Tatra Mt. Math. Publ. **36** (2007), 71–107

**Tatra Mountains**
*Mathematical Publications*

# ON COMPLETE TRIPARTITE GRAPHS ARBITRARILY DECOMPOSABLE INTO CLOSED TRAILS

Mirko Horňák — Zuzana Kocková

ABSTRACT. It is proved that any complete tripartite graph $K_{r,r,r}$, where $r = 5 \cdot 2^n$ and $n$ is a nonnegative integer, has the following property: Whenever $(l_1, \dots, l_p)$ is a sequence of integers $\geqslant 3$ adding up to $|E(K_{r,r,r})|$, there is a sequence $(T_1, \dots, T_p)$ of edge-disjoint closed trails in $K_{r,r,r}$ such that $T_i$ is of length $l_i$, $i = 1, \dots, p$.

## 1. Introduction

Any simple finite nonoriented graph $G$ with $\delta(G) \geqslant 2$ contains a cycle. Therefore, if $G$ is *even* (if all vertices of $G$ are of even degrees), it is an edge-disjoint union of cycles. Several authors investigated edge decompositions of complete multipartite graphs into cycles of equal lengths. The bipartite case has been completely solved by Sotteau [8]. For complete tripartite graphs some partial results are known, see Billington and Cavenagh [3], [4], Cavenagh [5]. In the general case a reader can consult Cockayne and Hartnell [6].

A connected edge-disjoint union of cycles is an Eulerian graph and has a closed Eulerian trail. So, an even graph can be expressed as an edge-disjoint union of closed trails, and there are many possibilities how to do it. Balister [1] has proved that if $n$ is odd and $l_1, \dots, l_p$ are integers $\geqslant 3$ adding up to $\left|E(K_n)\right|$, there are edge-disjoint closed trails $T_1, \dots, T_p$ in $K_n$ such that $T_i$ is of length $l_i$, $i = 1, \dots, p$. In the same paper a similar result has been reached for the graph $K_n - M_n$ (the complete graph of even order without a perfect matching). Further, Balister [2] has shown that there are positive constants $n$ and $\varepsilon$ such that any even graph $G$ with $\left|V(G)\right| \geqslant n$ and $\delta(G) \geqslant (1-\varepsilon)\left|V(G)\right|$

2000 Mathematics Subject Classification: 05C45.
Keywords: closed trail, edge decomposition, complete tripartite graph.
The authors wish to thank for the support of the Slovak grant VEGA 1/0424/03.

can be (edge-)decomposed in the above manner. Another graphs with analogous properties concerning closed trails are complete bipartite graphs $K_{m,n}$ with $m, n$ even, as proved by Horňák and Woźniak [7] (note that in that case all $l_i$'s have to be even). In this paper we do concentrate on complete tripartite graphs.

Let us now specify the problem we are going to deal with. For integers $p, q$ we use the notation $[p, q] := \{z \in \mathbb{Z} : p \leqslant z \leqslant q\}$ and $[p, \infty) := \{z \in \mathbb{Z} : p \leqslant z\}$. The *concatenation* of finite sequences $A = (a_1, \dots, a_m)$ and $B = (b_1, \dots, b_n)$ is the sequence $AB = (a_1, \dots, a_m, b_1, \dots, b_n)$. The concatenation is associative, and so we can use the symbol $\prod_{i=1}^{k} A_i$ for the result of concatenation of finite sequences $A_1, \dots, A_k$ (independently from the order in which "partial" concatenations are realised). If $k \in [0, \infty)$ and $A_i = A$ for any $i \in [1, k]$, we write $A^k$ instead of $\prod_{i=1}^{k} A_i$ (hence $A^0 = (\ )$ is the empty sequence).

A *closed trail* of length $n \in \big[3, |E(G)|\big]$ (an *$n$-trail* for short) in a graph $G$ is a sequence $T = \prod_{i=0}^{n} (v_i)$ of vertices of $G$ such that $v_0 = v_n$, $v_i v_{i+1} \in E(G)$ and $v_i v_{i+1} \neq v_j v_{j+1}$ for any $i, j \in [0, n-1]$, $i \neq j$. The set $\{v_i v_{i+1} : i \in [0, n-1]\}$ of edges of $T$ induces an Eulerian subgraph of $G$ and throughout the whole paper we shall identify $T$ with this subgraph. Deleting the edges of a closed trail from an even graph $G$ results in an even graph with smaller number of edges. This process can be repeated until the edgeless graph $\big(V(G), \emptyset\big)$ is reached. If $(T_1, \dots, T_p)$ is the sequence of (closed) trails occurring in the process, then $1 \leqslant p \leqslant \big\lfloor |E(G)|/3 \big\rfloor$, $\{E(T_i) : i \in [1, p]\}$ is a decomposition of $E(G)$ and $\sum_{i=1}^{p} |E(T_i)| = |E(G)|$. Let $\operatorname{Lct}(G)$ be the set of all lengths of closed trails in $G$ and let

$$\operatorname{Sct}(G) := \bigcup_{p=1}^{\lfloor |E(G)|/3 \rfloor} \left\{ (l_1, \dots, l_p) \in \big(\operatorname{Lct}(G)\big)^p : \sum_{i=1}^{p} l_i = |E(G)| \right\}.$$

A sequence $L = (l_1, \dots, l_p) \in \operatorname{Sct}(G)$ is said to be *$G$-realisable* if there is a *$G$-realisation* of $L$, that is, a sequence $(T_1, \dots, T_p)$ of edge-disjoint closed trails in $G$ (so that $\bigcup_{i=1}^{p} E(T_i) = E(G)$). A graph $G$ is said to be *arbitrarily decomposable into closed trails* (ADCT for short) if $\operatorname{Sct}(G) \neq \emptyset$ and any sequence from $\operatorname{Sct}(G)$ is $G$-realisable. Evidently, if a graph is ADCT, it is even.

A sequence $A = (a_1, \dots, a_m)$ is said to be *changeable* to a sequence $B = (b_1, \dots, b_m)$ if there is a bijection $\pi : [1, m] \to [1, m]$ such that $b_i = a_{\pi(i)}$ for any $i \in [1, m]$; if $A$ is changeable to $B$, we write $A \sim B$. For $I \subseteq [1, m]$ let $A\langle I \rangle$ be the subsequence of $A$ created by deleting all entries $a_i$ with $i \in [1, m] - I$

from $A$. If $A \in \mathbb{R}^m$ and $r \in \mathbb{R}$, we denote by $\mathrm{nd}(A)$ the (unique) nondecreasing sequence that is changeable to $A$ and by $f_r(A)$ the *frequency* of $r$ in $A$. For $l \in \mathbb{Z}$ let $(l)_4$ be the unique $m \in [0,3]$ such that $l \equiv m \pmod 4$.

Let $G, H$ be isomorphic graphs and let $\varphi\colon V(G) \to V(H)$ be an isomorphism from $G$ onto $H$. If $T = \prod_{i=0}^{n} (v_i)$ is a closed trail in $G$, then $\varphi(T) := \prod_{i=0}^{n} \bigl(\varphi(v_i)\bigr)$ is a closed trail in $H$. Further, if $\mathcal{T} = \prod_{i=1}^{q} (T_i)$ is a sequence of edge-disjoint closed trails in $G$, then $\varphi(\mathcal{T}) := \prod_{i=1}^{q} \bigl(\varphi(T_i)\bigr)$ is a sequence of edge-disjoint closed trails in $H$.

Consider edge-disjoint closed trails $T_1, T_2$ in a graph $G$ and let $T_1 + T_2$ denote the set of all closed trails $T$ in $G$ with $E(T) = E(T_1) \cup E(T_2)$. Clearly, $T_1 + T_2$ is nonempty if and only if $V(T_1) \cap V(T_2) \neq \emptyset$.

A sequence is said to be *simple* if no two its terms at distinct positions are the same. Let $G$ be a graph and let $m \in \bigl[1, |V(G)|\bigr]$. A simple sequence $(v_1, \ldots, v_m)$ of vertices of $G$ is *similar* to a simple sequence $(w_1, \ldots, w_m)$ of vertices of $G$ provided that there is an automorphism $\varphi$ of $G$, such that $\varphi(v_i) = w_i$ for any $i \in [1, m]$; if $m = 1$, we say for short that vertex $v_1$ is similar to vertex $w_1$. The relation of similarity of vertices of a graph $G$ is an equivalence and a *similarity class* of $G$ is a class of this equivalence.

A set $S$ of edges of a graph $G$ is said to be *complementary bipartite* in $G$ provided that the graph $G - S$ is bipartite. Let $\mathrm{Cb}(G)$ denote the system of all sets $S \subseteq E(G)$ that are complementary bipartite in $G$ and let $\mathrm{mcb}(G)$ be the minimum cardinality of a set $S \in \mathrm{Cb}(G)$.

Let $r \in [1, \infty)$ and $(n_1, \ldots, n_r) \in [1, \infty)^r$. Throughout the whole paper we shall suppose that the $r$-partition of the complete $r$-partite graph $K_{n_1, \ldots, n_r}$ is $\bigl\{\{v_{i,j}\colon i \in [1, n_j]\}\colon j \in [1, r]\bigr\}$. If $(n_1, \ldots, n_r) = (n)^r$, we write for short $K_{(n)r}$ instead of $K_{n_1, \ldots, n_r}$.

In the paper we adopt the following general strategy for proving that a union of two edge-disjoint even ADCT graphs $G_1$ and $G_2$ with $V(G_1) \cap V(G_2) \neq \emptyset$ is ADCT. Of course, there is no problem to find a $(G_1 \cup G_2)$-realisation of any sequence $L_1 L_2 \in \mathrm{Sct}(G_1 \cup G_2)$ such that $L_i \in \mathrm{Sct}(G_i)$, $i = 1, 2$. If $L \in \mathrm{Sct}(G_1 \cup G_2)$, but $L$ is not changeable to a sequence $L_1 L_2$ with $L_i \in \mathrm{Sct}(G_i)$, $i = 1, 2$, it may happen that $L \sim \hat{L}(\hat{l}_1 + \hat{l}_2)$ and $\hat{L} \sim \hat{L}_1 \hat{L}_2$, where $\hat{l}_1, \hat{l}_2 \geqslant 3$ and $\hat{L}_i(\hat{l}_i) \in \mathrm{Sct}(G_i)$, $i = 1, 2$. If we are able to find a $G_i$-realisation of the sequence $\hat{L}_i(\hat{l}_i)$, $i = 1, 2$, in which a trail $\hat{T}_1$ of length $\hat{l}_1$ in $G_1$ has a common vertex with a trail $\hat{T}_2$ of length $\hat{l}_2$ in $G_2$, then clearly there is also a $G$-realisation of $L$ (using a trail of length $\hat{l}_1 + \hat{l}_2$ formed from $\hat{T}_1$ and $\hat{T}_2$). The described "term-cutting" may also apply to more than one term of a sequence $L \in \mathrm{Sct}(G_1 \cup G_2)$.

