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PERTURBATION OF DISCRETE QUADRATIC FUNCTIONALS

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ABSTRACT. The nonnegativity of a discrete quadratic functional $\mathcal{F}(x,u)$, associated with discrete symplectic systems, with zero, separated, and general endpoints, respectively, on admissible pairs (x,u) with corresponding boundary conditions is equivalent to the nonnegativity of certain perturbed functionals, e.g., of the functional $\mathcal{F}(x,u)+\alpha\|x_0\|^2+\beta\|x_{N+1}\|^2$ on admissible pairs (x,u) such that $\left(\begin{smallmatrix} x_0 \\ x_{N+1}\end{smallmatrix}\right)$ is restricted to a (larger) subspace. Similar results hold also in case of the positivity.

1. Notation and introduction

Let $N \in \mathbb{N}$ and let $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^n$ be real n-vectors for $k \in \{0,1,\dots,N+1\} := [0,N+1]$, and $\mathcal{S}_k = \left(\begin{smallmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k\end{smallmatrix}\right)$ be a symplectic $2n\times 2n$ matrix for $k\in[0,N]$, i.e.,

$$\mathcal{A}_k^T\mathcal{C}_k=\mathcal{C}_k^T\mathcal{A}_k,\quad \mathcal{B}_k^T\mathcal{D}_k=\mathcal{D}_k^T\mathcal{B}_k,\quad \mathcal{A}_k^T\mathcal{D}_k-\mathcal{C}_k^T\mathcal{B}_k=I,\quad k\in[0,N]. \tag{1}$$

The system

$$x_{k+1}=\mathcal{A}_kx_k+\mathcal{B}_ku_k,\quad u_{k+1}=\mathcal{C}_kx_k+\mathcal{D}_ku_k,\quad k\in[0,N], \tag{S}$$

is called a *discrete symplectic system*.

Vector solutions of (S) are pairs of sequences $(x,u):=\big(\{x_k\}_{k=0}^{N+1},\{u_k\}_{k=0}^{N+1}\big)$, and matrix solutions of (S) are pairs of $n\times n$ matrix sequences $(X,U):=\big(\{X_k\}_{k=0}^{N+1},\{U_k\}_{k=0}^{N+1}\big)$. Further, throughout the paper we denote by $(\hat X,\hat U)$ the matrix solution of (S) with $(\hat X_0,\hat U_0)=(0,I)$ (called the *principal solution* at

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$k=0$), and by $(\tilde{X},\tilde{U})$ the solution of (S) with $(\tilde{X}_0,\tilde{U}_0)=(I,0)$ (its *associated solution*).

Solutions $(\tilde{X},\tilde{U})$ and $(\hat{X},\hat{U})$ are *normalized conjoined bases*, that is,

$$\tilde{X}_k^T\tilde{U}_k=\tilde{U}_k^T\tilde{X}_k,\quad \hat{X}_k^T\hat{U}_k=\hat{U}_k^T\hat{X}_k,\quad \tilde{X}_k^T\hat{U}_k-\tilde{U}_k^T\hat{X}_k=I,\quad k\in[0,N+1]. \tag{2}$$

A pair (x,u) satisfying

$$x_{k+1}=\mathcal{A}_kx_k+\mathcal{B}_ku_k \quad \text{ for } k\in[0,N], \tag{3}$$

is called *admissible*.

The *discrete quadratic functional* on admissible pair (x,u) is defined by

$$\mathcal{F}_0(x,u):=\sum_{k=0}^{N}\Big\{x_k^T\mathcal{A}_k^T\mathcal{C}_kx_k+2\,x_k^T\mathcal{C}_k^T\mathcal{B}_ku_k+u_k^T\mathcal{B}_k^T\mathcal{D}_ku_k\Big\}. \tag{4}$$

This functional corresponds to the second variation in discrete calculus of variations and control problems, and definiteness of such functionals is studied in [3,4,10] (positivity), [5,7] (nonnegativity), and also in many other publications.

In this work, we complete characterization of the definiteness of discrete quadratic functionals via perturbations. We show that the nonnegativity (or positivity) of the functionals on admissible pairs with given boundary conditions is equivalent to the nonnegativity (or positivity) of certain perturbed functionals on a larger set of admissible pairs, which is again defined by a restriction on endpoints.

Condition of this type appeared for the first time as a part of the proof of [2, Theorem 4], (here (iii') in Theorem 2). A perturbation condition for the case of nonnegativity is contained in [8], (here (v) in Theorem 1). Time scales versions of conditions corresponding to (v), (vi) in Theorem 1, and (v'), (vi') in Theorem 2 are proven in [9].

We show that all these previously known conditions are special cases of a more general one, where in addition the final endpoint x_{N+1} appears in the perturbation. Further, in subsections 3.1, 3.2 we derive perturbation conditions for the functional with separated endpoints and the functional with general endpoints.

Some of these new conditions can be immediately extended to time scales. A paper summarizing results about perturbation of time scale quadratic functionals is in preparation.

2. Auxiliary identities

In this section we present identities describing the relation between the value of a functional $\mathcal{F}_0$ on a pair (x, u) and the value of the same functional $\mathcal{F}_0$ on another pair $(\bar{x}, \bar{u})$ with $\bar{x}_0 = \bar{x}_{N+1} = 0$.

