Tatra Mt. Math. Publ. **38** (2007), 201–213

Tatra Mountains
Mathematical Publications

COMPARISON THEOREMS FOR THIRD ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

Ivan Mojsej

ABSTRACT. Differential equations are often classified according to oscillatory and nonoscillatory properties of their solutions as equations having property A or property B. The aim of this paper is to present comparison results on properties A and B for the third order nonlinear differential equations with deviating argument. The obtained results are compared with existing ones in the framework of the papers.

1. Introduction

We consider the third order nonlinear differential equations with deviating argument of the form:

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}x'(t)\right)'\right)' + q(t)f\big(x(h(t))\big) = 0, \qquad t \geq 0 \tag{N, h}$$

and

$$\left(\frac{1}{r(t)}\left(\frac{1}{p(t)}z'(t)\right)'\right)' - q(t)f\big(z(h(t))\big) = 0, \qquad t \geq 0 \tag{$\mathrm{N}^{\mathcal{A}}$, h}$$

where

$$r, p, q, h \in C\big([0,\infty), \mathbb{R}\big),\ r(t) > 0,\ p(t) > 0,\ q(t) > 0 \text{ on } [0,\infty), \tag{H1}$$

$$f \in C(\mathbb{R}, \mathbb{R}), \quad f(u)u > 0 \qquad \text{for} \quad u \neq 0, \tag{H2}$$

2000 Mathematics Subject Classification: 34K11.
Keywords: comparison theorem, oscillation theory, nonlinear equation, deviating argument, quasiderivatives.
This research was supported by the grant 1/3005/06 of the Grant Agency of Slovak Republic (VEGA).

$$\int_0^\infty r(t)\,\mathrm{d}t = \int_0^\infty p(t)\,\mathrm{d}t = \infty, \tag{H3}$$

$$\lim_{t\to\infty} h(t) = \infty. \tag{H4}$$

Without mentioning them again, we shall assume the validity of conditions (H1)–(H4) throughout the paper.

The notation $(N^{\mathcal{A}}, h)$ is suggested by the fact that for linear equation without deviating argument, i.e., for the equation

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}x'(t)\right)'\right)' + q(t)x(t) = 0, \tag{L}$$

the *adjoint equation* is

$$\left(\frac{1}{r(t)}\left(\frac{1}{p(t)}z'(t)\right)'\right)' - q(t)z(t) = 0. \tag{$\mathrm{L}^{\mathcal{A}}$}$$

For the sake of brevity we put

$$x^{[0]} = x,\ \ x^{[1]} = \frac{1}{r}x',\ \ x^{[2]} = \frac{1}{p}\left(\frac{1}{r}x'\right)' = \frac{1}{p}\left(x^{[1]}\right)',\ \ x^{[3]} = \frac{1}{q}\left(x^{[2]}\right)'.$$

The functions $x^{[i]}$, i=0, 1, 2, 3 we call the *quasiderivatives* of x. For $(N^{\mathcal{A}}, h)$ we can proceed in a similar way. For simplicity, when $h(t) \equiv t$, we will denote (N, h) and $(N^{\mathcal{A}}, h)$ with (N) and $(N^{\mathcal{A}})$, respectively. In addition to (H1)–(H4) we will sometimes assume

$$\liminf_{|u|\to\infty} \frac{f(u)}{u} > 0 \tag{H5}$$

or

$$f \text{ is nondecreasing in } \mathbb{R}, \quad |u| \ge |f(u)| \qquad \text{for all} \quad u \in \mathbb{R}. \tag{H6}$$

By a *solution* of an equation of the form (N, h) $\left[(N^{\mathcal{A}}, h)\right]$ we mean a function $w \in C^1\big([0,\infty), \mathbb{R}\big)$ such that $w^{[1]}(t)$, $w^{[2]}(t) \in C^1\big([0,\infty), \mathbb{R}\big)$ satisfying equation (N, h) $\left[(N^{\mathcal{A}}, h)\right]$ for all $t \ge 0$. Any solution of (N, h) or $(N^{\mathcal{A}}, h)$ is said to be *proper* if it is defined on the interval $[0, \infty)$ and is nontrivial in any neighborhood of infinity. A proper solution is said to be *oscillatory* if it has a sequence of zeros converging to ∞ otherwise it is said to be *nonoscillatory.* In addition, equation (N, h) $\left[(N^{\mathcal{A}}, h)\right]$ is called *oscillatory* if it has at least one nontrivial oscillatory solution and *nonoscillatory* if all its solutions are nonoscillatory. The study of asymptotic behavior of solutions, in the ordinary case as well as in the case with deviating argument, is often connected by introducing the concepts of *equation with property A* and *equation with property B.*