## 2. Some preparatory results

**Lemma 1.** *If $G$ is a graph, $L_1, L_2 \in \mathrm{Sct}(G)$ and $L_1 \sim L_2$, then $L_1$ is $G$-realisable if and only if $L_2$ is $G$-realisable.*

Proof. If $i \in [1,2]$, $L_i = (l_1, \dots, l_p)$ has a $G$-realisation $(T_1, \dots, T_p)$ and $\pi_i \colon [1,p] \to [1,p]$ is such a bijection that $L_{3-i} = (l_{\pi(1)}, \dots, l_{\pi(p)})$, then $(T_{\pi(1)}, \dots, T_{\pi(p)})$ is a $G$-realisation of $L_{3-i}$. □

**Proposition 2.** *If $G$ is a graph, $S \in \mathrm{Cb}(G)$ and $T$ is a closed trail in $G$ of an odd length, then $E(T) \cap S \neq \emptyset$.*

Proof. $T$ is a non-bipartite graph, so it cannot be a subgraph of the bipartite graph $G - S$. □

**Proposition 3.** *If $G$ is a graph and a sequence $L = (l_1, \dots, l_p) \in \mathrm{Sct}(G)$ is $G$-realisable, then $L$ contains at most $\mathrm{mcb}(G)$ odd terms.*

Proof. Let $S \in \mathrm{Cb}(G)$. Suppose that $(T_1, \dots, T_p)$ is a $G$-realisation of the sequence $L$ in $G$ and put $I := \{i \in [1,p] \colon l_i \equiv 1 \pmod 2\}$. If $i \in I$, by Proposition 2 there exists $e_i \in E(T_i) \cap S$. Since trails in $(T_1, \dots, T_p)$ are edge-disjoint, we have $|I| = \left|\bigcup_{i \in I} \{e_i\}\right| \leqslant \left|\bigcup_{i \in I} (E(T_i) \cap S)\right| \leqslant |S| = \mathrm{mcb}(G)$. □

Let $\mathrm{Sct}^*(G)$ be the subset of $\mathrm{Sct}(G)$ consisting of sequences with at most $\mathrm{mcb}(G)$ odd terms. From Proposition 3 it follows that if a sequence $L \in \mathrm{Sct}(G)$ is $G$-realisable, then $L \in \mathrm{Sct}^*(G)$. So, if $\mathrm{Sct}(G) - \mathrm{Sct}^*(G) \neq \emptyset$, the graph $G$ is not ADCT.

**Proposition 4.** *If $n \in [1, \infty)$, $\{G^{(i)} \colon i \in [1,n]\}$ is a set of pairwise edge-disjoint graphs and $G = \bigcup_{i=1}^{n} G^{(i)}$, then $\mathrm{mcb}(G) \geqslant \sum_{i=1}^{n} \mathrm{mcb}(G^{(i)})$.*

Proof. Suppose that $S \subseteq E(G)$ and put $S^{(i)} := S \cap E(G^{(i)})$ for $i \in [1,n]$. If $|S| < s := \sum_{i=1}^{n} \mathrm{mcb}(G^{(i)})$, there is $j \in [1,n]$ such that $|S^{(j)}| < \mathrm{mcb}(G^{(j)})$. The graph $G - S$ is a supergraph of the graph $G^{(j)} - S^{(j)}$ that is not bipartite, hence $S \notin \mathrm{Cb}(G)$. Thus, $\mathrm{mcb}(G)$ cannot be smaller than $s$. □

**Proposition 5.** *If a sequence $(n_1, n_2, n_3) \in [1, \infty)^3$ is nondecreasing, then $\mathrm{mcb}(K_{n_1,n_2,n_3}) = n_1 n_2$.*

Proof. Let $V_1, V_2, V_3$ be parts of $G := K_{n_1,n_2,n_3}$ with $|V_i| = n_i$, $i = 1,2,3$. Then $E(G) = E_{1,2} \cup E_{2,3} \cup E_{3,1}$, where $E_{i,j} := \{xy \colon x \in V_i, y \in V_j\}$, $i,j \in [1,3]$, $i \neq j$. The graph $G - E_{1,2}$ is bipartite, hence $\mathrm{mcb}(G) \leqslant |E_{1,2}| = n_1 n_2$. Assume there is $S \in \mathrm{Cb}(G)$ with $|S| < n_1 n_2$. Then $S = S_{1,2} \cup S_{2,3} \cup S_{3,1}$, where $S_{i,j} := S \cap E_{i,j}$. If $i,j,k \in [1,3]$ and $\{i,j,k\} = [1,3]$, then the deletion of $e \in S_{i,j}$ from $G$ destroys exactly $n_k$ triangles of $G$. Therefore, the deletion of $S$ from $G$ destroys at most $|S_{1,2}|n_3 + |S_{2,3}|n_1 + |S_{3,1}|n_2 \leqslant (|S_{1,2}| + |S_{2,3}| + |S_{3,1}|)n_3 = |S|n_3 < n_1 n_2 n_3$ triangles of $G$ and $G - S$ has a triangle in contradiction with the fact that $G - S$ is bipartite. □

Let $p_{(n)r}\colon V(K_{(n)r}) \to [1,r]$ be the function defined by $p_{(n)r}(v_{i,j}) = j$ for any $i \in [1,n]$ and $j \in [1,r]$ (a vertex $x$ of $K_{(n)r}$ is assigned the number of the part containing $x$).

**Proposition 6.** *Let $n, r \in [1,\infty)$, $m \in [1, rn]$ and let $v = (v_1, \dots, v_m)$, $w = (w_1, \dots, w_m)$ be simple sequences of vertices of $K_{(n)r}$ such that there is a permutation $\pi\colon [1,r] \to [1,r]$ satisfying $\pi\big(p_{(n)r}(v_i)\big) = p_{(n)r}(w_i)$ for any $i \in [1,m]$. Then $v$ is similar to $w$.*

P r o o f. Consider a bijection $\varphi\colon V(K_{(n)r}) \to V(K_{(n)r})$ such that $\varphi\big(\{v_{i,j}\colon i \in [1,n]\}\big) = \{v_{i,\pi(j)}\colon i \in [1,n]\}$ for any $j \in [1,r]$ and $\varphi(v_i) = w_i$ for any $i \in [1,m]$. Clearly, $\varphi$ is an automorphism of $K_{(n)r}$. □

**Theorem 7.** *If a graph $K_{n_1,n_2,n_3}$ with $n_1 \leqslant n_2 \leqslant n_3$ is ADCT, then either $(n_1,n_2,n_3) \in \{(1,1,3),(1,1,5)\}$ or $n_1 = n_2 = n_3$.*

P r o o f. Let $G := K_{n_1,n_2,n_3}$ and $e := |E(G)|$. Vertices of $G$ are of even degrees $n_1 + n_2$, $n_2 + n_3$ and $n_3 + n_1$, hence $n_1, n_2$ and $n_3$ are of the same parity.

(1) Suppose that $n_1 = n_2 = 1$ and $n_3 = 6n + k$, where $n \geqslant 1$ and $k \in \{1,3,5\}$. The set $\mathrm{Lct}(G)$ contains $3, 4, 8$ since the following sequences are closed trails in $G$: $(v_{1,1}, v_{1,2}, v_{1,3}, v_{1,1})$, $(v_{1,1}, v_{1,3}, v_{1,2}, v_{2,3}, v_{1,1})$ and $(v_{1,1}, v_{1,3}, v_{1,2}, v_{2,3}, v_{1,1}, v_{3,3}, v_{1,2}, v_{4,3}, v_{1,1})$. As $e = 2(6n+k) + 1 = 3 \cdot 4n + 2k + 1$ and $2k + 1 \in \{3, 7, 11\}$, the set $\mathrm{Sct}(G)$ contains one of the sequences $(3)^{4n+1}$, $(3)^{4n+1}(4)$, $(3)^{4n+1}(8)$, having $4n + 1 \geqslant 5$ odd terms. By Propositions 3 and 5 $G$ is not ADCT.

(2) Suppose $2 \leqslant n_1 < n_3$. The set $\mathrm{Lct}(G)$ contains $3, 4, 5$ since the following sequences are closed trails in $G$: $(v_{1,1}, v_{1,2}, v_{1,3}, v_{1,1})$, $(v_{1,1}, v_{1,2}, v_{2,1}, v_{1,3}, v_{1,1})$ and $(v_{1,1}, v_{1,2}, v_{2,1}, v_{2,2}, v_{1,3}, v_{1,1})$. If $k := \lfloor \frac{e-3}{3} \rfloor$, then $e = 3k + r$, where $r \in [3,5]$, and the sequence $(3)^k(r) \in \mathrm{Sct}(G)$ has at least $k$ odd terms. Since $\mathrm{mcb}(G) = n_1 n_2$ (Proposition 5), the inequality $n_1 n_2 < k$ implies that $G$ is not ADCT. Let us show that $n_1 n_2 < k$. Indeed, this inequality is a consequence of the first from the following two equivalent inequalities: $3n_1 n_2 + 5 < e = n_1 n_2 + n_1 n_3 + n_2 n_3$, $5 < n_1(n_3 - n_2) + n_2(n_3 - n_1)$. Now, since $n_3 - n_1 \geqslant 2$

and $n_3 - n_2 \geqslant 0$, the latter inequality is true; note that if $n_2 = 2$, then $n_1 = 2$ and on the right-hand side we have $4(n_3 - n_1) \geqslant 8$. $\square$

**LEMMA 8.** *If $G$ is an even graph, then* $\mathrm{Lct}(G) \subseteq \big[3, |E(G)| - 3\big] \cup \big\{|E(G)|\big\}$.

P r o o f. Let $T$ be a closed trail in $G$. Clearly, $T$ has at least three edges (a subgraph of $G$ induced by $m$ edges, $m \in [1,2]$, has at least two vertices of degree one). If $E(T) \neq E(G)$, then $G - T$ is a nonempty even subgraph of $G$. As any component of $G - T$ has at least three edges, we have $\big|E(T)\big| = \big|E(G)\big| - \big|E(G-T)\big| \leqslant \big|E(G)\big| - 3$. $\square$

**PROPOSITION 9.** *Let $G$ be a tripartite graph with tripartition $\{V_1, V_2, V_3\}$ and let $T$ be a closed trail in $G$ such that there are $i, j \in [1,3]$, $i \neq j$, and $x, y \in V(T)$, satisfying $V(T) \cap V_i = \{x\}$ and $V(T) \cap V_j = \{y\}$. Then either $\big(|E(T)|\big)_4 = 3$ and $xy \in E(T)$ or $\big(|E(T)|\big)_4 = 0$.*

P r o o f. With $k := 6 - i - j$ we have $\{i,j,k\} = [1,3]$. If $z \in V(T) \cap V_k$, then $xz, yz \in E(T)$.

(1) If $xy \notin E(T)$, then $T$ is a closed trail in the bipartite graph with bipartition $\big\{\{x,y\}, V_k\big\}$ and $\big(|E(T)|\big)_4 = 0$ since the vertices $x$ and $y$ must alternate in $T$.