By using the last identity in (1), we can write $\mathcal{F}_0(x, u)$ in the equivalent form

$$\mathcal{F}_0(x,u) = \sum_{k=0}^{N} \Big\{ (\mathcal{A}_k x_k + \mathcal{B}_k u_k)^T (\mathcal{C}_k x_k + \mathcal{D}_k u_k) - x_k^T u_k \Big\}. \tag{5}$$

Lemma 1. *If (x, u) is a solution of* (S) *and $(\bar{x}, \bar{u})$ is admissible, then*

$$\mathcal{F}_0(x,u) = x_{N+1}^T u_{N+1} - x_0^T u_0 =: x_k^T u_k \big|_0^{N+1}, \tag{6}$$

$$\mathcal{F}_0(x+\bar{x}, u+\bar{u}) = x_k^T u_k \big|_0^{N+1} + 2\, \bar{x}_k^T u_k \big|_0^{N+1} + \mathcal{F}_0(\bar{x}, \bar{u}). \tag{7}$$

P r o o f. Identity (6) can be directly seen from (5). For (7), we have

$$\mathcal{F}_0(x+\bar{x}, u+\bar{u}) = \mathcal{F}_0(x,u) + \mathcal{F}_0(x,\bar{u}) + \mathcal{F}_0(\bar{x},u) + \mathcal{F}_0(\bar{x},\bar{u}).$$

Now the two middle terms are equal, because

$$\begin{aligned}(\mathcal{A}_k \bar{x}_k + \mathcal{B}_k \bar{u}_k)^T &(\mathcal{C}_k x_k + \mathcal{D}_k u_k) - \bar{x}_k^T u_k \\ &= (\mathcal{A}_k x_k + \mathcal{B}_k u_k)^T (\mathcal{C}_k \bar{x}_k + \mathcal{D}_k \bar{u}_k) - x_k^T \bar{u}_k,\end{aligned}$$

and together with the first term they can be simplified as in (6). □

Lemma 2. *Let (x, u) be admissible with*

$$x_{N+1} - \tilde{X}_{N+1} x_0 \in \operatorname{Im} \hat{X}_{N+1}. \tag{8}$$

Then the pair $(\bar{x}, \bar{u})$, defined by

$$\begin{gathered}\bar{x}_k := x_k - \tilde{X}_k x_0 - \hat{X}_k c, \quad \bar{u}_k := u_k - \tilde{U}_k x_0 - \hat{U}_k c, \\ \text{where } c := \hat{X}_{N+1}^{\dagger} (x_{N+1} - \tilde{X}_{N+1} x_0),\end{gathered} \tag{9}$$

is admissible and $\bar{x}_0 = 0 = \bar{x}_{N+1}$.

P r o o f. The admissibility of $(\bar{x}, \bar{u})$ follows from the fact that it is a sum of the admissible pairs (x, u), $(-\tilde{X} x_0, -\tilde{U} x_0)$, and $(-\hat{X} c, -\hat{U} c)$. Furthermore, $\bar{x}_0 = x_0 - \tilde{X}_0 x_0 - \hat{X}_0 c = 0$ and $\bar{x}_{N+1} = x_{N+1} - \tilde{X}_{N+1} x_0 - \hat{X}_{N+1} c = 0$, by (8) and properties of Moore-Penrose generalized inverse (see [1]). □

Lemma 3. *Let (x,u) be admissible and such that (8) holds. Then*

$$\mathcal{F}_0(x,u) = \begin{pmatrix} x_0 \\ x_{N+1} \end{pmatrix}^T Q^* \begin{pmatrix} x_0 \\ x_{N+1} \end{pmatrix} + \mathcal{F}_0(\bar{x},\bar{u}), \tag{10}$$

where $(\bar{x},\bar{u})$ is defined by (9) and Q^* is the symmetric $2n \times 2n$ matrix defined by

$$Q^* := \begin{pmatrix} \tilde{X}^T_{N+1} Q \tilde{X}_{N+1} - \tilde{X}^T_{N+1}\tilde{U}_{N+1} & \tilde{U}^T_{N+1} - \tilde{X}^T_{N+1} Q \\ \tilde{U}_{N+1} - Q\tilde{X}_{N+1} & Q \end{pmatrix}, \tag{11}$$

$$\text{with} \quad Q := \hat{X}_{N+1}\hat{X}^\dagger_{N+1}\hat{U}_{N+1}\hat{X}^\dagger_{N+1}. \tag{12}$$

P r o o f. From identity (7) we have

$$\begin{aligned} \mathcal{F}_0(x,u) = &(\tilde{X}x_0 + \hat{X}c)^T(\tilde{U}_k x_0 + \hat{U}_k c)\big|_0^{N+1} \\ &+ 2\,\bar{x}_k^T(\tilde{U}_k x_0 + \hat{U}_k c)\big|_0^{N+1} + \mathcal{F}_0(\bar{x},\bar{u}). \end{aligned}$$

This is equal to

$$\begin{aligned} \mathcal{F}_0(x,u) &= (\tilde{X}_{N+1}x_0 + \hat{X}_{N+1}c)^T(\tilde{U}_{N+1}x_0 + \hat{U}_{N+1}c) - x_0^T c + \mathcal{F}_0(\bar{x},\bar{u}) \\ &= x_0^T\tilde{X}^T_{N+1}\tilde{U}_{N+1}x_0 + 2\,x_0^T\tilde{U}^T_{N+1}\hat{X}_{N+1}c + c^T\hat{X}^T_{N+1}\hat{U}_{N+1}c + \mathcal{F}_0(\bar{x},\bar{u}) \\ &= \begin{pmatrix} c \\ x_0 \end{pmatrix}^T \begin{pmatrix} \hat{X}^T_{N+1}\hat{U}_{N+1} & \hat{X}^T_{N+1}\tilde{U}_{N+1} \\ \tilde{U}^T_{N+1}\hat{X}_{N+1} & \tilde{X}^T_{N+1}\tilde{U}_{N+1} \end{pmatrix} \begin{pmatrix} c \\ x_0 \end{pmatrix} + \mathcal{F}_0(\bar{x},\bar{u}). \end{aligned} \tag{13}$$