Definition 1.1. Equation (N,h) is said to have *property A* if any proper solution x of (N,h) is either oscillatory or satisfies

$$|x^{[i]}(t)| \downarrow 0 \quad \text{as } t \to \infty \qquad \text{for} \quad i = 0,1,2$$

and equation $(N^{\mathcal{A}},h)$ is said to have *property B* if any proper solution z of $(N^{\mathcal{A}},h)$ is either oscillatory or satisfies

$$|z^{[i]}(t)| \uparrow \infty \quad \text{as } t \to \infty \qquad \text{for} \quad i = 0,1,2.$$

The notations $u(t) \downarrow 0$ and $u(t) \uparrow \infty$ mean that function u monotonically decreases to zero as $t \to \infty$ or monotonically increases to infinity as $t \to \infty$, respectively.

Denote by $\mathcal{N}\big[(N,h)\big]$, $\mathcal{N}\big[(N^{\mathcal{A}},h)\big]$ the sets of all proper nonoscillatory solutions of (N,h), $(N^{\mathcal{A}},h)$, respectively. From slight modification of the well-known lemma of Kiguradze (see, e.g., [8], [13]) it follows that nonoscillatory solutions x of (N,h) can be divided into the following two classes in the same way as in [3]:

$$\mathcal{N}_0 = \Big\{x \in \mathcal{N}\,\big[(N,h)\big],\ \exists\, T_x : x(t)x^{[1]}(t) < 0,\ x(t)x^{[2]}(t) > 0 \qquad \text{for} \quad t \ge T_x\Big\},$$

$$\mathcal{N}_2 = \Big\{x \in \mathcal{N}\,\big[(N,h)\big],\ \exists\, T_x : x(t)x^{[1]}(t) > 0,\ x(t)x^{[2]}(t) > 0 \qquad \text{for} \quad t \ge T_x\Big\}.$$

Similarly nonoscillatory solutions z of $(N^{\mathcal{A}},h)$ can be divided into the following two classes:

$$\mathcal{M}_1 = \Big\{z \in \mathcal{N}\,\big[(N^{\mathcal{A}},h)\big],\ \exists\, T_z : z(t)z^{[1]}(t) > 0,\ z(t)z^{[2]}(t) < 0 \qquad \text{for} \quad t \ge T_z\Big\},$$

$$\mathcal{M}_3 = \Big\{z \in \mathcal{N}\,\big[(N^{\mathcal{A}},h)\big],\ \exists\, T_z : z(t)z^{[1]}(t) > 0,\ z(t)z^{[2]}(t) > 0 \qquad \text{for} \quad t \ge T_z\Big\}.$$

If solution $x \in \mathcal{N}_0$ $[z \in \mathcal{M}_3]$, then its quasiderivatives satisfy the inequality $x^{[i]}(t)x^{[i+1]}(t) < 0$ $\big[z^{[i]}(t)z^{[i+1]}(t) > 0\big]$ for $i = 0,1,2$, for all sufficiently large t and using name as in [1], [2], [4], [9] we call it *Kneser solution* [*strongly monotone solution*].

Let us remark that we consider differential equations in so called *canonical form*, i.e., in the case if $\int\limits_0^\infty r(t)\,\mathrm{d}t = \int\limits_0^\infty p(t)\,\mathrm{d}t = \infty$.

The comparison results regarding the oscillatory and asymptotic behavior of differential equations of the third order with or without deviating argument have been largely investigated in [2]–[5], [7], [9], [12]. In particular, many papers have been devoted to the study of the oscillatory and asymptotic properties of solutions of differential equations of the nth order with quasiderivatives. Among the wide literature on this field, we refer the reader to [6], [8], [10], [11], [13], [14] and to the references contained therein.