(2) If $xy \in E(T)$, then $T' := T - \{xy, yz, zx\}$ is an even graph and either $E(T') = \emptyset$ or $T'$ is connected. In the latter case we can proceed as in (1) with $T'$ instead of $T$. Therefore, in both cases $\big(|E(T')|\big)_4 = 0$ and $\big|E(T)\big| = \big|E(T')\big| + 3 \equiv 3 \pmod 4$. $\square$

**PROPOSITION 10.** *The graphs $K_{1,1,3}$ and $K_{1,1,5}$ are* ADCT.

P r o o f. Let $n \in \{3,5\}$ and let $V_1, V_2, V_3$ be parts of $K_{1,1,n}$ with $|V_1| = |V_2| = 1$. If $T$ is a closed trail in $K_{1,1,n}$, then clearly $V(T) \cap V_i = V_i$, $i = 1,2$. Therefore, by Proposition 9, $\big(|E(T)|\big)_4 \in \{0,3\}$. Further, if $T'$ is a closed trail in $K_{1,1,n}$ that is edge-disjoint with $T$, then $T + T' \neq \emptyset$. Thus, using edge-disjoint closed trails $(v_{1,1}, v_{1,2}, v_{1,3}, v_{1,1})$, $(v_{1,1}, v_{2,3}, v_{1,2}, v_{3,3}, v_{1,1})$ in $K_{1,1,3} \subseteq K_{1,1,5}$ and $(v_{1,1}, v_{4,3}, v_{1,2}, v_{5,3}, v_{1,1})$ in $K_{1,1,5}$, we can see that $\mathrm{Lct}(K_{1,1,3}) = \{3,4,7\}$, $\mathrm{Lct}(K_{1,1,5}) = \{3,4,7,8,11\}$ and, for both $n = 3,5$, any sequence from $\mathrm{Sct}(K_{1,1,n})$ is $K_{1,1,n}$-realisable. $\square$

## 3. The graph $K_{5,5,5}$ is ADCT

In the main theorem of our paper we shall prove by induction on $n$ that the graph $G_n := K_{(5\cdot 2^n)3}$, $n \in [0,\infty)$, is ADCT. For brevity we denote the part $\big\{v_{i,j} \colon i \in [1, 5\cdot 2^n]\big\}$, $j \in [1,3]$, of $K_{(5\cdot 2^n)3}$, by $V_{n,j}$. Consider three mutually isomorphic 16-edge graphs $F_1$ (full lines in Fig. 1), $F_2$ (short-dashed lines), $F_3$ (long-dashed lines), and put $F^j := \bigcup_{i=1}^{j} F_i$, $j = 1,2,3$; then $F^3 = K_{5,5,5} - K_{3,3,3}$. Let $i, j \in [1,3]$. The tripartition of $F_i$ is $\{V_i^3, V_{i+1}^1 \cup V_{i+1}^2, V_{i+2}^2\}$, where lower indices are taken modulo 3 in the set $[1,3]$ (such a convention will be used throughout the whole paper without explicitly mentioning it), $V_i^1 \subseteq V_i^3$, $V_i^2 \cup V_i^3 = V_{0,i}$ and $|V_i^j| = j$. The mapping $\iota_i \colon V(F_1) \to V(F_i)$, determined by $\iota_i(v_{j,k}) = v_{j,k-1+i}$ is a natural isomorphism from $F_1$ onto $F_i$ with $\iota_i(V_l^m) = V_{l-1+i}^m$ (for all four meaningful pairs $(l,m)$).

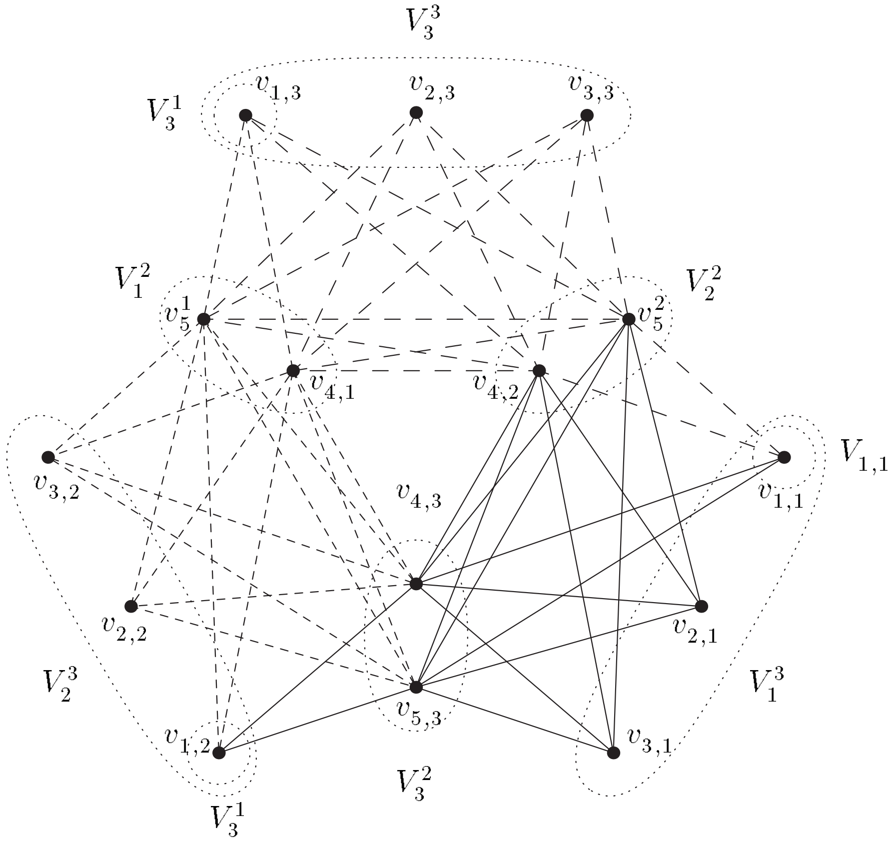


FIGURE 1. The graph $F^3$.

**PROPOSITION 11.** *If $j \in [1,3]$, then* $\mathrm{mcb}(F^j) = 4j$.

Proof. Putting $E_i := \{xy \in E(F_i) \colon x \in V_{i+1}^2, y \in V_{i+2}^2\}$, $i = 1,2,3$, it is easy to see that $\bigcup_{k=1}^{j} E_k \in \mathrm{Cb}(F^j)$, and so $\mathrm{mcb}(F^j) \leqslant 4j$. On the other hand, the sets $\{v_{2,1}, v_{4,2}, v_{4,3}\}$, $\{v_{2,1}, v_{5,2}, v_{5,3}\}$, $\{v_{3,1}, v_{4,2}, v_{5,3}\}$ and $\{v_{3,1}, v_{4,3}, v_{5,2}\}$ induce in $F_1$ four pairwise edge-disjoint $K_3$'s. Therefore, by Proposition 4, $\mathrm{mcb}(F^j) \geqslant j\,\mathrm{mcb}(F_1) \geqslant j \cdot 4\mathrm{mcb}(K_3) = 4j$. □

A closed trail $T$ in $F^3$ is said to be $F^3$-*extendable* if $V(T) \cap V^3 \neq \emptyset$, where $V^3 := V_1^3 \cup V_2^3 \cup V_3^3$. If $i \in [1,3]$, the graph $F_i - V^3$ is isomorphic to $C_4$, hence any closed trail in $F_i$ of length $\neq 4$ is $F^3$-extendable. An $F^3$-extendable closed trail $T$ is said to be $F^3$-*good* if $V(T) \cap V_{0,j} \neq \emptyset$, $j = 1,2,3$. Since a graph, induced in $G_0$ by two of its parts $V_{0,1}, V_{0,2}, V_{0,3}$, is bipartite, a closed trail in $F^3$ of an odd length is $F^3$-good. Moreover, from the structure of the graph $F_i$, $i \in [1,3]$, it is easy to see that any closed trail in $F_i$ of length $\neq 4$ is $F^3$-good.

For $i \in [1,3]$, a closed trail $T$ in $F_i$ is said to be $F_i$-*good* if $V(T) \cap V_i^3 \neq \emptyset$, $V(T) \cap V_{i+1}^2 \neq \emptyset$ and $V(T) \cap V_{i+2}^2 \neq \emptyset$. Evidently, an $F_i$-good closed trail in $F_i$ is also $F^3$-good.

Let $i \in [1,3]$, $l \in \mathrm{Lct}(F^3)$ and let $\mathcal{T}$ be a sequence of closed trails in $F^3$. Consider the following five conditions:

(C1.$i$) Any trail of $\mathcal{T}$ length $\neq 4$ is $F_i$-good.

(C2) Any trail of $\mathcal{T}$ is $F^3$-extendable.

(C3.$i.l$) $\mathcal{T}$ has an $l$-trail $T$ such that $V(T) \cap V_{i+1}^1 \neq \emptyset$, $V(T) \cap V_{i+1}^2 \neq \emptyset$ and $V(T) \cap V_{i+2}^2 \neq \emptyset$.

(C4) Any trail of $\mathcal{T}$ of length $\neq 4$ is $F^3$-good.

(C5.$l$) $\mathcal{T}$ has an $l$-trail $T$ such that either $V(T) \cap V_1^2 \neq \emptyset$ and $V(T) \cap V_3^1 \neq \emptyset$ or $V(T) \cap V_2^2 \neq \emptyset$.

For $i \in [1,3]$, an $F_i$-realisation $\mathcal{T}$ of a sequence $L \in \mathrm{Sct}(F_i)$ is said to be *good* [$l$-*good*] if $\mathcal{T}$ satisfies (C1.$i$) and (C2) [(C1.$i$), (C2) and (C3.$i.l$)].

**Theorem 12.** *Let $i \in [1,3]$, $L = (l_1, \dots, l_p) \in \mathrm{Sct}(F_i)$ and $j \in [1,p]$. Then the following holds:*

1. *There is a good $F_i$-realisation of $L$.*
2. *If $\mathrm{nd}(L) \neq (3)^4(4)$ and $l_j \geqslant 4$, there is an $l_j$-good $F_i$-realisation of $L$.*

P r o o f . We are able to prove the statement of our theorem for any sequence $L = (l_1, \dots, l_p)$ of integers from $[3,13] \cup \{16\}$ such that $\sum_{k=1}^{p} l_k = 16$. This in turn implies that $\mathrm{Lct}(F_i) = [3,13] \cup \{16\}$. We proceed in this way throughout the whole paper: when working with a graph $G$, we do not take care of the structure of $\mathrm{Lct}(G)$, but in all cases it turns out that $\mathrm{Lct}(G)$ is of maximal extent, i.e., $\mathrm{Lct}(G) = [3, |E(G)| - 3] \cup \{|E(G)|\}$ (cf. Lemma 8).

Using Lemma 1 we may suppose without loss of generality that $L$ is non-decreasing (i.e., $\mathrm{nd}(L) = L$). Examples of appropriate $F_1$-realisations of $L$ are presented at http://umv.science.upjs.sk/adct and can be seen by clicking on Graph $F_1$. To pass to $F_i$-realisations, $i \in [2,3]$, consider the isomorphism $\iota_i$. □

The next proposition shows that the exclusion of the sequence $(3)^4(4)$ in Theorem 12.2 is unavoidable.

**Proposition 13.** *If $i \in [1,3]$ and $T$ is a 4-trail of an $F_i$-realisation of $(3)^4(4)$, then $V(T) = V_i^1 \cup V_{i+1}^1 \cup V_{i+2}^2$.*

Proof. The statement follows from the fact that both $v_{1,i}$ and $v_{1,i+1}$ belong in $F_i$ only to trails of length $\geqslant 4$. $\square$

Let $L \in \mathrm{Sct}^*(F^2)$ and let $l$ be a term of $L$. An $F^2$-realisation $\mathcal{T}$ of $L$ is said to be *$l$-good* if $\mathcal{T}$ satisfies (C2), (C4) and (C5.$l$).

