] renamed such a set *sparse*. Clearly sparse sets are both of first-category and of measure zero; indeed, they are σ-porous. In [10] Young observed that a set S with the property that every point of S has a triangular void (a triangle with the point as a vertex and no other point of S inside the triangle) is a sparse set. We shall exploit this fact in proving the following.

Theorem 4.1. *The set of points of discontinuity of a radially first-return approachable function $f\colon \mathbb{I}^2 \to \mathbb{R}$ is sparse.*

Proof. Let $f\colon \mathbb{I}^2 \to \mathbb{R}$ be radially first-return approachable via the trajectory $\overline{x}$ and let $D = \{x : f \text{ is not continuous at } x\}$. If $x \in D$, then the oscillation of f at x is positive and thus we can find a natural number k_x, and two open intervals J_x and K_x, having rational endpoints, such that the distance between J_x and K_x, $\operatorname{dist}(J_x, K_x) > 1/k_x$, $f(x) \in J_x$, and every neighborhood of x contains points which map to K_x. If for each natural number k we let $\mathcal{I}_k$ denote the collection of all pairs (J, K) of open intervals having rational endpoints with

$\operatorname{dist}(J,K) > 1/k$, and we let $D_{k,(J,K)} = \{x\colon f(x) \in J$ and every neighborhood of x contains points which map to $K\}$, then

$$D = \bigcup_{k\in\mathbb{N}} \bigcup_{(J,K)\in\mathcal{I}_k} D_{k,(J,K)}\,.$$

Next, if $x \in D_{k,(J,K)}$, then, using Theorem 3.2, we see that for a co-countable set of θ's in $[0,2\pi)$ there exists an envelope $G_{x,\theta}$ for $R(x,\theta)$ and an $i_{x,\theta} \in \mathbb{N}$ such that for $z \in G_{x,\theta} \cap B(x, 1/i_{x,\theta})$, $f(z) \in J$. Thus, there will exist an integer i_x and a Stolz angle $S_{\alpha_x,\beta_x}(x)$ at x, where α_x and β_x are rational with $0 < \alpha_x < \beta_x < 2\pi$, such that for an open dense set of points $y \in S_{\alpha_x,\beta_x}(x) \cap B(x, 1/i_x)$ we have $f(y) \in J$. We may clearly further assume that $\beta_x - \alpha_x < \pi/2$. Let $\mathcal{A}$ denote the collection of all rational pairs (α,β) of numbers in $[0,2\pi)$ with $0 < \beta - \alpha < \pi/2$. Let $D_{k,(J,K),i,(\alpha,\beta)}$ denote the set of x's in $D_{k,(J,K)}$ such that for an open dense set of y's in $S_{\alpha_x,\beta_x}(x) \cap B(x,1/i)$, we have $f(y) \in J$. Then

$$D = \bigcup_{k\in\mathbb{N}} \bigcup_{(J,K)\in\mathcal{I}_k} \bigcup_{i\in\mathbb{N}} \bigcup_{(\alpha,\beta)\in\mathcal{A}} D_{k,(J,K),i,(\alpha,\beta)}\,.$$

We shall show that D is sparse by showing that each set $D_{k,(J,K),i,(\alpha,\beta)}$ is sparse.

To this end, fix $k,(J,K),i,(\alpha,\beta)$ and set $D^* = D_{k,(J,K),i,(\alpha,\beta)}$. If $x \in D^*$ and $y \in S_{\alpha,\beta}(x) \cap B(x,R)$ we note that $f(y) \notin K$. This is because f is radially first-return approachable at y and for all but countably many directions θ, condition (A.) of Theorem 3.2 applies. Now, fix $x_o \in D^*$ and $0 < R < 1/i$. With this last observation we may next show that $S_{\pi+\alpha,\pi+\beta}(x_0) \cap B(x_0,R) \cap D^* = \emptyset$. To see this, suppose $x' \in S_{\pi+\alpha,\pi+\beta}(x_0) \cap B(x_0,R) \cap D^*$. Note that $S_{\alpha,\beta}(x') \cap B(x',R)$ is an open set containing x_0. This implies there must be points in $S_{\alpha,\beta}(x') \cap B(x',R)$ arbitrarily close to x_0 which f maps to K. However, no point of $S_{\alpha,\beta}(x') \cap B(x',R)$ can map to K. Thus, $S_{\pi+\alpha,\pi+\beta}(x_0) \cap B(x_0,R) \cap D^* = \emptyset$. Using the aforementioned result of Young, we conclude that D^* is sparse, completing our proof. $\square$

Next, we show that the set of points of discontinuity of a radially first-return approachable function is also small in the sense that it cannot contain a second category subset of a line segment.