The aim of this paper is to study the oscillatory and asymptotic properties of solutions of differential equations of the form (N, h) and $(N^{\mathcal{A}}, h)$. More precisely, we will establish some comparison results on properties A and B. Our research is based on a study of asymptotic behavior of nonoscillatory solutions of (N, h) and $(N^{\mathcal{A}}, h)$ as well as on known comparison results (see [8], [9]). Such comparison criteria, in the forms here used, are quoted in section 2. The paper is organized as follows: The second section summarizes some known results which will be useful in our later consideration. In the section 3 we prove a comparison result concerning property A for delay equation (N, g) and property B for delay equation $(N^{\mathcal{A}}, g)$. As a consequence we obtain several interesting results for special cases of equations (N, g), $(N^{\mathcal{A}}, g)$. Here we also state a comparison theorem on properties A and B between nonlinear differential equations with delay and with advanced argument. The obtained results extend some other ones which have been given in [1], [3]. Several examples illustrating the main theorems are also provided.

2. Preliminary results

We introduce the following notation:

$$I(u_i) = \int_0^\infty u_i(t)\,\mathrm{d}t, \qquad I(u_i, u_j) = \int_0^\infty u_i(t) \int_0^t u_j(s)\,\mathrm{d}s\,\mathrm{d}t, \qquad i, j = 1, 2,$$

$$I(u_i, u_j, u_k) = \int_0^\infty u_i(t) \int_0^t u_j(s) \int_0^s u_k(b)\,\mathrm{d}b\,\mathrm{d}s\,\mathrm{d}t, \qquad i, j, k = 1, 2, 3,$$

where u_i, $i = 1, 2, 3$ are continuous positive functions on $[0, \infty)$.
For simplicity, sometimes we will write $u(\infty)$ instead of $\lim\limits_{t\to\infty} u(t)$.

In the recent papers [1], [2], [4] there are studied oscillatory and asymptotic properties of solutions of linear equations (L) and $(L^{\mathcal{A}})$.

Theorem 2.1. ([1], Theorem 2.2) *The following assertions are equivalent:*

- (i) *(L) has property A.*
- (i') *$(L^{\mathcal{A}})$ has property B.*
- (ii) *(L) is oscillatory and $I(q, p, r) = \infty$.*
- (ii') *$(L^{\mathcal{A}})$ is oscillatory and $I(q, p, r) = \infty$.*

The following comparison theorems and sufficient integral criteria on properties A and B will be needed in the sequel.

Theorem 2.2. ([8], Theorem 1) *Assume (H6), $g \in C\big([0,\infty),\mathbb{R}\big)$, $g(t) \le t$ and $\lim\limits_{t\to\infty} g(t) = \infty$.*

a) *If equation (N, g) has property A, then (L) has property A.*

b) *If equation $(N^{\mathcal{A}}, g)$ has property B, then $(L^{\mathcal{A}})$ has property B.*

Theorem 2.3. ([9], Theorem 3.1) *Assume (H5), $h(t) \ge t$ and let the following condition be satisfied:*

$$\begin{aligned} \textit{either } I(q,r) &= \infty \\ \textit{or } \limsup_{t\to\infty} \int_0^t p(s)\,\mathrm{d}s \int_t^\infty q(s) \frac{\int_0^s r(u) \int_0^u p(v)\,\mathrm{d}v\,\mathrm{d}u}{\int_0^s p(u)\,\mathrm{d}u}\,\mathrm{d}s &= \infty. \end{aligned} \tag{1}$$

If for some $K > 0$ the equation

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}x'(t)\right)'\right)' + Kq(t)x(t) = 0 \tag{$\mathrm{L_K}$}$$

has property A, then (N, h) has property A and $(N^{\mathcal{A}}, h)$ has property B.

Theorem 2.4. ([9], Theorem 5.6) *Assume (H5).*

a) *If $I(q,p,r) = \infty$ and $\displaystyle\int_T^\infty q(t) \int_T^{h(t)} r(s)\,\mathrm{d}s\,\mathrm{d}t = \infty$, then equation (N, h) has property A.*

b) *If $I(q,r) = \infty$ and $\displaystyle\int_T^\infty q(t) \int_T^{h(t)} p(s)\,\mathrm{d}s\,\mathrm{d}t = \infty$, then equation $(N^{\mathcal{A}}, h)$ has property B.*

Finally, we introduce some asymptotic properties of solutions of equations (N, h), $(N^{\mathcal{A}}, h)$ (for their proof see [3], [9]).