**Theorem 14.** *If $L = (l_1, \ldots, l_p) \in \mathrm{Sct}^*(F^2)$ and $i \in [1,p]$, there exists an $l_i$-good $F^2$-realisation of $L$.*

Proof. (1) Assume there is a decomposition $\{I^1, I^2\}$ of $[1,p]$ such that $\sum\limits_{k \in I^j} l_k = 16$, $j = 1,2$. We may suppose without loss of generality that $i \in I^1 \Leftrightarrow l_i = 3$. Let $\mathcal{T}^1$ be a good $F_1$-realisation of $L\langle I^1\rangle$ and, provided that $l_i \geqslant 4$ and $\mathrm{nd}(L\langle I^1\rangle) \neq (3)^4(4)$ $[l_i = 3$ or $\mathrm{nd}(L\langle I^1\rangle) = (3)^4(4)]$, let $\mathcal{T}^2$ be an $l_i$-good [a good] $F_2$-realisation of $L\langle I^2\rangle$; both realisations do exist by Theorem 12. Then $\mathcal{T} := \mathcal{T}^1\mathcal{T}^2$ is an $l_i$-good $F^2$-realisation of $L$. First note that any trail of $\mathcal{T}^j$, $j = 1,2$, satisfies (C2) and (C4) (as a consequence of (C1.$j$)). Further, $\mathcal{T}$ has an $l_i$-trail $T$ such that either $V(T) \cap V_1^2 \neq \emptyset$ and $V(T) \cap V_3^1 \neq \emptyset$ (if $l_i \geqslant 4$, see (C3.2.$l_i$) or Proposition 13) or $V(T) \cap V_2^2 \neq \emptyset$ (if $l_i = 3$, see (C1.1)).

(2) Now assume there is a decomposition $\{\{r\}, I^1, I^2\}$ of $[1,p]$ such that $\sum\limits_{k \in I^j} l_k \leqslant 13$, $j = 1,2$. Putting $l_r^j := 16 - \sum\limits_{k \in I^j} l_k$ and $L^j := L\langle I^j\rangle(l_r^j)$, we have $L^j \in \mathrm{Sct}(F_j)$, $j = 1,2$, and $l_r^1 + l_r^2 = 32 - \sum\limits_{k \in I^1 \cup I^2} l_k = l_r$. We may suppose without loss of generality that $i \neq r \Rightarrow (i \in I^1 \Leftrightarrow l_i = 3)$.

If $l_r^1 \geqslant 4$ and $\mathrm{nd}(L^1) \neq (3)^4(4)$ $[l_r^1 = 3$ or $\mathrm{nd}(L^1) = (3)^4(4)]$, let $\mathcal{T}^1$ be an $l_r^1$-good [a good] $F_1$-realisation of $L^1$.

If $i \in I^2$ and $\mathrm{nd}(L^2) \neq (3)^4(4)$ $[i = r$, $l_r^2 \geqslant 4$ and $\mathrm{nd}(L^2) \neq (3)^4(4)]$ {otherwise}, let $\mathcal{T}^2$ be an $l_i$-good [an $l_r^2$-good] {a good} $F_2$-realisation of $L^2$.

If $i \neq r$ and $l_i = 3$, let $T_i$ be any $l_i$-trail of $\mathcal{T}^1$; it satisfies $V(T_i) \cap V_2^2 \neq \emptyset$ (see (C1.1)). If $i \neq r$ and $l_i \geqslant 4$, let $T_i$ be an $l_i$-trail of $\mathcal{T}^2$ satisfying $V(T_i) \cap V_1^2 \neq \emptyset$ and $V(T_i) \cap V_3^1 \neq \emptyset$ (see (C3.2.$l_i$) or Proposition 13). Further, $\mathcal{T}^j$ contains an $l_r^j$-trail $T_r^j \neq T_i$ (if $T_i$ is defined at all, i.e., if $i \neq r$), $j = 1,2$, with the following two properties: either $V(T_r^1) \cap V_2^1 \neq \emptyset$ and $V(T_r^1) \cap V_3^2 \neq \emptyset$ (if $l_r^1 \geqslant 4$, see (C3.1.$l_r^1$) or Proposition 13) or $V(T_r^1) \cap V_3^2 \neq \emptyset$ (if $l_r^1 = 3$, see (C1.1)); either $V(T_r^2) \cap V_2^1 \neq \emptyset$ (if $\mathrm{nd}(L^2) = (3)^4(4)$, see Proposition 13) or $V(T_r^2) \cap V_3^2 \neq \emptyset$ (otherwise, see (C1.2) or (C3.2.$l_r^2$)).

We may suppose without loss of generality that $l_r^1 = 3 \Rightarrow \mathrm{nd}(L^2) \neq (3)^4(4)$ (otherwise, if $m \in I^2$ and $l_m = 3$, then $l_m + \sum\limits_{k \in I^1} l_k = 16$ and the case (1) applies). Therefore, we have either $V(T_r^1) \cap V(T_r^2) \supseteq V_2^1 \neq \emptyset$ or $V(T_r^j) \cap V_3^2 \neq \emptyset$,

$j = 1, 2$. In the latter case, since $V_3^2$ is a similarity class in $F_1$, we may suppose without loss of generality that $V(T_r^1) \cap V(T_r^2) \neq \emptyset$. Thus, in both cases there is a trail $T_r \in T_r^1 + T_r^2$.

Denote by $\hat{\mathcal{T}}^j$ the sequence obtained by deleting $T_r^j$ from $\mathcal{T}^j$, $j = 1, 2$. Then $\mathcal{T} := \hat{\mathcal{T}}^1 \hat{\mathcal{T}}^2 (T_r)$ is an $l_i$-good $F^2$-realisation of $L\langle I^1 \rangle L \langle I^2 \rangle (l_r) \sim L$. First, the only trail of $\mathcal{T}$, that is neither in $\mathcal{T}^1$ nor in $\mathcal{T}^2$, is an $F^3$-extendable trail $T_r$ (of length $\geq 6$) with $V(T_r) \supseteq V(T_r^j)$, $j = 1, 2$. If $T_r^1$ is $F_1$-good, it is also $F^3$-good, hence so is $T_r$. In the opposite case $l_r^1 = 4$, $\mathrm{nd}(L^1) = (3)^4(4)$ and, by Proposition 13, both $T_r^1$ and $T_r$ are $F^3$-good. If $i \neq r$, the $l_i$-trail $T_i$ satisfies (C5.$l_i$). Finally, if $i = r$, the $l_r$-trail $T_r$ satisfies (C5.$l_r$): with $l_r^2 \geqslant 4$ we have $V(T_r) \cap V_1^2 \supseteq V(T_r^2) \cap V_1^2 \neq \emptyset$ and $V(T_r) \cap V_3^1 \supseteq V(T_r^2) \cap V_3^1 \neq \emptyset$ (by (C3.2.$l_r^2$) or Proposition 13), while $l_r^2 = 3$ implies $V(T_r) \cap V_2^2 \supseteq V(T_r^1) \cap V_2^2 \neq \emptyset$ (by (C1.1) or (C3.$l_r^1$); note that here we may suppose without loss of generality that $\mathrm{nd}(L^1) \neq (3)^4(4)$).

(3) In what follows, we suppose that the sequence $L$ is nondecreasing and the assumptions of (1) and (2) are not fulfilled. Let $q \in [1, p]$ be such that $\sum\limits_{i=1}^{q-1} l_i \leqslant 13$ and $\sum\limits_{i=1}^{q} l_i > 13$. Then $\sum\limits_{i=1}^{q} l_i \in \{14, 15, 17, 18\}$ (if $\sum\limits_{i=1}^{q} l_i \geqslant 19$, then $\sum\limits_{i=q+1}^{p} l_i \leqslant 13$ and (2) is fulfilled with $I^1 := [1, q-1]$ and $I^2 := [q+1, p]$). Let $M_k$ be the set of all nondecreasing sequences with terms from $[3, \infty)$ adding up to $k$ and let $S_k$ be the set of all nondecreasing sequences $L = (l_1, \dots, l_p) \in \mathrm{Sct}^*(F^2)$ such that $\sum\limits_{i=1}^{q} l_i = k$ for some $q \in [1, p-1]$ and $L$ violates the assumptions of both (1) and (2). We are going to determine the structure of $S_k$, $k = 14, 15, 17, 18$.

(31) If $L \in S_{14}$, then $l_j \in \bigcap\limits_{i=1}^{q} \{l_i, l_i + 1, l_i + 3, l_i + 4\}$ for any $j \in [q+1, p] \neq \emptyset$. Indeed, if $l_j = l_i + 2$ for some $i \in [1, q]$ and $j \in [q+1, p]$, then (1) is fulfilled with $I^1 := [1, q] - \{i\} \cup \{j\}$ and $I^2 := [q+1, p] - \{j\} \cup \{i\}$. If $l_j \geqslant l_i + 5$ for some $i \in [1, q]$ and $j \in [q+1, p]$, then with $I^1 := [2, q]$, $I^2 := \{1\} \cup [q+1, p] - \{j\}$, we have $\sum\limits_{k \in I^1} l_k \leqslant 11$ and $\sum\limits_{k \in I^2} l_k = 18 - (l_j - l_1) \leqslant 13$. By analysing all sequences $(l_1, \dots, l_q)$ from $M_{14}$ such that $l_q - l_1 \leqslant 4$ (the above intersection must be nonempty), we obtain $S_{14} = \big\{(3)^3(5)(6)^3,\ (3)^2(4)^3(7)^2,\ (3,5)(6)^4,\ (5)(9)^3,\ (6,8)(9)^2,\ (7)^3(11),\ (7)^2(8,10),\ (14,18)\big\}$. As (somewhat typical) examples of this analysis consider the following:

(a) If $(l_1, l_2, l_3, l_4) = (3)^3(5)$, then $l_j \in \{3, 4, 6, 7\} \cap \{5, 6, 8, 9\} = \{6\}$ for any $j \in [5, p]$, hence $p = 7$ and $L = (3)^3(5)(6)^3$.

(b) If $(l_1, l_2, l_3) = (4)^2(6)$, then $l_j \in \{4, 5, 7, 8\} \cap \{6, 7, 9, 10\} = \{7\}$ for any $j \in [4, p]$, and, since $7 \nmid 18$, there is no such $L$.

(c) If $(l_1, l_2) = (7)^2$, then $l_j \in \{7, 8, 10, 11\}$ for any $j \in [3, p]$, hence $p = 4$ and $(l_3, l_4)$ is either $(7, 11)$ or $(8, 10)$, which yields $L = (7)^3(11)$ or $L = (7)^2(8, 10)$.

(32) If $L \in S_{15}$, then $l_j \in \bigcap_{i=1}^{q} \{l_i, l_i + 2, l_i + 3\}$ for any $j \in [q+1, p]$, hence $l_q - l_1 \leqslant 3$. In the same way as above we obtain the set $S_{15} = \{(3)^7(5,6), (3)^5(5)(6)^2, (5)^5(7), (15,17)\}$; notice that $(3)^9(5) \notin \mathrm{Sct}^*(F^2)$ and, consequently, $(3)^9(5) \notin S_{15}$.

(33) If $L \in S_{17}$, then $l_q = 17 - \sum_{i=1}^{q-1} l_i \geqslant 4$. Further, $l_j \in \{l_q, l_q + 1\}$ for any $j \in [q+1, p]$, otherwise with $I^1 := [1, q-1]$ and $I^2 := [q, p] - \{j\}$ we have $\sum_{i \in I^1} l_i \leqslant 13$ and $\sum_{i \in I^2} l_i = 15 - (l_j - l_q) \leqslant 13$. We can easily find that $S_{17} = \{(3)^4(5)^4, (3)^2(4)(7)^2(8), (3,4)(5)^5, (4)^3(5)^4, (3)(7)^3(8), (4,6)(7)^2(8), (5)^2(7)^2(8), (3,14,15)\}$.