Now, since

$$\begin{pmatrix} c \\ x_0 \end{pmatrix} = \begin{pmatrix} -\hat{X}^\dagger_{N+1}\tilde{X}_{N+1} & \hat{X}^\dagger_{N+1} \\ I & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_{N+1} \end{pmatrix},$$

we have

$$\begin{pmatrix} c \\ x_0 \end{pmatrix}^T \begin{pmatrix} \hat{X}^T_{N+1}\hat{U}_{N+1} & \hat{X}^T_{N+1}\tilde{U}_{N+1} \\ \tilde{U}^T_{N+1}\hat{X}_{N+1} & \tilde{X}^T_{N+1}\tilde{U}_{N+1} \end{pmatrix} \begin{pmatrix} c \\ x_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ x_{N+1} \end{pmatrix}^T Q^* \begin{pmatrix} x_0 \\ x_{N+1} \end{pmatrix},$$

where Q^* is the $2n \times 2n$ matrix defined in (11). Symmetry of Q^* follows from identities in (2). □

Remark 1. There is another proof of this lemma (see [11]), which uses augmented symplectic system (S*), defined here in Subsection 3.2. Moreover, it shows that identity (10) holds for any symmetric matrix Q^* with $\hat{X}^{*T}_{N+1}Q^*\hat{X}^*_{N+1} = \hat{X}^{*T}_{N+1}\hat{U}^*_{N+1}$, where $(\hat{X}^*,\hat{U}^*)$ is a solution of (S*), defined later in (21).

Lemma 4. *Let (x,u) be admissible with $x_{N+1} - \tilde{X}_{N+1}x_0 = 0$. Then*

$$\mathcal{F}_0(x,u) = x_0^T\tilde{X}^T_{N+1}\tilde{U}_{N+1}x_0 + \mathcal{F}_0(\bar{x},\bar{u}), \tag{14}$$

where $(\bar{x}, \bar{u})$ *is defined by*

$$\bar{x}_k := x_k - \tilde{X}_k x_0, \quad \bar{u}_k := u_k - \tilde{U}_k x_0. \tag{15}$$

P r o o f. It follows from (13), because $c = 0$ in this case. □

Lemma 5. *Let* (x, u) *be admissible and let* $\operatorname{Ker} \hat{X}_{k+1} \subseteq \operatorname{Ker} \hat{X}_k$ *for all* $k \in [0, N]$. *Then identity* (10) *from Lemma 3 holds.*

P r o o f. We use the fact that the condition $\operatorname{Ker} \hat{X}_{k+1} \subseteq \operatorname{Ker} \hat{X}_k$ for all $k \in [0, N]$ implies $x_k - \tilde{X}_k x_0 \in \operatorname{Im} \hat{X}_k$ for all $k \in [0, N+1]$. (For details, see, e.g., [11, page 37].) □

3. Main results

In this section we present theorems with equivalent conditions for the nonnegativity and for the positivity of discrete quadratic functionals. By $\mathcal{F}(x, u) \geq 0$ (or > 0) we mean, that the functional $\mathcal{F}$ takes nonnegative (or positive) values on all admissible pairs (x, u) satisfying the given boundary conditions.

Theorem 1. *The following statements are equivalent.*

(i) $\mathcal{F}_0(x, u) \geq 0$ *over* $x_0 = 0 = x_{N+1}$.

(ii) *There exist* $\alpha > 0$ *and* $\beta > 0$ *such that*

$$\mathcal{F}_0(x, u) + \alpha \|x_0\|^2 + \beta \|x_{N+1}\|^2 \geq 0 \quad \textit{over} \quad x_{N+1} - \tilde{X}_{N+1} x_0 \in \operatorname{Im} \hat{X}_{N+1}.$$

(iii) *There exists* $\alpha > 0$ *such that*

$$\mathcal{F}_0(x, u) + \alpha \|x_0\|^2 \geq 0 \quad \textit{over} \quad \tilde{X}_{N+1} x_0 \in \operatorname{Im} \hat{X}_{N+1}, \quad x_{N+1} = 0.$$

(iv) *There exists* $\beta > 0$ *such that*

$$\mathcal{F}_0(x, u) + \beta \|x_{N+1}\|^2 \geq 0 \quad \textit{over} \quad x_{N+1} \in \operatorname{Im} \hat{X}_{N+1}, \quad x_0 = 0.$$

(v) *There exists* $\alpha > 0$ *such that*

$$\mathcal{F}_0(x, u) + \alpha \|x_0\|^2 \geq 0 \quad \textit{over} \quad \tilde{X}_{N+1} x_0 = x_{N+1}.$$

(vi) $\mathcal{F}_0(x, u) \geq 0$ *over* $\tilde{X}_{N+1} x_0 = x_{N+1} = 0$.

Theorem 2. *The following statements are equivalent.*

(i') $\mathcal{F}_0(x, u) > 0$ *over* $x_0 = 0 = x_{N+1}, \quad x \not\equiv 0$.

(ii') *There exist* $\alpha > 0$ *and* $\beta > 0$ *such that*

$$\mathcal{F}_0(x, u) + \alpha \|x_0\|^2 + \beta \|x_{N+1}\|^2 > 0 \quad \textit{over} \quad x \not\equiv 0.$$

(iii') *There exists* $\alpha > 0$ *such that*

$$\mathcal{F}_0(x,u) + \alpha \|x_0\|^2 > 0 \quad \textit{over} \quad x_{N+1} = 0, \quad x \not\equiv 0.$$

(iv') *There exists* $\beta > 0$ *such that*

$$\mathcal{F}_0(x,u) + \beta \|x_{N+1}\|^2 > 0 \quad \textit{over} \quad x_0 = 0, \quad x \not\equiv 0.$$

(v') *There exists* $\alpha > 0$ *such that*

$$\mathcal{F}_0(x,u) + \alpha \|x_0\|^2 > 0 \quad \textit{over} \quad \tilde{X}_{N+1} x_0 = x_{N+1}, \quad x \not\equiv 0.$$

(vi') $\mathcal{F}_0(x,u) > 0$ *over* $\tilde{X}_{N+1} x_0 = x_{N+1} = 0, \quad x \not\equiv \tilde{X} x_0$.