Lemma 2.5.

a) *Any solution x of (N, h) from $\mathcal{N}_0$ satisfies $\lim\limits_{t\to\infty} x^{[i]}(t) = 0,\ i = 1, 2.$*

b) *Any solution z of $(N^{\mathcal{A}}, h)$ from $\mathcal{M}_3$ satisfies $\lim\limits_{t\to\infty} |z^{[i]}(t)| = \infty,\ i = 0, 1.$*

c) *Any solution x of (N,h) from $\mathcal{N}_2$ satisfies* $\lim_{t\to\infty} |x(t)| = \infty$.

d) *Any solution z of $(N^{\mathcal{A}},h)$ from $\mathcal{M}_1$ satisfies* $\lim_{t\to\infty} z^{[2]}(t) = 0$.

e) *If $I(q,r,p) = \infty$, then every solution z of $(N^{\mathcal{A}},h)$ from $\mathcal{M}_1$ satisfies* $\lim_{t\to\infty} |z(t)| = \infty$.

f) *If $I(q,p,r) = \infty$, then every solution x of (N,h) from $\mathcal{N}_0$ satisfies* $\lim_{t\to\infty} x^{[i]}(t) = 0,\ i = 0,1,2$.

3. Main results

In this section we will always assume that $g \in C\bigl([0,\infty),\mathbb{R}\bigr)$, $\lim_{t\to\infty} g(t) = \infty$.

We start our consideration with the following lemma.

Lemma 3.1. *If equation (L) has property A, then every solution x of (N,h) from the class $\mathcal{N}_0$ satisfies $\lim_{t\to\infty} x^{[i]}(t) = 0$ for i=0, 1, 2.*

P r o o f. Because equation (L) has property A, from Theorem 2.1 we have $I(q,p,r) = \infty$. Now, Lemma 2.5, part f) immediately gives the assertion. $\square$

Now, we prove a comparison theorem concerning property A for delay equation (N,g) and property B for delay equation $(N^{\mathcal{A}},g)$.

Theorem 3.2. *Assume (H5), (H6) and $g(t) \le t$.*

a) *Let* $\displaystyle\int_0^\infty q(t) \int_0^{g(t)} r(s)\,\mathrm{d}s\,\mathrm{d}t = \infty.$

If equation $(N^{\mathcal{A}},g)$ has property B, then (N,g) has property A.

b) *Let* $\displaystyle I(q,r) = \infty,\ \int_0^\infty q(t) \int_{g(t)}^{t} p(s) \int_0^{s} r(u)\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t < \infty.$

If equation (N,g) has property A, then $(N^{\mathcal{A}},g)$ has property B.

P r o o f. a) Let x be a proper nonoscillatory solution of (N,g). We assume that there exists $T \ge 0$ such that $x(t) > 0$ for all $t \ge T$. The case $x(t) < 0$ for all $t \ge T^*$ may be treated by using similar arguments. We know that $x \in \mathcal{N}_0 \cup \mathcal{N}_2$. Now we assume that (N,g) has not property A. By Lemma 2.5, part a) there are two possibilities:

I. $x \in \mathcal{N}_0$ such that $\lim\limits_{t\to\infty} x(t) = \alpha > 0$,

II. $x \in \mathcal{N}_2$.

But case I is not allowed to occur. Really, by Theorem 2.2 equation $(L^{\mathcal{A}})$ has property B and so by Theorem 2.1 equation (L) has property A. And now, Lemma 3.1 gives that every solution $x \in \mathcal{N}_0$ of equation (N, g) satisfies $\lim\limits_{t\to\infty} x(t) = 0$.