(34) If $L \in S_{18}$, then $l_q \geqslant 5$ and $l_j = l_q$ for any $j \in [q+1, p]$, for otherwise $\sum_{i=q}^{p-1} l_i \leqslant 13$. Thus $S_{18} = \{(3)^2(5)(7)^3, (3)(4)^2(7)^3, (4)(7)^4, (5,6)(7)^3, (4)(14)^2\}$.

Examples of $l_i$-good $F^2$-realisations of sequences $L \in S_{14} \cup S_{15} \cup S_{17} \cup S_{18}$ can be seen by clicking on Graph $F^2$ (see the address presented in the proof of Theorem 12). $\square$

An $F^3$-realisation $\mathcal{T}$ of a sequence $L \in \mathrm{Sct}^*(F^3)$ is said to be *good* if $\mathcal{T}$ satisfies (C2) and (C4).

**THEOREM 15.** *If $L \in \mathrm{Sct}^*(F^3)$, there exists a good $F^3$-realisation of $L$.*

P r o o f. Let $L = (l_1, \ldots, l_p)$.

(1) Assume there is a decomposition $\{I^1, I^2\}$ of $[1, p]$ such that $\sum_{i \in I^1} l_i = 16$, $\sum_{i \in I^2} l_i = 32$, and $\mathrm{nd}(L\langle I^2 \rangle) \neq (3)^9(5)$ (so that $L\langle I^2 \rangle \in \mathrm{Sct}^*(F^2)$). Consider an arbitrary $i \in I^2$. By Theorems 12 and 14 there exists a good $F_3$-realisation $\mathcal{T}^1$ of $L\langle I^1 \rangle$ and an $l_i$-good $F^2$-realisation $\mathcal{T}^2$ of $L\langle I^2 \rangle$. Then $\mathcal{T}^1\mathcal{T}^2$ is a good $F^3$-realisation of $L\langle I^1 \rangle L\langle I^2 \rangle \sim L$.

(2) Let there exist a decomposition $\{\{r\}, I^1, I^2\}$ of $[1, p]$ such that $\sum_{i \in I^1} l_i \leqslant 13$, $\sum_{i \in I^2} l_i \leqslant 29$ and $\mathrm{nd}(L\langle I^2 \rangle) \notin \{(3)^8(5), (3)^9\}$. If $l_r^j := 16j - \sum_{i \in I^j} l_i$, $j = 1, 2$, then $l_r^1 + l_r^2 = 48 - \sum_{i \in I^1 \cup I^2} l_i = l_r$. Put $L^j := L\langle I^j \rangle (l_r^j)$, $j = 1, 2$. If $l_r^1 \geqslant 4$ and

$\mathrm{nd}(L^1) \neq (3)^4(4)$ $\left[l_r^1 = 3 \text{ or } \mathrm{nd}(L^1) = (3)^4(4)\right]$, let $\mathcal{T}^1$ be an $l_r^1$-good [a good] $F_3$-realisation of $L^1$. Then $\mathcal{T}^1$ has an $l_r^1$-trail $T_r^1$ such that $V(T_r^1) \cap V_2^2 \neq \emptyset$ and either $V(T_r^1) \cap V_3^1 \neq \emptyset$ (if $\mathrm{nd}(L^1) = (3)^4(4)$) or $V(T_r^1) \cap V_1^2 \neq \emptyset$ (otherwise). Further, let $\mathcal{T}^2$ be an $l_r^2$-good $F^2$-realisation of $L^2$ and let $T_r^2$ be an $l_r^2$-trail of $\mathcal{T}^2$ satisfying (C5.$l_r^2$). Then either $V(T_r^1) \cap V(T_r^2) \supseteq V_3^1 \neq \emptyset$ or there is $k \in [1,2]$ such that $V(T_r^j) \cap V_k^2 \neq \emptyset$, $j = 1,2$. In the latter case, as both $V_1^2$ and $V_2^2$ are similarity classes in $F_3$, we may suppose without loss of generality that $V(T_r^1) \cap V(T_r^2) \neq \emptyset$. Thus, in any case there is a trail $T_r \in T_r^1 + T_r^2$. If $\hat{\mathcal{T}}^j$ results from $\mathcal{T}^j$ by deleting $T_r^j$, $j = 1,2$, then $\hat{\mathcal{T}}^1\hat{\mathcal{T}}^2(T_r)$ is a good $F^3$-realisation of $L\langle I^1\rangle L\langle I^2\rangle(l_r) \sim L$.

(3) Suppose that $L$ is nondecreasing and violates the assumptions of both (1) and (2). Let $q \in [1,p]$ be defined by the inequalities $\sum_{i=1}^{q-1} l_i \leqslant 13$ and $s := \sum_{i=1}^{q} l_i > 13$. Then $s \neq 16$, for otherwise $l_i \geqslant l_q \geqslant 4$ for any $i \in [q+1,p]$, and with $I^1 := [1,q]$ and $I^2 := [q+1,p]$ we are in the case (1). Also, $s \leqslant 18$, since $s \geqslant 19$ leads to $l_i \geqslant l_q \geqslant 6$ for any $i \in [q+1,p]$, and then with $I_1 := [1,q-1]$ and $I_2 := [q+1,p]$ we are in the case (2). Thus, $s \in \{14,15,17,18\}$ and, similarly as in the proof of Theorem 14, we conclude that $L$ is one of the following sequences: $(3)^{12}(6)^2$, $(3)^{10}(6)^3$, $(3)^8(6)^4$, $(3)^6(6)^5$, $(3)^4(6)^6$, $(3)^2(6)^7$, $(3)(5)^9$, $(4)^2(5)^8$, $(5)^8(8)$, $(6)^8$, $(14)(17)^2$; for good $F^3$-realisations of these sequences a reader is referred to Graph $F^3$. □

Put $P := \{(0,0),\ (0,3),\ (3,0)\}$.

**PROPOSITION 16.** *Let $n \in [2,\infty)$ and let $T_1$, $T_2$ be edge-disjoint closed trails in the graph $K_{n,n,n}$ with the tripartition $\{V_1, V_2, V_3\}$. Then the following holds:*

1. *There is $p \in [1,3]$ such that $\{V(T_1), V(T_2)\}$ has a system of distinct representatives in $V_p$.*

2. *If there is $m \in [1,2]$ such that $V(T_m) \cap V_i \neq \emptyset$, $i = 1,2,3$, and $\left(\left(|E(T_1)|\right)_4, \left(|E(T_2)|\right)_4\right) \notin P$, then there are $q, r \in [1,3]$, $q \neq r$, such that $\{V(T_1), V(T_2)\}$ has a system of distinct representatives in both $V_q$ and $V_r$.*

P r o o f. Set $c_i := \left(|E(T_i)|\right)_4$ and let $\rho_i\colon [1,3] \to [1,3]$ be a bijection satisfying $x(i,1) \geqslant x(i,2) \geqslant x(i,3)$, where $x(i,j) := \left|V(T_i) \cap V_{\rho_i(j)}\right|$, $i = 1,2$, $j = 1,2,3$. Clearly, $x(i,2) \geqslant 1$, $i = 1,2$.

1. If there is $i \in [1,2]$ such that $x(i,2) \geqslant 2$, then there exists $j \in [1,2]$ such that $V(T_{3-i}) \cap V_{\rho_i(j)} \neq \emptyset$, and we are done with $p := \rho_i(j)$. In the opposite case $x(i,2) = x(i,3) = 1$, $i = 1,2$. Let $i \in [1,2]$ be such that $x(i,1) = \max\bigl(x(1,1), x(2,1)\bigr)$. If $x(i,1) \geqslant 2$, then, since $V(T_{3-i}) \cap V_{\rho_i(1)} \neq \emptyset$, we can take $p := \rho_i(1)$. So, suppose that $x(k,j) = 1$, $k = 1,2$, $j = 1,2,3$. As $E(T_1) \cap E(T_2) = \emptyset$, we have $V(T_1) \neq V(T_2)$, and the existence of $p$ follows.

2. Here we have $x(m,3) \geqslant 1$. If $x(3-m,3) = 0$, then $T_{3-m}$ is a bipartite graph, $x(3-m,2) \geqslant 2$ and we are done with $q := \rho_{3-m}(1)$ and $r := \rho_{3-m}(2)$. So, suppose that $x(3-m,3) \geqslant 1$ and let $\rho\colon [1,3] \to [1,3]$ be a bijection satisfying $x(1) \geqslant x(2) \geqslant x(3)$, where $x(j) := |X_j|$ and $X_j := \big(V(T_1) \cup V(T_2)\big) \cap V_{\rho(j)}$, $j = 1,2,3$. If $x(2) = x(3) = 1$, then $X_j = V(T_1) \cap V_{\rho(j)} = V(T_2) \cap V_{\rho(j)} = \{x_j\}$, $j = 2,3$. From Proposition 9 it follows that $c_i \in \{0,3\}$ and $c_i = 3 \Rightarrow x_2x_3 \in E(T_i)$, $i = 1,2$. Since $E(T_1) \cap E(T_2) = \emptyset$, we obtain $(c_1, c_2) \in P$, a contradiction. Thus, $x(2) \geqslant 2$, and it suffices to take $q := \rho(1)$ and $r := \rho(2)$. $\square$

If $l \in [3,5]$, then any closed trail of length $l$ is in fact a cycle. Let us identify all $2l$ closed trails of length $l$ in $K_{3,3,3}$ having the same edge set and denote by $\mathcal{C}_l$ the set of all (representatives of) closed trails of length $l$ in $K_{3,3,3}$. Then $c_l := |\mathcal{C}_l|$ is equal to the number of $l$-element subsets of $E(K_{3,3,3})$ inducing a cycle of length $l$.

Let $T$ be a closed trail of length $l \in [3,5]$, let $V_i := V(T) \cap \{v_{1,i}, v_{2,i}, v_{3,i}\}$ and $v_i := |V_i|$, $i = 1,2,3$. Let $i,j,k \in [1,3]$ be such that $\{i,j,k\} = [1,3]$ and $v_i \leqslant v_j \leqslant v_k$. If $l \in \{3,5\}$, $T$ is a non-bipartite subgraph of $K_{3,3,3}$, therefore $v_i \geqslant 1$. If $l = 3$, then $v_1 = v_2 = v_3 = 1$, and so $c_3 = \binom{3}{1}^3 = 27$. If $l = 4$, then either $v_i = 0$ and $v_j = v_k = 2$ or $v_i = v_j = 1$ and $v_k = 2$, so that $c_4 = 3 \cdot \binom{3}{2}^2 + 3 \cdot \binom{3}{1}^2 \cdot \binom{3}{2} = 108$. If $l = 5$, then from Proposition 9 it follows that $v_i = 1$, $v_j = v_k = 2$. The edge set of $T$ is uniquely determined by a simple sequence $\prod_{i=1}^{5}(x_i)$ such that $x_1, x_3 \in V_j$, $x_2, x_4 \in V_k$ and $x_5 \in V_i$. Therefore, $c_5 = 3 \cdot (3 \cdot 3 \cdot 2 \cdot 2 \cdot 3) = 324$.