Remark 2. In Theorem 1, (ii), the restriction for endpoints x_0 and x_{N+1} is defined by the condition $x_{N+1} - \tilde{X}_{N+1} x_0 \in \operatorname{Im} \hat{X}_{N+1}$. We see that in comparison with the previously known results, where the difference $x_{N+1} - \tilde{X}_{N+1} x_0$ had to be zero, it suffices in fact to be in the image of the matrix $\hat{X}_{N+1}$.

This restriction, that does not appear in conditions for the positivity, is necessary in case of the nonnegativity. We refer to [8] or [11] for an illustrating example.

Proof of Theorem 1. Conditions (ii)–(vi) imply condition (i) trivially, and condition (ii) implies (iii) and (iv). It remains to prove that (i) implies (ii), (v) and (vi).

(i) $\Rightarrow$ (ii): Let (x,u) be an admissible pair with $x_{N+1} - \tilde{X}_{N+1} x_0 \in \operatorname{Im} \hat{X}_{N+1}$. Then, by Lemma 3, we have that identity (10) holds, where $(\bar{x}, \bar{u})$ is an admissible pair with $\bar{x}_0 = \bar{x}_{N+1} = 0$ and Q^* is a symmetric $2n \times 2n$ matrix. Let λ_0 be the smallest eigenvalue of Q^*. Then from (10) and (i) we get

$$\mathcal{F}_0(x,u) \geq \lambda_0 \big(\|x_0\|^2 + \|x_{N+1}\|^2 \big) \geq -|\lambda_0| \big(\|x_0\|^2 + \|x_{N+1}\|^2 \big).$$

Hence, the inequality in (ii) holds for any $\alpha, \beta \geq |\lambda_0|$.

(i) $\Rightarrow$ (v): If $\tilde{X}_{N+1} x_0 = x_{N+1}$, then, by Lemma 4, we get

$$\mathcal{F}_0(x,u) \geq x_0^T \tilde{X}_{N+1}^T \tilde{U}_{N+1} x_0.$$

The inequality in (v) holds for any $\alpha \geq |\lambda_1|$, where λ_1 is the smallest eigenvalue of $\tilde{X}_{N+1}^T \tilde{U}_{N+1}$.

(i) $\Rightarrow$ (vi): As in the previous case, we have $\mathcal{F}_0(x,u) \geq x_0^T \tilde{X}_{N+1}^T \tilde{U}_{N+1} x_0$, and $\tilde{X}_{N+1} x_0 = 0$ further implies $\mathcal{F}_0(x,u) \geq 0$. □

Proof of Theorem 2. Again, conditions (ii')–(vi') imply condition (i') trivially, and condition (ii') implies (iii') and (iv'). It remains to prove that (i') implies (ii'), (v') and (vi'). Conditions (i)–(vi) refer in this proof to Theorem 1.

(i') $\Rightarrow$ (ii'): From [4, Theorem 1] we have that $\operatorname{Ker} \hat{X}_{k+1} \subseteq \operatorname{Ker} \hat{X}_k$ for all $k \in [0, N]$ and hence, by Lemma 5, identity (10) holds for all admissible (x, u). Thus, similarly as in the proof of Theorem 1, we get that there exist $\bar{\alpha} > 0$ and $\bar{\beta} > 0$ such that $\mathcal{F}_0(x, u) + \bar{\alpha}\, \|x_0\|^2 + \bar{\beta}\, \|x_{N+1}\|^2 \geq 0$ for all admissible (x, u). Now, for $\alpha := \bar{\alpha} + 1$ and $\beta := \bar{\beta} + 1$ we have $\mathcal{F}_0(x, u) + \alpha\, \|x_0\|^2 + \beta\, \|x_{N+1}\|^2 \geq \|x_0\|^2 + \|x_{N+1}\|^2$, and $\|x_0\|^2 + \|x_{N+1}\|^2 > 0$, except when $x_0 = 0 = x_{N+1}$. But if $x_0 = 0 = x_{N+1}$, then, since $x \not\equiv 0$, condition (i') directly implies that the inequality in (ii') holds for such (x, u) with any α, β.

(i') $\Rightarrow$ (v'): We use (i') $\Rightarrow$ (i) $\Rightarrow$ (v) and get that there exists $\bar{\alpha} > 0$ such that $\mathcal{F}_0(x, u) + \bar{\alpha}\, \|x_0\|^2 \geq 0$ over $\tilde{X}_{N+1} x_0 = x_{N+1}$. As in the previous case, for $\alpha := \bar{\alpha} + 1$ we get $\mathcal{F}_0(x, u) + \alpha \|x_0\|^2 \geq \|x_0\|^2$, and as $\tilde{X}_{N+1} x_0 = x_{N+1}$, $\|x_0\|^2 > 0$, except when $x_0 = 0 = x_{N+1}$. But if $x_0 = 0 = x_{N+1}$, then, since $x \not\equiv 0$, condition (i') directly implies that the inequality in (ii') holds for such (x, u) with any α.

(i') $\Rightarrow$ (vi'): Define $(\bar{x}, \bar{u})$ as in (15). Then $\bar{x} \not\equiv 0$, because $x \not\equiv \tilde{X} x_0$, and from (14) we get $\mathcal{F}_0(x, u) = \mathcal{F}_0(\bar{x}, \bar{u}) > 0$. $\square$

Remark 3. There are other possible proofs of Theorem 2, using a generalized Picone-type identity, or results for the positivity of functional with general endpoints.