THEOREM 4.2. *The set of points of discontinuity of a radially first-return approachable function $f\colon \mathbb{I}^2 \to \mathbb{R}$ cannot contain a second category subset of a line segment.*

Proof. Let H denote the set of points at which f is discontinuous and let $\overline{x}$ be an enumeration of a support set D such that f is radially first-return approachable with respect to $\overline{x}$. It suffices to show that if $I = [0,1] \times \{0\}$, then

$F = I \cap H$ is not second category in I. Suppose, to the contrary that F is of the second category in I, and let $F_n = \{x \in F\colon \operatorname{oscx}(f,x) \geq \frac{1}{n}\}$ where $\operatorname{oscx}(f,x) = \limsup_{y \to x} |f(y) - f(x)|$. If $x_o \in F_n$, then there are open intervals J_1 and J_2, having rational end points, and a sequence $\overline{x_o} = \{y_k(x_o)\} \to x_o$ such that $f(x_o) \in J_1$, for every k, $f(y_k(x_o)) \in J_2$ and $\operatorname{dist}(J_1, J_2) > \frac{1}{2n}$. Assign the triple (n, J_1, J_2) to x_o and let $F_{n,J_1,J_2} = \{x \in F\colon n,\ J_1,\ \text{and } J_2 \text{ have been assigned to } x\}$. As F is second category in I, there is a fixed index (n, J_1, J_2) such that $F^* = F_{n,J_1,J_2}$ is of the second category in I. Note that there is no loss in generality in assuming that for each $x_o \in F^*$, the corresponding sequence $\{y_k(x_o)\}$ lies in $[0,1] \times [0,1]$. Next, let I_o be a subinterval of I in which F^* is of second category and dense. Let $C = f^{-1}(\mathbb{R} \setminus J_1)$.

Let $x_o \in F^*$ and $y_k(x_o) \in \overline{x_o}$. Let $R(y_k(x_o), \theta)$ be a ray at $y_k(x_o)$ where $\frac{\pi}{4} < \theta < \frac{\pi}{2}$ and $D \cap R(y_k(x_o), \theta) = \emptyset$. It follows from Lemma 3.2 that there is an envelope $E(y_k(x_o), \theta)$ of $R(y_k(x_o), \theta)$ with the property that $f|_{E(y_k(x_o),\theta)}$ is continuous at $y_k(x_o)$. Let $\delta > 0$ be sufficiently small that for each $x \in U(y_k(x_o)) \equiv E(y_k(x_o), \theta) \cap B(y_k, \delta)$, $|f(x) - f(y_k)| < \frac{1}{10n}$ and $\delta < |y_k - x_o|$. It follows that $f(u) \notin J_1$ whenever $u \in U(y_k(x_o))$; that is, the open set

$$B \equiv \bigcup_{x_o \in F^*} \bigcup_{k=1}^{\infty} U(y_k(x_o)) \subseteq C.$$

For each natural number j, let $C_j = C \cap (I_o \times (0, 1/j))$, and $A_j = \{x \in I_o\colon R(x, \frac{\pi}{2}) \cap C_j \neq \emptyset\}$. Then each A_j contains an open dense subset of I_o and, consequently, the set $A \equiv \cap_{j=1}^{\infty} A_j$ is (linearly) residual in I_o. Consequently, $A \cap F^*$ is of second category in I_o. If $x* \in A \cap F^*$ is any point such that $R(x*, \frac{\pi}{2}) \cap D = \emptyset$, then it follows from Lemma 3.2 that $f|_{R(x*,\frac{\pi}{2})}$ is continuous at $x*$. But, $f(x*) \in J_1$ and there is a sequence $\{u_i\} \subset R(x*, \frac{\pi}{2})$ converging to $x*$ such that $f(u_i) \notin J_1$. This contradiction completes the proof. $\square$

Although the set of points of discontinuity can only intersect a line segment in a first category subset of that segment, that intersection can be large in terms of linear measure, as the next two examples illustrate.