A $K_{3,3,3}$-realisation $(T_1, \ldots, T_p)$ of a sequence $(l_1, \ldots, l_p) \in \mathrm{Sct}(K_{3,3,3})$ is said to be *good* if, for any $i \in [1,p]$, $E(T_i)$ is a union of edge sets of some (edge-disjoint) trails from $\mathcal{C}_3 \cup \mathcal{C}_4 \cup \mathcal{C}_5$.

**Theorem 17.** *If $L = (l_1, \ldots, l_p) \in \mathrm{Sct}(K_{3,3,3})$, there exists a good $K_{3,3,3}$-realization of $L$.*

P r o o f. Good $K_{3,3,3}$-realisations of sequences from $\mathrm{Sct}(K_{3,3,3})$ have been found by a computer; to see them, click on Graph $K_{3,3,3}$. $\square$

If $n \in [0,\infty)$, we have $e_n := \big|E(G_n)\big| = 3 \cdot (5 \cdot 2^n)^2$. A closed trail $T$ in $G_n$ is said to be $G_n$-*good* if $V(T) \cap V_{n,i} \neq \emptyset$, $i = 1,2,3$. Clearly, if $T$ is of an odd length, it is $G_n$-good. A $G_n$-realisation $\mathcal{T}$ of a sequence from $\mathrm{Sct}(G_n)$ is said to be *good* if any trail of $\mathcal{T}$ of length $\not\equiv 0 \pmod 4$ is $G_n$-good. The graph $G_n$ is said to be *strongly* ADCT provided that (i) for any $L \in \mathrm{Sct}(G_n)$ there is a good $G_n$-realisation of $L$ and (ii) if $(t,1)$, $t \in [1,5]$, is a position of Table 1 (regarded as a $5 \times 2$ matrix) containing a sequence $L = (l_1, \ldots, l_p) \in \mathrm{Sct}(G_n)$ ($n$ has to be in a specified congruence class modulo 3 such that the exponent

TAB. 1. Required properties of $G_n$-realisations.

| | |
|---|---|
| $(3)^4(7)^{\frac{e_n-12}{7}}$ | $v_{i,1}\in V(T_i)$, $i=1,2,3$ |
| $(3)^3(4)(7)^{\frac{e_n-13}{7}}$ | $v_{i,1}\in V(T_i)$, $i=1,2,3$ |
| $(4)^6(7)^{\frac{e_n-24}{7}}$ | $v_{i,1}\in V(T_i)$, $i=1,2,3$, $v_{i,2}\in V(T_{3+i})$, $i=1,2$, $v_{1,3}\in V(T_6)$ |
| $(4)^5(7)^{\frac{e_n-20}{7}}$ | $v_{i,1}\in V(T_i)$, $i=1,2,3$, $v_{i,2}\in V(T_{3+i})$, $i=1,2$ |
| $(4)^3(7)^{\frac{e_n-12}{7}}$ | $v_{i,1}\in V(T_i)$, $i=1,2,3$ |

of (7) is an integer), there is a good $G_n$-realisation $(T_1,\dots,T_p)$ of $L$ satisfying the conditions presented in the position $(t,2)$ of Table 1.

**THEOREM 18.** *The graph $G_0$ is strongly* ADCT.

Proof. Consider a sequence $L=(l_1,\dots,l_p)\in \mathrm{Sct}(G_0)$.

(1) Suppose there is a decomposition $\{I^1,I^2\}$ of $[1,p]$ such that $\sum\limits_{i\in I^1} l_i=48$ and $\sum\limits_{i\in I^2} l_i=27$. If, moreover, $L\langle I^1\rangle\in \mathrm{Sct}^*(F^3)$, then, by Theorem 15, there exists a good $F^3$-realisation $\mathcal{T}^1$ of $L\langle I^1\rangle$ and, by Theorem 17, there exists a good $K_{3,3,3}$-realisation $\mathcal{T}^2$ of $L\langle I^2\rangle$. The above additional assumption is true, if $f_3(L)\leqslant 8$, since in such a case $L\langle I^1\rangle$ contains at most twelve odd terms (note that $\lfloor\frac{48-8\cdot 3}{5}\rfloor=4$ and, by Proposition 11, $\mathrm{mcb}(F^3)=12$). Then $\mathcal{T}^1\mathcal{T}^2$ is a good $G_0$-realisation of the sequence $L\langle I^1\rangle L\langle I^2\rangle$. Indeed, trails of $\mathcal{T}^1$ of length $\neq 4$ are $G_0$-good since they are $F^3$-good, and trails of $\mathcal{T}^2$ of length $\not\equiv 0 \pmod 4$ are $G_0$-good as they are composed of at least one trail of length 3 or 5. So, we may suppose that $f_3(L)\geq 9$ and choose $I^2$ so that $L\langle I^2\rangle=(3)^9$. If, additionally $L\langle I^1\rangle\in \mathrm{Sct}(F_3)-\mathrm{Sct}^*(F_3)$, then $L\langle I^1\rangle$ has more than 12 odd terms and it is easy to see that for $\mathrm{nd}(L)$ we have only six possibilities, namely $(3)^{25}$, $(3)^{23}(6)$, $(3)^{22}(4,5)$, $(3)^{22}(9)$, $(3)^{21}(5,7)$ and $(3)^{20}(5)^3$. Examples of corresponding good $G_0$-realisations are available by clicking on Graph $G_0$.

(2) Let there be a decomposition $\{\{r\},I^1,I^2\}$ of $[1,p]$ such that $\sum\limits_{i\in I^1} l_i\leqslant 45$, $\sum\limits_{i\in I^2} l_i\leqslant 24$. If $l_r^1:=48-\sum\limits_{i\in I^1} l_i$ and $l_r^2:=27-\sum\limits_{i\in I^2} l_i$, then $l_r=l_r^1+l_r^2$. We may suppose that $f_3(L)\leqslant 8$ (otherwise we are in the case (1)), and then $L^1:=L\langle I^1\rangle(l_r^1)\in \mathrm{Sct}^*(F^3)$. By Theorem 15 there exists a good $F^3$-realisation $\mathcal{T}^1(T_r^1)$ of $L^1$ with $V(T_r^1)\cap V(K_{3,3,3})=V(T_r^1)\cap V^3\neq\emptyset$ ($T_r^1$ is $F^3$-extendable). Further, by Theorem 17 there is a $K_{3,3,3}$-realisation $\mathcal{T}^2(T_r^2)$ of the sequence $L^2:=\langle I^2\rangle(l_r^2)\in \mathrm{Sct}(K_{3,3,3})$. By Proposition 6 we may suppose without loss of generality that $V(T_r^1)\cap V(T_r^2)\neq\emptyset$, hence there is a trail $T_r\in T_r^1+T_r^2$. If $(l_r)_4\neq$

$0$, then $(l_r^j)_4 \neq 0$ for some $j \in [1,2]$, and so $V(T_r) \cap V_{0,i} \supseteq V(T_r^j) \cap V_{0,i} \neq \emptyset$, $i = 1,2,3$. Thus, $\mathcal{T}^1\mathcal{T}^2(T_r)$ is a good $G_0$-realisation of $L\langle I^1\rangle L\langle I^2\rangle(l_r) \sim L$.

(3) Now suppose that $L$ is a nondecreasing sequence violating the assumptions of both (1) and (2). Let $q \in [1,p]$ be defined by the inequalities $\sum_{i=1}^{q-1} l_i \leqslant 45$ and $\sum_{i=1}^{q} l_i > 45$. Then $\sum_{i=1}^{q} l_i \in \{46, 47, 49, 50\}$. Proceeding analogously as in the proof of Theorem 14, we exhibit five possibilities for $L$, namely $(4)^3(7)^9$, $(4)(7)^9(8)$, $(5)^{15}$, $(5)(7)^{10}$, $(25)^3$. For examples of remaining appropriate $G_0$-realisations see again Graph $G_0$ (note that for $n = 0$ the “real” sequences of Table 1 are $(4)^3(7)^9$ and $(3)^4(7)^9$). $\square$

## 4. The main theorem

Our main Theorem will be proved by induction on $n$. Therefore, we need to know how to construct $G_{n+1}$ from $G_n$. Consider two copies $G_n^1$, $G_n^2$ of the graph $G_n$ such that the tripartition of $G_n^k$ is $\{V_{n,j}^k : j \in [1,3]\}$ with $V_{n,j}^k = \{v_{i,j}^k : i \in [1, 5\cdot 2^n]\}$, where $v_{i,1}^1 = v_{i,1}^2$ for any $i \in [1, 5\cdot 2^n]$ and $V_{n,j_1}^1 \cap V_{n,j_2}^2 = \emptyset$ for any $j_1, j_2 \in [2,3]$. Then $\varphi_n^k : V(G_n) \to V(G_n^k)$, with $\varphi_n^k(v_{i,j}) = v_{i,j}^k$ for any $i \in [1, 5\cdot 2^n]$ and $k \in [1,3]$, is a natural isomorphism from $G_n$ onto $G_n^k$, $k = 1,2$. Let $H_n := G_n^1 \cup G_n^2$ (see Fig. 2); the subsets $V_{n,j}^k, j \in [2,3], k \in [1,2]$, are called *eccentric parts* of $H_n$, while $V_{n,1}^1 = V_{n,1}^2$ is the *central part* of $H_n$. The graph $H_n$ is tripartite with one possible tripartition $\{W_{n,1}, W_{n,2}, W_{n,3}\}$, where $W_{n,1} := V_{n,1}^1$ and $W_{n,j} := V_{n,j}^1 \cup V_{n,j}^2$, $j = 2,3$.

Suppose that $L = (l_1, \dots, l_p) \in \mathrm{Sct}(H_n)$ is an $H_n$-realisable sequence and $\mathcal{T} = (T_1, \dots, T_p)$ is an $H_n$-realisation of $L$. Consider two terms $l, l'$ of $L$. The $H_n$-realisation $\mathcal{T}$ of $L$ is said to be *$(l,l')$-global* if there is an $l$-trail $T_i$ and an $l'$-trail $T_j$ of $\mathcal{T}$, $i \neq j$, such that $V(T_i) \cap (V_{n,2}^k \cup V_{n,3}^k) \neq \emptyset$ and $V(T_j) \cap (V_{n,2}^{3-k} \cup V_{n,3}^{3-k}) \neq \emptyset$ for some $k \in [1,2]$ (or, equivalently, $V(T_i) \cup V(T_j) \nsubseteq V(G_n^k)$, $k = 1,2$).

**Proposition 19.** *Let $n \in [0,\infty)$, $m \in [1, 5\cdot 2^n]$, and let $v = (v_1, \dots, v_m)$, $w = (w_1, \dots, w_m)$ be simple sequences of vertices of the graph $H_n$. Then $v$ is similar to $w$ whenever one of the following conditions is fulfilled:*

1. *$\{v_i : i \in [1,m]\} \subseteq V_{n,j_1}^{k_1}$ and $\{w_i : i \in [1,m]\} \subseteq V_{n,j_2}^{k_2}$ for some $j_1, j_2 \in [2,3]$ and $k_1, k_2 \in [1,2]$;*
2. *$m = 2$, $v_1 \in V_{n,j_1}^{k_1}$, $v_2 \in V_{n,5-j_1}^{k_1}$, $w_1 \in V_{n,j_2}^{k_2}$ and $w_2 \in V_{n,5-j_2}^{k_2}$ for some $j_1, j_2 \in [2,3]$, $k_1, k_2 \in [1,2]$;*
3. *$m = 2$, $v_1 \in V_{n,j_1}^{k_1}$, $v_2 \in V_{n,j_1}^{3-k_1}$, $w_1 \in V_{n,j_2}^{k_2}$ and $w_2 \in V_{n,j_2}^{3-k_2}$ for some $j_1, j_2 \in [2,3]$, $k_1, k_2 \in [1,2]$;*

4. $m = 2$, $v_1 \in V^{k_1}_{n,j_1}$, $v_2 \in V^{3-k_1}_{n,5-j_1}$, $w_1 \in V^{k_2}_{n,j_2}$ *and* $w_2 \in V^{3-k_2}_{n,5-j_2}$ *for some* $j_1, j_2 \in [2,3]$, $k_1, k_2 \in [1,2]$.