Case II. Since x is a positive nonoscillatory solution of (N, g) in the class $\mathcal{N}_2$, there exists $T_1 \ge T$ such that $x(t) > 0$, $x^{[1]}(t) > 0$, $x^{[2]}(t) > 0$ for all $t \ge T_1$. Let $T_2 > T_1$ be such that $g(t) \ge T_1$ for all $t \ge T_2$. Because $\bigl(x^{[2]}(t)\bigr)' = -q(t)f\bigl(x(g(t))\bigr) < 0$ for all $t \ge T_2$, $x^{[2]}(t)$ is a positive decreasing function and thus $0 \le x^{[2]}(\infty) < \infty$. Integrating (N, g) in $[T_2, \infty)$, we obtain

$$x^{[2]}(T_2) - x^{[2]}(\infty) = \int_{T_2}^{\infty} q(t)f\bigl(x(g(t))\bigr)\,\mathrm{d}t\,.$$

In view of the fact $x^{[2]}(\infty) < \infty$, there exists a positive constant c such that

$$c = \int_{T_2}^{\infty} q(t)f\bigl(x(g(t))\bigr)\,\mathrm{d}t\,. \tag{2}$$

Assertion c) of Lemma 2.5 allows us to use assumption (H5) which implies that there exists a positive constant K_1 such that $f\bigl(x(g(t))\bigr) > K_1 x\bigl(g(t)\bigr)$ for all $t \ge T_3 \ge T_2$ and thus from (2) we get

$$c > K_1 \int_{T_3}^{\infty} q(t)x\bigl(g(t)\bigr)\,\mathrm{d}t\,. \tag{3}$$

As $x^{[1]}(t)$ is a positive increasing function, we have $x^{[1]}(t) \ge x^{[1]}(T_1)$ for all $t \ge T_1$. Integrating this inequality in $[T_1, t]$, we get

$$x(t) \ge x(T_1) + x^{[1]}(T_1) \int_{T_1}^{t} r(s)\,\mathrm{d}s > x^{[1]}(T_1) \int_{T_1}^{t} r(s)\,\mathrm{d}s \qquad \text{for all} \quad t \ge T_1$$

and so

$$x\bigl(g(t)\bigr) > x^{[1]}(T_1) \int_{T_1}^{g(t)} r(s)\,\mathrm{d}s > x^{[1]}(T_1) \int_{T_3}^{g(t)} r(s)\,\mathrm{d}s$$

for all t sufficiently large. Using the last inequality in (3) we obtain

$$c > K_1 x^{[1]}(T_1) \int\limits_{T_3}^{\infty} q(t) \int\limits_{T_3}^{g(t)} r(s)\,\mathrm{d}s\,\mathrm{d}t\,,$$

which is a contradiction.

b) Let z be a proper nonoscillatory solution of $(N^{\mathcal{A}}, g)$. We assume that there exists $T \geq 0$ such that $z(t) > 0$ for all $t \geq T$. The case $z(t) < 0$ for all $t \geq T^*$ may be treated by using similar arguments. We know that $z \in \mathcal{M}_1 \cup \mathcal{M}_3$. Now we assume that $(N^{\mathcal{A}}, g)$ has not property B. By Lemma 2.5, part b) there are two possibilities:

I. $z \in \mathcal{M}_3$ such that $\lim\limits_{t\to\infty} z^{[2]}(t) = \beta < \infty$,

II. $z \in \mathcal{M}_1$.

Case I. Since z is a positive nonoscillatory solution of $(N^{\mathcal{A}}, g)$ in the class $\mathcal{M}_3$, there exists $T_1 \geq T$ such that $z(t) > 0$, $z^{[1]}(t) > 0$, $z^{[2]}(t) > 0$ for all $t \geq T_1$. Let $T_2 > T_1$ be such that $g(t) \geq T_1$ for all $t \geq T_2$. Because $\big(z^{[2]}(t)\big)' = q(t) f\big(z(g(t))\big) > 0$ for all $t \geq T_2$, $z^{[2]}(t)$ is a positive increasing function. By integrating $(N^{\mathcal{A}}, g)$ in $[T_2, \infty)$, we obtain

$$z^{[2]}(\infty) - z^{[2]}(T_2) = \int\limits_{T_2}^{\infty} q(t) f\big(z(g(t))\big)\,\mathrm{d}t\,.$$

In view of the facts that $z^{[2]}(\infty) < \infty$ and $z^{[2]}(t)$ is an increasing function, there exists a positive constant c such that

$$c = \int\limits_{T_2}^{\infty} q(t) f\big(z(g(t))\big)\,\mathrm{d}t\,. \tag{4}$$