EXAMPLE 4.1. Let P be any perfect subset of $[0,1]$. Then there is a radially first-return approachable function $f\colon \mathbb{I}^2 \to [0,1]$ such that the set of points at which f is not continuous is $P \times \{0\}$.

Proof. Enumerate the component intervals of $[0,1] \setminus P$ as $\{I_i = (a_i, b_i)\}_{i=1}^{\infty}$. For each i let T_i denote the open isosceles triangular region in the upper half plane determined by the segment (a_i, b_i) and the rays $R(a_i, \pi/4^i)$ and $R(b_i, \pi - \pi/4^i)$. It is easily seen that if $x \in [0,1] \setminus (a_i, b_i)$ and $R(x, \theta)$ is a ray at x such that both θ and $\pi - \theta$ are greater than $\pi/4^i$, then $R(x, \theta) \cap T_i = \emptyset$.

Thus, for each $x \in P$ and each $\theta \in [0, 2\pi)$, $R(x,\theta)$ can intersect at most finitely many of the T_i's. For each i let B_i be an open disk lying in T_i such that the distance from B_i to the x-axis is at least half the height of T_i. Let $g_i \colon \overline{B_i} \to \mathbb{R}$ be the conical-shaped continuous function that has value 1 at the center and the value 0 on the boundary of B_i. Define $f \colon \mathbb{I}^2 \to \mathbb{R}$ by

$$f(x) = \begin{cases} g_i(x)\,, & x \in B_i\,,\\ 0\,, & \text{otherwise}\,. \end{cases}$$

Note that the set of points of discontinuity of f is P.

Let $\overline{x}$ be any enumeration of the set of points in $\mathbb{I}^2$ having rational coordinates. To show that f is radially first-return approachable on $\mathbb{I}^2$ via $\overline{x}$ we need only check that it is radially first-return approachable via $\overline{x}$ at each point of P. To this end, let $x \in P$ and $R(x,\theta)$ be a ray at x. First, if $\pi < \theta < 2\pi$ then the open lower half-plane serves as an envelope for $R(x,\theta)$ on which f is constantly 0 and since $f(x) = 0$, f is obviously radially first-return approachable at x via $\overline{x}$ in the direction θ. If θ is either 0 or π, then f is constantly 0 on $R(x,\theta)$ and hence first return approachable at x via the sequence $\{x_n \in \overline{x} \colon x_n \in R(x,\theta)\}$. If $\theta = \pi/2$, then no point in the Stolz angle at x determined by the rays $R(x,\pi/4)$ and $R(x, 3\pi/4)$ belongs to any T_i and hence f is constantly 0 in that Stolz angle. Next, consider the situation $0 < \theta < \pi/2$. We choose θ_1 and θ_2 such that $0 < \theta_1 < \theta < \theta_2 < \pi/2$. We let k be the smallest natural number such that $\theta_1 > \pi/4^k$. Then the Stolz angle S at x determined by the rays $R(x,\theta_1)$ and $R(x,\theta_2)$ has the feature that S intersects no T_i for which $i \geqslant k$. Thus there is a $\delta > 0$ such that $f(z) = 0$ for all $z \in B(x,\delta) \cap S$. The case where $\pi/2 < \theta < \pi$ may be handled symmetrically and, thus, by Lemma 3.2 f is radially first-return approachable at x, completing the proof. ☐

EXAMPLE 4.2. There is a radially first-return approachable function $f \colon \mathbb{I}^2 \to \mathbb{R}$ such that the set of points at which f is not continuous is of full linear measure in $I = [0,1] \times \{0\}$.

Proof. For each $n \in \mathbb{N}$, let $P_n \subset I$ be a Cantor set of measure μ_n where $\{\mu_n\} \to 1$. Let $f_n \colon \mathbb{I}^2 \to [0,1]$ be as in Example 4.1 and define

$$f(x) = \sum_{n=1}^{\infty} \frac{f_n(x)}{2^n}.$$

It is easy to see that f fulfills the conditions of Example 4.2 ☐

In summary, radially first-return approachable functions present themselves as a possible candidate for the analog of first-return continuous functions on the line since they are Baire one and have the intermediate value property in that they send line segments to line segments. However, they are almost too nice in terms of their continuity properties—first-return continuous functions on $\mathbb{I}$ can be discontinuous on sets of full measure.