Proof. There is a bijection $\varphi \colon V(H_n) \to V(H_n)$ such that $\varphi(V^1_{n,1}) = V^1_{n,1}$, any eccentric part of $H_n$ is mapped under $\varphi$ to an eccentric part of $H_n$, $\varphi(V^k_{n,2} \cup V^k_{n,3}) \in \{V^1_{n,2} \cup V^1_{n,3}, V^2_{n,2} \cup V^2_{n,3}\}$, $k = 1,2$, and $\varphi(v_i) = w_i$ for any $i \in [1,m]$; clearly, $\varphi$ is an automorphism of $H$. □

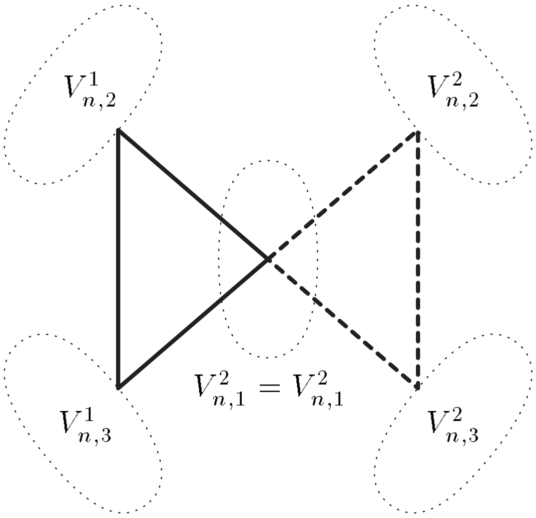


FIGURE 2. The graph $H^n$.

Now consider two copies $H^1_n$, $H^2_n$ of the graph $H_n$ such that, for both $l = 1,2$, $H^l_n$ has parts $V^{k,l}_{n,j} = \{v^{k,l}_{i,j} : i \in [1, 5 \cdot 2^n]\}$, $j = 1,2,3$, $k = 1,2$, where $V^{1,l}_{n,1} = V^{2,l}_{n,1}$, with $v^{1,l}_{i,1} = v^{2,l}_{i,1}$ for any $i \in [1, 5\cdot 2^n]$, is the central part, $V^{1,1}_{n,1} \cap V^{1,2}_{n,1} = \emptyset$, and eccentric parts are chosen so that $v^{k,1}_{i,2} = v^{k,2}_{i,2}$ and $v^{k,1}_{i,3} = v^{3-k,2}_{i,3}$ for any $i \in [1, 5\cdot 2^n]$ and $k \in [1,2]$. Then $\psi^l_n \colon V(H_n) \to V(H^l_n)$, with $\psi^l_n(v^k_{i,j}) = v^{k,l}_{i,j}$ for any $i \in [1, 5\cdot 2^n]$, $j \in [1,3]$ and $k \in [1,2]$, is a natural isomorphism from $H_n$ onto $H^l_n$, $l = 1,2$. In the graph $H^l_n$ the eccentric part $V^{k,l}_{n,2}$ is "joined" to the eccentric part $V^{k,l}_{n,3}$, $k = 1,2$ (see Fig. 3). Clearly, the graph $H^1_n \cup H^2_n$ is isomorphic to $G_{n+1}$. We shall suppose that if $i \in [1, 5.2^n]$, then $v^{1,1}_{i,j} = v_{i,j}$ (recall the notation of vertices of the graph $G_n = K_{(5\cdot 2^n)3}$) for any $j \in [1,3]$, $v^{1,2}_{i,1} = v_{5\cdot 2^n+i,1}$ and $v^{2,1}_{i,j} = v_{5\cdot 2^n+i,j}$ for any $j \in [2,3]$.

Let $n \in [0,\infty)$. A closed trail $T$ in $H_n$ is said to be $H_n$-*good* if $V(T) \cap W_{n,i} \neq \emptyset$, $i = 1,2,3$. Evidently, if $T$ is of an odd length, it is $H_n$-good. An $H_n$-realisation $\mathcal{T}$ of a sequence from $\mathrm{Sct}(H_n)$ is said to be *good* if any trail of $\mathcal{T}$ of length $\not\equiv 0 \pmod 4$ is $H_n$-good. The graph $H_n$ is said to be *strongly*

ADCT provided that (i) for any $L \in \mathrm{Sct}(H_n)$ there is a good $H_n$-realisation of $L$ and (ii) if $t \in [1,8]$ and $L = (l_1, \dots, l_p) \in \mathrm{Sct}(H_n)$ is "a real $(t,1)$-sequence" of Table 2 (so that the exponent of (7) is an integer), there is a good $H_n$-realisation $(T_1, \dots, T_p)$ of $L$ satisfying "$(t,2)$-conditions" of Table 2.

TAB. 2. Required properties of $H_n$-realisations.

| | |
|---|---|
| $(3)^4(7)^{\frac{2e_n-12}{7}}$ | $v^1_{i,2} \in V(T_i)$, $i = 1,2,3$ |
| $(3)^2(4)(7)^{\frac{2e_n-10}{7}}$ | $v^1_{i,2} \in V(T_i)$, $i = 1,2$ |
| $(3)^2(6)(7)^{\frac{2e_n-12}{7}}$ | $v^1_{i,2} \in V(T_i)$, $i = 1,2$ |
| $(3)^2(7)^{\frac{2e_n-6}{7}}$ | $v^1_{i,2} \in V(T_i)$, $i = 1,2$ |
| $(4)^6(7)^{\frac{2e_n-24}{7}}$ | $v^1_{i,2} \in V(T_i)$, $v^2_{i,3} \in V(T_{3+i})$, $i = 1,2,3$ |
| $(4)^5(7)^{\frac{2e_n-20}{7}}$ | $v^1_{i,2} \in V(T_i)$, $i = 1,2,3$, $v^1_{i,3} \in V(T_{3+i})$, $i = 1,2$ |
| $(4)^3(7)^{\frac{2e_n-12}{7}}$ | $v^1_{i,2} \in V(T_i)$, $i = 1,2,3$ |
| $(4)^3(7)^{\frac{2e_n-12}{7}}$ | $v^1_{i,2} \in V(T_i)$, $i = 1,2$, $v^1_{1,1} \in V(T_3)$ |

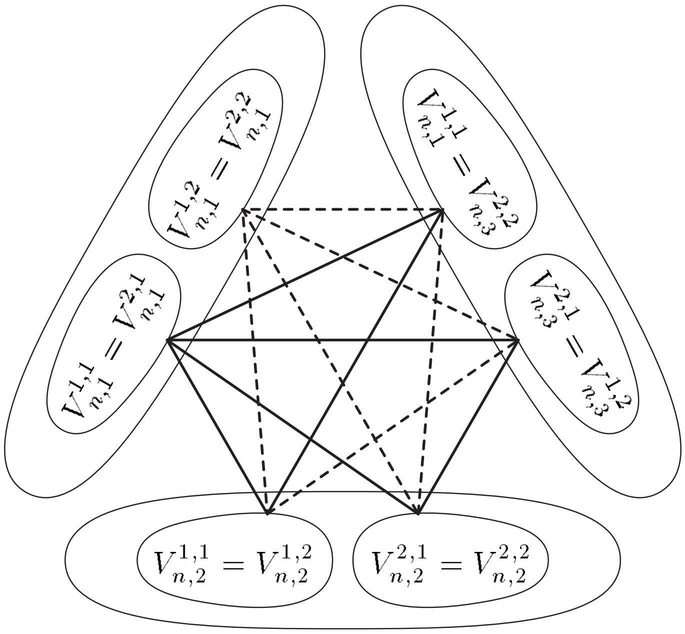


FIGURE 3. The graph $G^n$.

**THEOREM 20.** *For any $n \in [0,\infty)$, the graph $G_n$ is strongly* ADCT.

P r o o f. For $n \in [0,\infty)$ and $X \in \{G,H\}$ let $S(X_n)$ denote the following statement: The graph $X_n$ is strongly ADCT. We are going to prove the statement $\forall n \in [0,\infty)\ S(G_n)$ by induction on $n$. By Theorem 18, $S(G_0)$ is true.

So, suppose that $n \in [0,\infty)$ and $S(G_n)$ is true. We prove that the implication $S(G_n) \Rightarrow S(G_{n+1})$ is true by proving that both implications $S(G_n) \Rightarrow S(H_n)$ and $\big(S(G_n) \wedge S(H_n)\big) \Rightarrow S(G_{n+1})$ are true.

(a) $S(G_n) \Rightarrow S(H_n)$

**Claim 1.** A sequence $L = (l_1,\dots,l_p) \in \mathrm{Sct}(H_n)$ has a good $H_n$-realisation whenever one of the following conditions is fulfilled:

1. There is a decomposition $\{I^1, I^2\}$ of $[1,p]$ such that $\sum_{i\in I^1} l_i = e_n$.

2. There is $m \in [1,p]$ and a decomposition $\{\{i_1,\dots,i_m\}, I^1, I^2\}$ of $[1,p]$ such that, for both $j=1,2$, there is a sequence $L^j = (l^j_{i_1},\dots,l^j_{i_m})$ such that $L^j L\langle I^j\rangle \in \mathrm{Sct}(G_n)$ and a good $G_n$-realisation $(T^j_{i_1},\dots,T^j_{i_m})\mathcal{T}^j$ of $L^j L\langle I^j\rangle$ satisfying $l_{i_k} = l^1_{i_k} + l^2_{i_k}$ and $V(T^1_{i_k}) \cap V(T^2_{i_k}) \cap V_{n,1} \neq \emptyset$ for any $k \in [1,m]$.

P r o o f. 1. With $I^2 := [1,p] - I^1$ we have $\sum_{i\in I^2} l_i = e_n$. By $S(G_n)$ there is a good $G_n$-realisation $\mathcal{T}^j$ of $L\langle I^j\rangle$, $j=1,2$. If $T$ is a trail of $\mathcal{T}^j$, $j\in[1,2]$, and $i \in [1,3]$, then $V\big(\varphi^j_n(T)\big) \cap W_{n,i} \supseteq V\big(\varphi^j_n(T)\big) \cap V^j_{n,i} = \varphi^j_n\big(V(T)\cap V_{n,i}\big)$. As $Y \neq \emptyset \Rightarrow \varphi^j_n(Y) \neq \emptyset$ for any $Y \subseteq V(G_n)$, it is clear that $\varphi^1_n(\mathcal{T}^1)\varphi^2_n(\mathcal{T}^2)$ is a good $H_n$-realisation of the sequence $L\langle I^1\rangle L\langle I^2\rangle \sim L$.