From Lemma 2.5, part b) we have $z(\infty) = \infty$ and so the assumption (H5) implies that there exists a positive constant K_2 such that $f\big(z(g(t))\big) > K_2 z\big(g(t)\big)$ for all $t \geq T_3 \geq T_2$ and thus from (4) we get

$$c > K_2 \int\limits_{T_3}^{\infty} q(t) z\big(g(t)\big)\,\mathrm{d}t\,. \tag{5}$$

As $z^{[2]}(t)$ is a positive increasing function, we have $z^{[2]}(t) \geq z^{[2]}(T_1)$ for all $t \geq T_1$. Integrating this inequality twice in $[T_1, t]$, we get

$$z(t) > z^{[2]}(T_1) \int_{T_1}^{t} p(s) \int_{T_1}^{s} r(u)\,\mathrm{d}u\,\mathrm{d}s \qquad \text{for all} \quad t \geq T_1$$

and hence

$$z\big(g(t)\big) > z^{[2]}(T_1) \int_{T_1}^{g(t)} p(s) \int_{T_1}^{s} r(u)\,\mathrm{d}u\,\mathrm{d}s > z^{[2]}(T_1) \int_{T_3}^{g(t)} p(s) \int_{T_3}^{s} r(u)\,\mathrm{d}u\,\mathrm{d}s$$

for all t, enough large. Using this inequality in (5) we get

$$c > K_2 z^{[2]}(T_1) \int_{T_3}^{\infty} q(t) \int_{T_3}^{g(t)} p(s) \int_{T_3}^{s} r(u)\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t\,,$$

which implies

$$\int_{0}^{\infty} q(t) \int_{0}^{g(t)} p(s) \int_{0}^{s} r(u)\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t < \infty\,.$$

So together with convergent integral in the assumption we have $I(q,p,r) < \infty$. But on the other hand, by Theorem 2.2 equation (L) has property A and so by Theorem 2.1 $I(q,p,r) = \infty$, which is a contradiction.

Case II. Since z is a positive nonoscillatory solution of $(N^{\mathcal{A}}, g)$ in the class $\mathcal{M}_1$, there exists $T_1 \geq T$ such that $z(t) > 0$, $z^{[1]}(t) > 0$, $z^{[2]}(t) < 0$ for all $t \geq T_1$. Taking into account that $z^{[1]}(t)$ is a positive decreasing function and assertion d) of Lemma 2.5, we have

$$0 \leq z^{[1]}(\infty) < \infty \quad \text{and} \quad z^{[2]}(\infty) = 0\,. \tag{6}$$

Since $z(t)$ is a positive increasing function, there exists a positive constant K_4 such that $z(t) \geq K_4$ for all $t \geq T_1$. Let $T_2 > T_1$ be such that $g(t) \geq T_1$ for all $t \geq T_2$. Now, we have that

$$z\big(g(t)\big) \geq K_4 \qquad \text{for all} \quad t \geq T_2. \tag{7}$$

Integrating (N^A, g) twice in $[t,\infty)$ and using (6) we obtain

$$z^{[1]}(t) = z^{[1]}(\infty) + \int_t^\infty r(s) \int_s^\infty q(u) f\bigl(z(g(u))\bigr)\,\mathrm{d}u\,\mathrm{d}s$$
$$\geq \int_t^\infty r(s) \int_s^\infty q(u) f\bigl(z(g(u))\bigr)\,\mathrm{d}u\,\mathrm{d}s\,.$$

As $I(q,r) = \infty$ yields $I(q,r,p) = \infty$, assertion e) of Lemma 2.5 allows us to use assumption (H5) which implies that there exists a positive constant K_3 such that $f\bigl(z(g(t))\bigr) > K_3 z\bigl(g(t)\bigr)$ for t sufficiently large. Moreover, taking (7) into account we get

$$z^{[1]}(t) > K_3 \int_t^\infty r(s) \int_s^\infty q(u) z\bigl(g(u)\bigr)\,\mathrm{d}u\,\mathrm{d}s$$
$$\geq K_3 \cdot K_4 \int_t^\infty r(s) \int_s^\infty q(u)\,\mathrm{d}u\,\mathrm{d}s$$
$$= K_3 \cdot K_4 \int_t^\infty q(s) \int_t^s r(u)\,\mathrm{d}u\,\mathrm{d}s\,,$$

which gives a contradiction with $I(q,r) = \infty$. □

Remark 3.3. If $f(u) = u$ in Theorem 3.2, we obtain Theorem 2.2 in [3] and so our theorem extends mentioned result to the nonlinear case.