5. The case for sectorially and radially sectorially first-return approachable functions

In [8], [9] it has been shown that sectorially first-return approachable functions are radially-sectorially first-return approachable and that sectorially first-return approachable functions can map a line segment to a disconnected set. Here we shall observe that radially-sectorially first-return approachable functions will map closed, convex, sets with nonempty interior to connected sets. Recall that Malý [7] showed that derivatives (gradients) have this type of "Darboux" property.

Theorem 5.1. *Let $f\colon \mathbb{I}^2 \to \mathbb{R}$ be radially-sectorially first-return approachable. Then, if $H \subseteq \mathbb{I}^2$ is closed, convex, and has nonempty interior, $f(H)$ is connected.*

Proof. Let $H \subseteq \mathbb{I}^2$ be closed, convex, and have nonempty interior. Suppose that $f(H)$ is not connected. Let A and B be two disjoint open sets in $\mathbb{R}$ such that $A \cap f(H) \neq \emptyset$, $B \cap f(H) \neq \emptyset$, and $f(H) \subseteq A \cup B$. Let $E = \overline{\left(f^{-1}(A) \cap H\right)} \cap \overline{\left(f^{-1}(B) \cap H\right)}$. Then $E \neq \emptyset$ since H is connected. Let $E_A = E \cap f^{-1}(A)$ and $E_B = E \cap f^{-1}(B)$. We claim that both E_A and E_B are dense in E. If so, this will yield a contraction since $f|_E$ will have no points of continuity; yet, as shown in [8], f must be a Baire class one function, being radially-sectorially first-return approachable.

To see that E_A and E_B are both dense in E, let $x_o \in E_A$. We shall produce a sequence of points from E_B converging to x_o. Since $x_o \in E$, there exists a sequence $\{s_n\}$ in $f^{-1}(B) \cap H$ converging to x_o. If infinitely many of the s_n's belong to E_B, then we have found our sequence. On the other hand, suppose there is an n_o such that for all $n \geqslant n_o$, $s_n \notin E_B$. Fix $n \geqslant n_o$. Find an open ball G_n within $1/n$ of s_n lying inside $f^{-1}(B) \cap H$. Let z_n be the center of G_n. Then we may find a closed isosceles triangular shaped region $T_n \subseteq H$ having z_n in its interior and on its altitude, with base in G_n, and with x_o as a vertex such that f is first-return approachable at x_o via $\overline{x}$ relative to T_n. For each $0 < \ell < \|z_n - x_o\|$, let S_ℓ be the closed similar triangular subregion of T_n having vertex x_o and height ℓ. Let ε be the largest positive number such that the set $U_\varepsilon \equiv T_n - S_{\|z_n - x_o\| - \varepsilon}$ lies in $f^{-1}(B)$. Then since f is radially-sectorially first-return approachable, each point on the boundary of U_ε also lies in $f^{-1}(B)$. Furthermore, since ε is maximal, at least one of these points must be a limit point of $f^{-1}(A)$. Thus, this point belongs to E_B. This completes our proof that each point of E_A is a limit point of E_B. Symmetrically, each point of E_B is a limit point of E_A and our proof is complete. □

6. Points of continuity of sectorially first-return recoverable functions

Unlike the situation with radially first-return approachable functions the set of points of discontinuity of sectorially first-return approachable function (and hence those of a radially-sectorially first-return approachable function) need not be of two-dimensional measure zero and can contain line segments.

EXAMPLE 6.1. There is a Cantor set $P \subset \mathbb{I}^2$ of positive two-dimensional Lebesgue measure containing denumerable many length one horizontal and vertical line segments and a sectorially first-return approachable function $f\colon \mathbb{I}^2 \to \mathbb{R}$ such that the set of points of discontinuity for f is P.