3.1. Functional with separated endpoints

In this subsection, we consider the quadratic functional

$$\mathcal{F}(x, u) := x_0^T \Gamma_0 x_0 + x_{N+1}^T \Gamma_1 x_{N+1} + \mathcal{F}_0(x, u),$$

with the boundary conditions

$$\mathcal{M}_0\, x_0 = 0, \mathcal{M}_1 x_{N+1} = 0, \tag{16}$$

where $\mathcal{M}_0, \mathcal{M}_1$ are real $n \times n$ projections and Γ_0, Γ_1 are symmetric $n \times n$ matrices satisfying $\Gamma_i = (I - \mathcal{M}_i)\, \Gamma_i\, (I - \mathcal{M}_i)$, $i = 0, 1$.

Let (X, U) be the matrix solution of (S) with $X_0 = I - \mathcal{M}_0$ and $U_0 = \Gamma_0 + \mathcal{M}_0$, (called the *natural conjoined basis*) and let $(\bar{X}, \bar{U})$ be the solution with $\bar{X}_0 = \mathcal{M}_0$ and $\bar{U}_0 = \mathcal{M}_0 - I$.

Theorem 3. *The following statements are equivalent.*

(i) $\mathcal{F}(x, u) \geq 0$ *over* $\mathcal{M}_0 x_0 = 0 = \mathcal{M}_1 x_{N+1}$.

(ii) *There exist* $\alpha > 0$ *and* $\beta > 0$ *such that*

$$\begin{aligned}\mathcal{F}(x,u) + \alpha\,\|\mathcal{M}_0 x_0\|^2 + \beta\,\|\mathcal{M}_1 x_{N+1}\|^2 \ge 0 \\ \textit{over}\quad \mathcal{M}_1 x_{N+1} - \big[(\Gamma_1 + \mathcal{M}_1)\bar{X}_{N+1} + (I - \mathcal{M}_1)\bar{U}_{N+1}\big]\mathcal{M}_0 x_0 \\ \in \operatorname{Im}\big[(\Gamma_1 + \mathcal{M}_1)X_{N+1} + (I - \mathcal{M}_1)U_{N+1}\big].\end{aligned}$$

(iii) *There exists* $\alpha > 0$ *such that*

$$\begin{aligned}\mathcal{F}(x,u) + \alpha\,\|\mathcal{M}_0 x_0\|^2 \ge 0 \\ \textit{over}\quad \big[(\Gamma_1 + \mathcal{M}_1)\bar{X}_{N+1} + (I - \mathcal{M}_1)\bar{U}_{N+1}\big]\mathcal{M}_0 x_0 \\ \in \operatorname{Im}\big[(\Gamma_1 + \mathcal{M}_1)X_{N+1} + (I - \mathcal{M}_1)U_{N+1}\big], \\ \mathcal{M}_1 x_{N+1} = 0.\end{aligned}$$

(iv) *There exists* $\beta > 0$ *such that*

$$\begin{aligned}\mathcal{F}(x,u) + \beta\,\|\mathcal{M}_1 x_{N+1}\|^2 \ge 0 \\ \textit{over}\quad \mathcal{M}_1 x_{N+1} \in \operatorname{Im}\big[(\Gamma_1 + \mathcal{M}_1)X_{N+1} + (I - \mathcal{M}_1)U_{N+1}\big], \\ \mathcal{M}_0 x_0 = 0.\end{aligned}$$

(v) *There exists* $\alpha > 0$ *such that*

$$\begin{aligned}\mathcal{F}(x,u) + \alpha\,\|\mathcal{M}_0 x_0\|^2 \ge 0 \\ \textit{over}\quad \mathcal{M}_1 x_{N+1} = \big[(\Gamma_1 + \mathcal{M}_1)\bar{X}_{N+1} + (I - \mathcal{M}_1)\bar{U}_{N+1}\big]\mathcal{M}_0 x_0.\end{aligned}$$

(vi) $\mathcal{F}(x,u) \ge 0$
over $\mathcal{M}_1 x_{N+1} = \big[(\Gamma_1 + \mathcal{M}_1)\bar{X}_{N+1} + (I - \mathcal{M}_1)\bar{U}_{N+1}\big]\mathcal{M}_0 x_0 = 0.$

Theorem 4. *The following statements are equivalent.*

(i') $\mathcal{F}(x,u) > 0$ *over* $\mathcal{M}_0 x_0 = 0 = \mathcal{M}_1 x_{N+1}$, $x \not\equiv 0$.

(ii') *There exist* $\alpha > 0$ *and* $\beta > 0$ *such that*

$$\mathcal{F}(x,u) + \alpha\,\|\mathcal{M}_0 x_0\|^2 + \beta\,\|\mathcal{M}_1 x_{N+1}\|^2 > 0 \quad \textit{over} \quad x \not\equiv 0.$$

(iii') *There exists* $\alpha > 0$ *such that*

$$\mathcal{F}(x,u) + \alpha\,\|\mathcal{M}_0 x_0\|^2 > 0 \quad \textit{over} \quad \mathcal{M}_1 x_{N+1} = 0, \quad x \not\equiv 0.$$

(iv') *There exists* $\beta > 0$ *such that*

$$\mathcal{F}(x,u) + \beta\,\|\mathcal{M}_1 x_{N+1}\|^2 > 0 \quad \textit{over} \quad \mathcal{M}_0 x_0 = 0, \quad x \not\equiv 0.$$