Theorem 3.2 yields the following:

Corollary 3.4. *Assume (H5), (H6), $g(t) \leq t$ and*

$$\int_0^\infty q(t) \int_0^{g(t)} r(s)\,\mathrm{d}s\,\mathrm{d}t = \infty, \qquad \int_0^\infty q(t) \int_{g(t)}^t p(s) \int_0^s r(u)\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t < \infty\,.$$

Then equation (N,g) has property A if and only if equation $(N^{\mathcal{A}}, g)$ has property B.

Proof. Taking into account that

$$\infty = \int_0^\infty q(t) \int_0^{g(t)} r(s)\,\mathrm{d}s\,\mathrm{d}t \leq \int_0^\infty q(t) \int_0^t r(s)\,\mathrm{d}s\,\mathrm{d}t = I(q,r), \tag{8}$$

Theorem 3.2 immediately gives the assertion. □

If $g(t) \equiv t$ in equations (N,g), $(N^{\mathcal{A}},g)$, Theorem 3.2 gives the following corollary.

Corollary 3.5. *Assume* $(H5)$, $(H6)$ *and* $I(q,r)=\infty$. *Then equation* (N) *has property A if and only if equation* $(N^{\mathcal{A}})$ *has property B.*

Remark 3.6. Corollary 3.5 extends result concerning property A for equation (N) and property B for equation $(N^{\mathcal{A}})$, see Theorem 4.1 in [1].

As a consequence of Theorem 3.2 we also obtain sufficient integral criterion ensuring property A for equation (N,g) and property B for equation $(N^{\mathcal{A}},g)$ as the following result shows.

Corollary 3.7. *Assume* $(H5)$, $(H6)$, $g(t)\le t$ *and*

$$\int\limits_0^\infty q(t)\int\limits_0^{g(t)} r(s)\,\mathrm{d}s\,\mathrm{d}t=\infty,\qquad \int\limits_0^\infty q(t)\int\limits_0^{g(t)} p(s)\,\mathrm{d}s\,\mathrm{d}t=\infty\,.$$

Then equation (N,g) *has property A and equation* $(N^{\mathcal{A}},g)$ *has property B.*

Proof. From (8) and Theorem 2.4, part b) immediately there follows the assertion that equation $(N^{\mathcal{A}},g)$ has property B. Now, by Theorem 3.2, part a) we also have that equation (N,g) has property A. □

The following examples illustrate the above results.

Example 3.8. The following equation of the form (N,g)

$$\left(t\left(\frac{1}{t}\cdot x'(t)\right)'\right)'+\frac{e^{2t}\left(t^2+t-1\right)}{t^2\left(e^{2t}-1\right)}\cdot\frac{x^2(t)-1}{x(t)}=0,\qquad t\ge 1$$

has property A (see, e.g., Theorem 2.4). The assumptions of Theorem 3.2, part b) hold and so we have that equation $(N^{\mathcal{A}},g)$ has property B. One nonoscillatory solution of equation $(N^{\mathcal{A}},g)$ such that $\lim\limits_{t\to\infty}|z^{[i]}(t)|=\infty$ for $i=0,1,2$ is the function $z(t)=e^t$.

Example 3.9. We consider the differential equation

$$\big(t(t\cdot x'(t))'\big)'+\frac{1}{t\cdot\ln t}\cdot x(t/\ln t)=0,\qquad t\ge e \tag{9}$$

This is the equation of the form (N,g), where $r(t)=p(t)=1/t$, $q(t)=1/(t\cdot\ln t)$, $g(t)=t/\ln t$ and $f(u)=u$. It is easy to verify that conditions of Corollary 3.7 are fulfilled and so we get that equation (9) has property A. One nonoscillatory solution of equation (9) such that $|x^{[i]}(t)| \downarrow 0$ as $t \to \infty$, $i = 0, 1, 2$ is the function $x(t) = \frac{1}{t}$.