Proof. We first describe the Cantor set P, which is constructed inductively in stages. It is convenient to introduce the sequence of integers $\{e_n\}$ defined recursively as $e_0 = 1$ and $e_{n+1} = \big((2n+1)^2-1\big)e_n$. At Stage 1 of the construction we partition $I \equiv [0,1] \times [0,1]$ into 9 equal area squares having horizontal and vertical sides. We delete the open central square, $S_{1,1}$, and let $F_1 = I \setminus S_{1,1}$. That is, F_1 is the union of the remaining $8 = e_1$ squares. At Stage 2 we similarly divide each of the the 8 squares comprising F_1 into 25 equal area squares and delete the middle open squares, $S_{2,k}$, $k = 1,2,\dots,8 = e_1$, from each of the 8 squares. We set $F_2 = F_1 \setminus \bigcup\limits_{k=1}^{e_1} S_{2,k}$. Assume that Stage n has been completed and that F_n has been defined, being the union of e_n closed squares. At Stage $n+1$, we divide each of these squares into $(2n+3)^2$ equal area squares, delete the central open square, $S_{n+1,k}$, $k = 1,2,\dots e_n$, from each of the e_n closed squares comprising F_n, and set $F_{n+1} = F_n \setminus \bigcup\limits_{k=1}^{e_n} S_{n+1,k}$. Finally, let $P = \bigcap\limits_{n=1}^{\infty} F_n$. It is easy to see that P has positive two-dimensional Lebesgue measure containing denumerable many length one horizontal and vertical line segments.

Next, we shall define the function $f\colon \mathbb{I}^2 \to \mathbb{R}$. For each $n \in \mathbb{N}$ and each $k = 1,2,\dots,e_{n-1}$, let $c_{n,k}$ denote the circle centered in the square $S_{n,k}$ with diameter $\frac{1}{3^n}$ times the side length of $S_{n,k}$. Likewise, let $C_{n,k}$ denote the circle centered in the square $S_{n,k}$ with diameter $\frac{3^n-1}{3^n}$ times the side length of $S_{n,k}$. Let $d_{n,k}$ denote the closed disk having $c_{n,k}$ as its boundary. Let $g_{n,k}\colon d_{n,k} \to \mathbb{R}$ be a conically shaped continuous function which is 1 at the center point of $d_{n,k}$, is 0 on $c_{n,k}$ and is the linear interpolation of these values elsewhere in $d_{n,k}$. Then define $f\colon \mathbb{I}^2 \to \mathbb{R}$ as

$$f(x) = \begin{cases} g_{n,k}(x) & \text{if } x \in d_{n,k} \\ 0 & \text{otherwise}. \end{cases}$$

It is immediate that the set of points of discontinuity of f is exactly P.

For each $n \in \mathbb{N}$ and each $k = 1, 2, \ldots, e_{n-1}$, let α_n denote the measure of the angle formed at the midpoint of the bottom edge of $S_{n,k}$ by the two rays emanating from that point which are tangent to $c_{n,k}$. Let ℓ_n denote the length of the smaller arc on $C_{n,k}$ determined by these two rays. Note that $\{\alpha_n\}$ decreases to 0 as $n \to \infty$.

For our support set D we take all points in $\mathbb{I}^2$ having rational coordinates along with a countable set E which is dense on each of the circles $C_{n,k}$, $n \in \mathbb{N}$, $k = 1, 2, \ldots e_{n-1}$. Let $\{t_j\}_{j=1}^{\infty}$ be an arbitrary, but fixed, ordering of this support set. We shall build the trajectory $\overline{x} = \{x_i\}$ inductively in stages as a rearrangement of $\{t_j\}$.

At Stage 1, we select a sufficiently large finite number of points from $E \cap C_{1,1}$ so that each arc of length ℓ_1 on $C_{1,1}$ contains at least one such point. We begin the trajectory $\overline{x}$ with this finite list of points, listed in any order. We then look at the point t_1. If t_1 has not yet been selected and if either $f(t_1) = 0$ or $t_1 \in d_{1,1}$, then we append t_1 to our initial segment for $\overline{x}$. Otherwise, we do not append it. We let i_1 denote the number of points in our initial segment to this point; that is, we have selected $\{x_1, x_2, \ldots, x_{i_1}\}$. This completes Stage 1.

Suppose that Stage n has been completed, resulting in the specification of the initial segment $\{x_1, x_2, \ldots, x_{i_n}\}$. For each $k = 1, 2, \ldots, e_n$, we select a sufficiently large finite number of points from $E \cap C_{n+1,k}$ so that each arc of length ℓ_{n+1} on $C_{n+1,k}$ contains at least one such point. We append these to our list in any order, beginning with x_{i_n+1}. Then we look at each of the points $t_1, t_2, \ldots, t_{n+1}$ which has not yet been appended. If such a point either belongs to $\bigcup_{m=1}^{n+1} \bigcup_{k=1}^{e_m} d_{m,k}$ or if f maps this point to 0, then we append it to our list. We let i_{n+1} be such that to this point we have specified the initial segment of $\overline{x}$ as $\left\{x_1, x_2, \ldots, x_{i_n}, x_{i_n+1}, x_{i_n+2}, \ldots, x_{i_{n+1}}\right\}$.