(v') *There exists* $\alpha > 0$ *such that*

$$\mathcal{F}(x,u) + \alpha\, \|\mathcal{M}_0 x_0\|^2 > 0$$
$$\text{over} \quad \mathcal{M}_1 x_{N+1} = \big[(\Gamma_1 + \mathcal{M}_1)\bar{X}_{N+1} + (I - \mathcal{M}_1)\bar{U}_{N+1}\big]\mathcal{M}_0 x_0, \qquad x \not\equiv 0.$$

(vi') $\mathcal{F}(x,u) > 0$

$$\text{over} \quad \mathcal{M}_1 x_{N+1} = \big[(\Gamma_1 + \mathcal{M}_1)\bar{X}_{N+1} + (I - \mathcal{M}_1)\bar{U}_{N+1}\big]\mathcal{M}_0 x_0 = 0, \qquad x \not\equiv \bar{X}\mathcal{M}_0 x_0.$$

We prove both theorems with the use of a transformed symplectic system

$$X_{k+1} = \tilde{\mathcal{A}}_k X_k + \tilde{\mathcal{B}}_k U_k, \quad U_{k+1} = \tilde{\mathcal{C}}_k X_k + \tilde{\mathcal{D}}_k U_k, \quad k \in [-1, N+1], \tag{$\tilde{\text{S}}$}$$

with $n \times n$ matrices $\tilde{\mathcal{A}}_k, \tilde{\mathcal{B}}_k, \tilde{\mathcal{C}}_k, \tilde{\mathcal{D}}_k$ defined on $[-1, N+1]$ by

$$\tilde{\mathcal{A}}_k := \begin{cases} \mathcal{M}_0, & \text{for } k = -1, \\ \mathcal{A}_k, & \text{for } k \in [0, N], \\ I, & \text{for } k = N+1, \end{cases}$$

$$\tilde{\mathcal{B}}_k := \begin{cases} I - \mathcal{M}_0, & \text{for } k = -1, \\ \mathcal{B}_k, & \text{for } k \in [0, N], \\ I - \mathcal{M}_1, & \text{for } k = N+1, \end{cases}$$

$$\tilde{\mathcal{C}}_k := \begin{cases} \mathcal{M}_0 - I, & \text{for } k = -1, \\ \mathcal{C}_k, & \text{for } k \in [0, N-1], \\ \mathcal{C}_N + \big[\Gamma_1 - (I - \mathcal{M}_1)\big]\mathcal{A}_N, & \text{for } k = N, \\ 0, & \text{for } k = N+1, \end{cases} \tag{17}$$

$$\tilde{\mathcal{D}}_k := \begin{cases} \Gamma_0 + \mathcal{M}_0, & \text{for } k = -1, \\ \mathcal{D}_k, & \text{for } k \in [0, N-1], \\ \mathcal{D}_N + \big[\Gamma_1 - (I - \mathcal{M}_1)\big]\mathcal{B}_N, & \text{for } k = N, \\ I, & \text{for } k = N+1. \end{cases}$$

In this transformed system, boundary conditions (16) at $k = 0$, $k = N+1$ are converted to zero boundary conditions at $k = -1$, $k = N+2$, and the corresponding functionals have equivalent definiteness. This is the contents of the following lemmas, stated here without proofs. The transformation is used for discrete Hamiltonian system in [6] and for discrete symplectic system in [10], only with different coefficients $\tilde{\mathcal{A}}_{-1}$ and $\tilde{\mathcal{C}}_{-1}$. (In symplectic case we do not need $\tilde{\mathcal{A}}_{-1}$ to be invertible.) Proofs for transformation with coefficients defined in (17) are included in [11].

Lemma 6. *Let (x,u) be an admissible pair w.r.t. $(\mathcal{A},\mathcal{B})$ on $[0,N+1]$. Then the pair $(\tilde{x},\tilde{u})$ defined by*

$$(\tilde{x},\tilde{u}) := \begin{cases} (\mathcal{M}_0 x_0, x_0), & \text{for } k=-1,\\ (x_k,u_k), & \text{for } k\in[0,N],\\ (x_{N+1}, -x_{N+1}), & \text{for } k=N+1,\\ (\mathcal{M}_1 x_{N+1}, \gamma), & \text{for } k=N+2, \end{cases} \tag{18}$$

where $\gamma\in\mathbb{R}^n$ is arbitrary, is admissible w.r.t. $(\tilde{\mathcal{A}},\tilde{\mathcal{B}})$ on $[-1,N+2]$ and

$$\mathcal{F}(x,u) = \tilde{\mathcal{F}}_0(\tilde{x},\tilde{u}) := \sum_{k=-1}^{N+1} \Big\{ \tilde{x}_k^T \tilde{\mathcal{A}}_k^T \tilde{\mathcal{C}}_k \tilde{x}_k + 2\,\tilde{x}_k^T \tilde{\mathcal{C}}_k^T \tilde{\mathcal{B}}_k \tilde{u}_k + \tilde{u}_k^T \tilde{\mathcal{B}}_k^T \tilde{\mathcal{D}}_k \tilde{u}_k \Big\}. \tag{19}$$

Lemma 7. *The functional $\mathcal{F}$ is nonnegative over $\mathcal{M}_0x_0 = 0 = \mathcal{M}_1 x_{N+1}$ if and only if the functional $\tilde{\mathcal{F}}_0$ defined in equation* (19) *is nonnegative over $\tilde{x}_{-1} = 0 = \tilde{x}_{N+2}$.*

Lemma 8. *The functional $\mathcal{F}$ is positive over $\mathcal{M}_0x_0 = 0 = \mathcal{M}_1 x_{N+1}$ and $x \not\equiv 0$ if and only if the functional $\tilde{\mathcal{F}}_0$ defined in equation* (19) *is positive over $\tilde{x}_{-1} = 0 = \tilde{x}_{N+2}$ and $\tilde{x}\not\equiv 0$.*

Proof of Theorem 3. Conditions (ii)–(vi) imply condition (i) trivially.