Our last result gives a relationship between nonlinear differential equations with delay and with advanced argument.

Theorem 3.10. *Assume* $(H5)$, $(H6)$, $g(t) \le t$, $h(t) \ge t$ *and let the condition* (1) *of* Theorem 2.3 *be satisfied.*

a) *If equation* (N, g) *has property A, then equation* (N, h) *has property A and equation* $(N^{\mathcal{A}}, h)$ *has property B.*

b) *If equation* $(N^{\mathcal{A}}, g)$ *has property B, then equation* (N, h) *has property A and equation* $(N^{\mathcal{A}}, h)$ *has property B.*

Proof. a) Because (N, g) has property A, by Theorem 2.2 equation (L) has property A. Now, Theorem 2.3 immediately gives the assertion.

b) Because $(N^{\mathcal{A}}, g)$ has property B, by Theorem 2.2 equation $(L^{\mathcal{A}})$ has property B and so by Theorem 2.1 equation (L) has property A. Now, again Theorem 2.3 gives the assertion. ☐

Remark 3.11. Similar comparison result as our Theorem 3.10 is proved in the paper [3] (Theorem 2.1) for the linear case.

REFERENCES

[1] CECCHI, M.—DOŠLÁ, Z.—MARINI, M.: *On nonlinear oscillations for equations associated to disconjugate operators*, Nonlinear Anal. **30**, (1997), 1583–1594.

[2] CECCHI, M.—DOŠLÁ, Z.—MARINI, M.: *Comparison theorems for third order differential equations.* In: Proceedings of the 2nd international conference, May 24–27, Morehouse College, Atlanta, GA, USA, 1995 (Ladde, G. S. et al., eds.), Dynam. Systems App., Vol. 2, Dynamic Publishers, Atlanta, GA, 1996, pp. 99–106.

[3] CECCHI, M.—DOŠLÁ, Z.—MARINI, M.: *Asymptotic behavior of solutions of third-order delay differential equations*, Arch. Math. (Brno) **33** (1997), 99–108.

[4] CECCHI, M.—DOŠLÁ, Z.—MARINI, M.: *An equivalence theorem on properties A, B for third order differential equations*, Ann. Mat. Pura Appl. (4), **173** (1997), 373–389.

[5] CECCHI, M.—DOŠLÁ, Z.—MARINI, M.: *On third order differential equations with property A and B*, J. Math. Anal. Appl. **231** (1999), 509–525.

[6] DŽURINA, J.: *Property (A) of n-th order ODE's*, Math. Bohem. **122**, (1997), 349–356.

[7] DŽURINA, J.: *Asymptotic properties of third order differential equations with deviating argument*, Czechoslovak Math. J. **44**, (1994), 163–172.

[8] KUSANO, T.—NAITO, M.: *Comparison theorems for functional differential equations with deviating arguments*, J. Math. Soc. Japan, **33**, (1981), 509–532.

[9] MOJSEJ, I.—OHRISKA, J.: *On solutions of third order nonlinear differential equations*, Cent. Eur. J. Math. **4**, (2006), 46–63.

[10] OHRISKA, J.: *Oscillation of differential equations and v-derivatives*, Czechoslovak Math. J. **39**, (1989), 24–44.
[11] OHRISKA, J.: *Adjoint differential equations and oscillation*, J. Math. Anal. Appl. **195** (1995), 778–796.
[12] PARHI, N.—PADHI, S.: *On asymptotic behavior of delay differential equations of third order*, Nonlinear Anal. **34** (1998), 391–403.
[13] ŠEDA, V.: *Nonoscillatory solutions of differential equations with deviating argument*, Czechoslovak Math. J. **36**, (1986), 93–107.
[14] ŠOLTÉS, V.: *Property A of the n-th order differential equations with deviating argument*, Arch. Math. (Brno) **31** (1995), 59–63.

Received November 9, 2006

Institute of Mathematics
Faculty of Science
P. J. Šafárik University
Jesenná 5
041 54 Košice
SLOVAK REPUBLIC
E-mail: ivan.mojsej@upjs.sk