Inductively, this completes our definition of $\overline{x} = \{x_i\}_{i=1}^{\infty}$. Clearly, $\overline{x}$ is a rearrangement of $\{t_j\}$ and is therefore a trajectory. We must yet show that f is sectorially first-return approachable via $\overline{x}$. Since f is continuous at each point not in P it will suffice to show that f is sectorially first-return approachable via $\overline{x}$ at each point in P. To this end, fix an $x \in P$ and a sector T having vertex at x. Let α be the size of the vertex angle of T. If there is a $\delta > 0$ such that $T \cap B(x, \delta)$ contains only points that f maps to 0, then we are done. If this is not the case, then for every $\delta > 0$, $T \cap B(x, \delta) \cap \left(\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{e_n} d_{n,k} \right) \neq \emptyset$. Choose n so large that $\alpha_n < \alpha$. Next, choose $\varepsilon > 0$ such that if $T \cap B(x, \varepsilon)$ intersects $d_{m,k}$, then $m \geqslant n$. Suppose $T \cap B(x, \varepsilon) \cap d_{m,k} \neq \emptyset$. Since $x \in P$ and $\alpha > \alpha_m$, $T \cap B(x, \varepsilon) \cap C_{m,k}$ contains an arc of length ℓ_m, every point of which is closer to x than is $d_{m,k}$. Furthermore, at least one such point was appended to the trajectory $\overline{x}$ during Stage m or earlier, and this means prior to the appending of any point from $d_{m,k}$. Thus, inside $T \cap B(x, \varepsilon)$ the first-approach to x contains only points that f maps to 0, completing our proof. $\square$

7. Summary

In conclusion, all three of the classes considered here have some right to be called the two-dimensional analog of first-return continuous functions of one variable, i.e, the Baire one, Darboux functions. But because the radially first-return approachable functions preserve the connectivity of line segments (whereas gradients do not) and have continuity properties stronger than Baire one, Darboux functions on the line, perhaps one of the other two classes is more "natural." At any rate, it would now be interesting to completely characterize these three classes, or at least characterize their sets of continuity points.

REFERENCES

[1] BLUMBERG, H.: *Exceptional sets*, Fund. Math. **32** (1939), 3–32.

[2] DARJI, U. B.—EVANS, M. J.: *Recovering Baire 1 functions*, Mathematika **42** (1995), 43–48.

[3] DARJI, U. B.—EVANS, M. J.—HUMKE, P. D.: *First return approachability*, J. Math. Anal. Appl. **199** (1996), 545–557.

[4] DARJI, U. B.—EVANS, M. J.—O'MALLEY, R. J.: *A first-return characterization of Baire 1 functions*, Real Anal. Exchange **19** (1993–1994), 510–515.

[5] DARJI, U. B.—EVANS, M. J.—O'MALLEY, R. J.: *First return path systems: Differentiability, continuity, and orderings*, Acta Math. Hungar. **66** (1995), 83–103.

[6] EVANS, M. J.—O'MALLEY, R. J.: *First-return limiting notions in real analysis*, Real Anal. Exchange **29** (2003–2004), 503–529.

[7] MALÝ, J.: *The Darboux property for gradients*, Real Anal. Exchange **22** (1996–1997), 167–173.

[8] SHELTON, J. D., Jr.: *First-return limits for functions of more than one variable*, Honors Thesis in Mathematics, Washington and Lee University, 2005.

[9] SHELTON, J. D., Jr.,: *First-return limits for functions of several variables*, Real Anal. Exchange **31** (2005-2006), 489–514.

[10] YOUNG, W. H.: *La symétrie de structure des fonctions de variables réeles*, Bull. Sci. Math. **52** (1928), 264–280.

[11] ZAHORSKI, Z.: *Sur la première dérivée*, Trans. Amer. Math. Soc. **69** (1950), 1–54.

Received October 10, 2005

Michael J. Evans
Paul D. Humke
Department of Mathematics
Washington and Lee University
204 Washington street
Lexington, VA 24450
U.S.A.
E-mail: evansm@wlu.edu

Paul D. Humke
Department of Mathematics
St. Olaf College
1520 St. Olaf Avenue
Northfield, MN 55057
U.S.A.
E-mail: humke@stolaf.edu