Note that if (X,U) and $(\bar{X},\bar{U})$ are the matrix solutions of $(\tilde{\mathrm{S}})$ with $X_{-1}=0$, $U_{-1}=I$, and $\bar{X}_{-1}=I$, $\bar{U}_{-1}=0$ then $X_0 = I-\mathcal{M}_0$, $U_0 = \Gamma_0+\mathcal{M}_0$, $\bar{X}_0 = \mathcal{M}_0$, $\bar{U}_0 = \mathcal{M}_0 - I$, $X_{N+2} = (\Gamma_1+\mathcal{M}_1)X_{N+1} + (I-\mathcal{M}_1)U_{N+1}$, and $\bar{X}_{N+2} = (\Gamma_1+\mathcal{M}_1)\bar{X}_{N+1} + (I-\mathcal{M}_1)\bar{U}_{N+1}$.

Let condition (i) hold. Then, by Lemma 7, $\tilde{\mathcal{F}}_0(\tilde{x},\tilde{u}) \ge 0$ over $\tilde{x}_{-1} = 0 = \tilde{x}_{N+2}$, and hence conditions (ii)–(vi) from Theorem 1 are satisfied, where we replace the interval $[0,N+1]$ by $[-1,N+2]$, the pair (x,u) by $(\tilde{x},\tilde{u})$, the functional $\mathcal{F}_0(x,u)$ by $\tilde{\mathcal{F}}_0(\tilde{x},\tilde{u})$, and the solutions $(\hat{X},\hat{U})$ and $(\tilde{X},\tilde{U})$ by (X,U) and $(\bar{X},\bar{U})$. Now we will show that (i) implies (ii) in details. The other implications (i) $\Rightarrow$ (iii)–(vi) are proven in a similar way.

Let (x,u) be an admissible pair w.r.t. $(\mathcal{A},\mathcal{B})$ with

$$\mathcal{M}_1 x_{N+1} - \big[(\Gamma_1+\mathcal{M}_1)\bar{X}_{N+1} + (I-\mathcal{M}_1)\bar{U}_{N+1}\big]\mathcal{M}_0 x_0 \in \operatorname{Im}\big[(\Gamma_1+\mathcal{M}_1)X_{N+1} + (I-\mathcal{M}_1)U_{N+1}\big]. \tag{20}$$

Then the pair $(\tilde{x},\tilde{u})$ defined by (18) is admissible w.r.t. $(\tilde{\mathcal{A}},\tilde{\mathcal{B}})$ on $[-1,N+2]$, $\tilde{x}_{-1} = \mathcal{M}_0x_0$, $\tilde{x}_{N+2} = \mathcal{M}_1 x_{N+1}$, and $\tilde{x}_{N+2} - \bar{X}_{N+2}\tilde{x}_{-1} \in \operatorname{Im} X_{N+2}$. Furthermore, by Lemma 6, $\mathcal{F}(x,u) = \tilde{\mathcal{F}}_0(\tilde{x},\tilde{u})$. Hence, by condition (ii) from Theorem 1, we have

$$\begin{aligned} 0 &\le \tilde{\mathcal{F}}_0(\tilde{x},\tilde{u}) + \alpha\,\|\tilde{x}_{-1}\|^2 + \beta\,\|\tilde{x}_{N+2}\|^2 \\ &= \mathcal{F}(x,u) + \alpha\,\|\mathcal{M}_0 x_0\|^2 + \beta\,\|\mathcal{M}_1 x_{N+1}\|^2, \end{aligned}$$

where α and β are the positive numbers that exist by condition (ii) from Theorem 1. □

Proof of Theorem 4. The proof is same as the proof of Theorem 3, only we use Lemma 8 and Theorem 2 instead of Lemma 7 and Theorem 1. □

3.2. Functional with general endpoints

In this subsection, we consider the quadratic functional

$$\mathcal{F}(x,u) := \begin{pmatrix} x_0 \\ x_{N+1} \end{pmatrix}^T \Gamma \begin{pmatrix} x_0 \\ x_{N+1} \end{pmatrix} + \mathcal{F}_0(x,u)$$

with boundary conditions

$$\mathcal{M}\begin{pmatrix} x_0 \\ x_{N+1} \end{pmatrix} = 0,$$

where $\mathcal{M}$ is real $2n\times 2n$ projection and Γ is symmetric $2n\times 2n$ matrix satisfying $\Gamma = (I-\mathcal{M})\,\Gamma\,(I-\mathcal{M})$.

Define the $2n\times 2n$ matrices

$$\hat{X}_k^* := \begin{pmatrix} 0 & I \\ \hat{X}_k & \tilde{X}_k \end{pmatrix}, \quad \hat{U}_k^* := \begin{pmatrix} -I & 0 \\ \hat{U}_k & \tilde{U}_k \end{pmatrix}, \quad k \in [0, N+1]. \tag{21}$$

Theorem 5. *The following statements are equivalent.*

(i) $\mathcal{F}(x,u) \ge 0$ *over* $\mathcal{M}\left(\begin{smallmatrix} x_0 \\ x_{N+1} \end{smallmatrix}\right) = 0$.

(ii) *There exists* $\alpha > 0$ *such that*

$$\begin{aligned} &\mathcal{F}(x,u) + \alpha\,\|\mathcal{M}\left(\begin{smallmatrix} x_0 \\ x_{N+1} \end{smallmatrix}\right)\|^2 \ge 0 \\ &\qquad\qquad \textit{over} \quad \mathcal{M}\left(\begin{smallmatrix} x_0 \\ x_{N+1} \end{smallmatrix}\right) \in \operatorname{Im}\big[(\Gamma+\mathcal{M})\hat{X}_{N+1}^* + (I-\mathcal{M})\hat{U}_{N+1}^*\big]. \end{aligned} \tag{22}$$

Theorem 6. *The following statements are equivalent.*

(i') $\mathcal{F}(x,u) > 0$ *over* $\mathcal{M}\left(\begin{smallmatrix} x_0 \\ x_{N+1} \end{smallmatrix}\right) = 0$, $x \not\equiv 0$.

(ii') *There exists* $\alpha > 0$ *such that*

$$\mathcal{F}(x,u) + \alpha\,\|\mathcal{M}\left(\begin{smallmatrix} x_0 \\ x_{N+1} \end{smallmatrix}\right)\|^2 > 0 \quad \textit{over} \quad x \not\equiv 0.$$

The proofs of Theorems 5, 6 are based on transforming system (S) and the quadratic functional $\mathcal{F}$ into augmented symplectic system (S*) with $2n\times 2n$ matrices defined on $[0,N]$ by $\mathcal{A}_k^* := \left(\begin{smallmatrix} I & 0 \\ 0 & \mathcal{A}_k \end{smallmatrix}\right)$, $\mathcal{B}_k^* := \left(\begin{smallmatrix} 0 & 0 \\ 0 & \mathcal{B}_k \end{smallmatrix}\right)$, $\mathcal{C}_k^* := \left(\begin{smallmatrix} 0 & 0 \\ 0 & \mathcal{C}_k \end{smallmatrix}\right)$, $\mathcal{D}_k^* := \left(\begin{smallmatrix} I & 0 \\ 0 & \mathcal{D}_k \end{smallmatrix}\right)$, and the corresponding functional $\mathcal{F}_0^*$. This transformation is commonly used to prove theorems for functional with general endpoints via theorems for functional with separated endpoints, e.g., in [6,10]. The following holds.

Lemma 9. *Let (x,u) be an admissible pair defined on $[0,N+1]$, $\beta_k \in \mathbb{R}^n$ and*

$$x_k^* := \begin{pmatrix} x_0 \\ x_k \end{pmatrix},\ k \in [0,N+1], \quad u_k^* := \begin{pmatrix} \beta_k \\ u_k \end{pmatrix},\ k \in [0,N].$$

Then

$$\mathcal{F}(x,u) = \mathcal{F}^*(x^*,u^*) := x_0^{*T}\Gamma_0^*\, x_0^* + x_{N+1}^{*T}\Gamma_1^*\, x_{N+1}^* + \mathcal{F}_0^*(x^*,u^*). \tag{23}$$

Further, (a) $\Leftrightarrow$ (b) and (c) $\Leftrightarrow$ (d), where

(a) $\mathcal{F}(x,u) \ge 0$ *over* $\mathcal{M}\left(\begin{smallmatrix} x_0 \\ x_{N+1}\end{smallmatrix}\right) = 0$,

(b) $\mathcal{F}^*(x^*,u^*) \ge 0$ *over* $\mathcal{M}_0^* x_0^* = 0 = \mathcal{M}_1^* x_{N+1}^*$,

(c) $\mathcal{F}(x,u) > 0$ *over* $\mathcal{M}\left(\begin{smallmatrix} x_0 \\ x_{N+1}\end{smallmatrix}\right) = 0, \quad x \not\equiv 0$,

(d) $\mathcal{F}^*(x^*,u^*) > 0$ *over* $\mathcal{M}_0^* x_0^* = 0 = \mathcal{M}_1^* x_{N+1}^*, \quad x^* \not\equiv 0$,

where $\mathcal{M}_0^* := \frac{1}{2}\left(\begin{smallmatrix} I & -I \\ -I & I\end{smallmatrix}\right)$, $\mathcal{M}_1^* := \mathcal{M}$, $\Gamma_0^* := 0$, $\Gamma_1^* := \Gamma$.

Proof of Theorem 5. Condition (ii) implies condition (i) trivially.

Let condition (i) hold. Then $\mathcal{F}^*(x^*,u^*) \ge 0$ over $\mathcal{M}_0^* x_0^* = 0 = \mathcal{M}_1^* x_{N+1}^*$, where $\mathcal{F}^*$ is the functional with separated endpoints defined in (23). Hence, condition (iv) from Theorem 3 holds for (x^*,u^*) and $\mathcal{F}^*(x^*,u^*)$ in place of (x,u) and $\mathcal{F}(x,u)$, and with the natural conjoined basis (X^*,U^*) of (S*) in place of the natural conjoined basis (X,U) of (S). This is equivalent to condition (ii), because $\mathcal{M}_0^* x_0^* = 0$ implies that (x^*,u^*) has the form from Lemma 9. Hence,

$$\begin{aligned} \mathcal{M}_1^* x_{N+1}^* &= \mathcal{M}\left(\begin{smallmatrix} x_0 \\ x_{N+1}\end{smallmatrix}\right), \\ \mathcal{F}^*(x^*,u^*) &= \mathcal{F}(x,u), \end{aligned}$$

and

$$(\hat{X}^*,\hat{U}^*) = (X^*,U^*)\left(\begin{smallmatrix} -I & I \\ I & I\end{smallmatrix}\right),$$

and

$$\begin{aligned} &\operatorname{Im}\big[(\Gamma+\mathcal{M})X_{N+1}^* + (I-\mathcal{M})U_{N+1}^*\big] \\ &\qquad = \operatorname{Im}\big\{\big[(\Gamma+\mathcal{M})X_{N+1}^* + (I-\mathcal{M})U_{N+1}^*\big]\left(\begin{smallmatrix} -I & I \\ I & I\end{smallmatrix}\right)\big\}. \end{aligned}$$

□

Proof of Theorem 6. The proof is the similar as the proof of Theorem 5, we only use Theorem 4 instead of Theorem 3. □

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