2. We have $V\big(\varphi^1_n(T^1_{i_k})\big) \cap V\big(\varphi^2_n(T^2_{i_k})\big) = \varphi^1_n\big(V(T^1_{i_k}) \cap V(T^2_{i_k}) \cap V_{n,1}\big) \neq \emptyset$, hence there is a trail $T_{i_k} \in \varphi^1_n(T^1_{i_k}) + \varphi^2_n(T^2_{i_k})$ for any $k \in [1,m]$. If $(l_{i_k})_4 \neq 0$, there exists $j \in [1,2]$ such that $(l^j_{i_k})_4 \neq 0$, and then $V(T_{i_k}) \cap W_{n,i} \supseteq V(T^j_{i_k}) \cap V^j_{n,i} = \varphi^j_n(V(T^j_{i_k}) \cap V_{n,i}) \neq \emptyset$ (as $T^j_{i_k}$ is $G_n$-good), $i=1,2,3$, so that $T_{i_k}$ is $H_n$-good. Therefore, $(T_{i_1},\dots,T_{i_m})\varphi^1_n(\mathcal{T}^1)\varphi^2_n(\mathcal{T}^2)$ is a good $H_n$-realisation of the sequence $(l_{i_1},\dots,l_{i_m})L\langle I^1\rangle L\langle I^2\rangle \sim L$. □

**Claim 2.** A sequence $(l_1,\dots,l_p) \in \mathrm{Sct}(H_n)$ has a good $H_n$-realisation whenever one of the following conditions is fulfilled:

1. There is a decomposition $\big\{\{i_1\}, I^1, I^2\big\}$ of $[1,p]$ such that $\sum_{i\in I^j} l_i \leqslant e_n - 3$, $j=1,2$.

2. There is a decomposition $\big\{\{i_1,i_2\}, I^1, I^2\big\}$ of $[1,p]$ and $l^1_{i_k} \in [3, l_{i_k} - 3]$, $k=1,2$, such that $l^1_{i_1} + l^1_{i_2} + \sum_{i\in I^1} l_i = e_n$.

P r o o f. 1. Put $l^j_{i_1} := e_n - \sum_{i\in I^j} l_i \in [3, l_{i_1} - 3]$, $j=1,2$. By $S(G_n)$ there is a good $G_n$-realisation $(T^j_{i_1})\mathcal{T}^j$ of $(l^j_{i_1})L\langle I^j\rangle$, $j=1,2$. By Proposition 6 we may suppose without loss of generality that $V(T^1_{i_1}) \cap V(T^2_{i_1}) \cap V_{n,1} \neq \emptyset$, and then it suffices to use Claim 1.2.

2. If $l^2_{i_k} := l_{i_k} - l^1_{i_k}$, then $l^2_{i_k} \in [3, l_{i_k} - 3]$, $k = 1,2$, and $l^2_{i_1} + l^2_{i_2} + \sum\limits_{i \in I^2} l_i = e_n$. By $S(G_n)$ there is a good $G_n$-realisation $(T^j_{i_1}, T^j_{i_2})\mathcal{T}^j$ of the sequence $(l^j_{i_1}, l^j_{i_2})L\langle I^j \rangle$, $j = 1,2$. Because of Propositions 6 and 16.1 we may suppose without loss of generality that $V(T^1_{i_k}) \cap V(T^2_{i_k}) \cap V_{n,1} \neq \emptyset$, $k = 1,2$, and we are done by Claim 1.2 again. $\square$

Consider $L \in \mathrm{Sct}(H_n)$ and suppose that $\mathrm{nd}(L) = (l_1, \ldots, l_p)$. Let $q \in [1,p]$ be defined by the inequalities $\sum\limits_{i=1}^{q-1} l_i \leqslant e_n - 3$ and $\sum\limits_{i=1}^{q-1} l_i + l_p > e_n - 3$.

Since $e_n = 75 \cdot 4^n$, the following statements are easy to be checked:

$$\forall m \in \{3,5,6\}\ e_n \equiv 0 \pmod{m}, \tag{G1}$$

$$n = 0 \Rightarrow e_n \equiv 3 \pmod{4}, \tag{G2}$$

$$n \in [1,\infty) \Rightarrow e_n \equiv 0 \pmod{4}, \tag{G3}$$

$$\exists r \in \{3,5,6\}\ e_n \equiv r \pmod{7}. \tag{G4}$$

(1) If $\sum\limits_{i=1}^{q-1} l_i + l_p = e_n$, use Claim 1.1. (In order to simplify the presentation, $\to i.j$ will mean that Claim $i.j$ guarantees the existence of a good $H_n$-realisation of a sequence $\sim L$. Moreover, we write for short $f_i$ instead of $f_i(L)$.)

(2) If $\sum\limits_{i=1}^{q-1} l_i + l_p \geqslant e_n + 3$, then $\sum\limits_{i=q}^{p-1} l_i = 2e_n - \left(\sum\limits_{i=1}^{q-1} l_i + l_p\right) \leqslant e_n - 3 \to 2.1$ (with $I^1 := [1, q-1]$ and $I^2 := [q, p-1]$).

(3) $\exists \delta \in \{-2,-1,1,2\}$, $\sum\limits_{i=1}^{q-1} l_i + l_p = e_n + \delta$.

(31) If $l_{p-1} \geqslant 8$, then, with $m := \max(3, 3-\delta)$, we have $m + (l_p - m - \delta) + \sum\limits_{i=1}^{q-1} l_i = e_n = (l_{p-1} - m) + (m + \delta) + \sum\limits_{i=q}^{p-2} l_i \to 2.2$.

(32) If $l_{p-1} \leqslant 7$, let $r \in [q,p]$ be defined by the inequalities $\sum\limits_{i=1}^{r-1} l_i \leqslant e_n - 3$ and $\sum\limits_{i=1}^{r} l_i > e_n - 3$.

(321) $\sum\limits_{i=1}^{r} l_i = e_n \to 1.1$.

(322) If $\sum\limits_{i=1}^{r} l_i \geqslant e_n + 3$, then $\sum\limits_{i=r+1}^{p} l_i \leqslant e_n - 3 \to 2.1$.

(323) $\sum_{i=1}^{r} l_i = e_n + \varepsilon$, $\varepsilon \in \{-2,-1,1,2\}$.

(3231) $\varepsilon \in [-2,-1]$.

(32311) If $l_p \geqslant l_1+3-\varepsilon$, then $\sum_{i=2}^{r} l_i = e_n + \varepsilon - l_1 \leqslant e_n - 4$ and $l_1 + \sum_{i=r+1}^{p-1} l_i = l_1 + e_n - \varepsilon - l_p \leqslant e_n - 3 \to 2.1$.

(32312) $l_p \leqslant l_1 + 2 - \varepsilon$.

(323121) If there is $j \in [1,r]$ and $k \in [r+1,p]$ such that $l_k = l_j - \varepsilon$, then $\sum_{i=1}^{j-1} l_i + \sum_{i=j+1}^{r} l_i + l_k = e_n \to 1.1$.

(323122) $\forall j \in [1,r]\ \ \forall k \in [r+1,p]\ \ l_k \neq l_j - \varepsilon$.

(3231221) If $\varepsilon = -2$, then $l_i \in [l_1, l_1+4]$ for any $i \in [1,p]$.

(32312211) If $l_r = l_1$, then $e_n - 2 \equiv 0 \pmod{l_1}$ in contradiction with $l_1 \leqslant 7$ and (G1)–(G4).

(32312212) If $l_r = l_1 + 1$, then $f_{l_1+2} = f_{l_1+3} = 0$.

(323122121) If $l_{p-1} = l_1 + 1$, then $e_n + 2 \leq (p-r)(l_1+1) + 3$, $p - r \geq \frac{e_n-1}{l_1+1} \geq \lfloor \frac{74}{7} \rfloor = 10$ and $l_{r+1} = l_{r+2} = l_1 + 1$.

(3231221211) If $l_2 = l_1$, then $\sum_{i=3}^{r+2} l_i = e_n \to 1.1$.

(3231221212) If $l_2 = l_1+1$, then $e_n - 2 = r(l_1+1) - 1$ and $e_n \equiv 1 \pmod{l_1+1}$ in contradiction with (G1)–(G4).

(323122122) If $l_{p-1} = l_1 + 4$, then $l_1 = 3$, there is $I \subseteq [1,r]$ such that $\sum_{i\in I} l_i = 12$ and $\sum_{i\in[1,r]-I} l_i + l_{p-1} + l_p = e_n \to 1.1$.

(32312213) If $l_r = l_1 + 2$, then $l_k = l_1 + 3$ for any $k \in [r+1,p]$, $e_n + 2 \equiv 0 \pmod{l_1+3}$, hence, because of $l_1 \leqslant 4$ and (G1), $l_1 = 4$. As $f_5 = 0$ (a consequence of $l_p = 7$), from $\sum_{i=1}^{r} l_i = e_n - 2 \geqslant 73$ it follows that there is $I \subseteq [1,r]$ such that $\sum_{i\in I} l_i = 12$. Therefore, $\sum_{i\in[1,r+2]-I} l_i = e_n \to 1.1$.

(32312214) $l_r = l_1 + 3$.

(323122141) If $l_{p-1} = l_1 + 3$, there is $m \in [0,1]$ such that $e_n + 2 \equiv m \pmod{l_1+3}$, and then $l_1 \leqslant 4$ together with (G1) imply $l_1 = 4$. We have also $f_5 = 0$ and $77 \leqslant e_n + 2 = 7(p-r-1) + l_p$, so that $l_{r+1} = l_{r+2} = 7$.

(3231221411) If $f_4 \geqslant 3$ or $f_6 \geqslant 2$, there is $I \subseteq [1,r]$ such that $\sum_{i\in[1,r+2]-I} l_i = e_n \to 1.1$.

(3231221412) $f_4 \leqslant 2 \wedge f_6 \leqslant 1$.

(32312214121) If $l_p = 7$, then $e_n + 2 = 7(p-r)$, $e_n - 2 \equiv 3 \pmod 7$, $\mathrm{nd}(L) = (4,6)(7)^{2r-2}$, $l_i^1 := 3 \leqslant l_i - 3$, $i = 3,4$, and $3+3+6+7(r-2) = e_n \to 2.2$ (with $i_1 := 3$, $i_2 := 4$, $I^1 := \{2\} \cup [5, r+2]$ and $I^2 := \{1\} \cup [r+3, 2r]$).

(32312214122) If $l_p = 8$, then $e_n + 2 = 7(p-r) + 1$, $e_n - 2 \equiv 4 \pmod 7$, $\mathrm{nd}(L) = (4)(7)^{2r-2}(8)$, $l_p^1 := 5 \leqslant l_p - 3$, $l_2^1 := 4 \leqslant l_2 - 3$ and $5+4+4+7(r-2) = e_n \to 2.2$.

(323122142) If $l_{p-1} = l_1 + 4$, then $l_1 = 3$, $f_5 = 0$, there is $I \subseteq [1,r]$ such that $\sum_{i \in I} l_i = 12$ and $\sum_{i \in [1,r]-I} l_i + l_{p-1} + l_p = e_n \to 1.1$.

(32312215) If $l_r = l_1 + 4$, then $l_1 = 3$, $f_5 = 0$, $e_n + 2 = 7(p-r)$ and $e_n - 2 \equiv 3 \pmod 7$.

(323122151) If $f_3 \geqslant 4$, $f_4 \geqslant 3$ or $f_6 \geqslant 2$, there is $I \subseteq [1,r]$ such that $\sum_{i \in [1,r+2]-I} l_i = e_n \to 1.1$.

(323122152) $f_3 \leqslant 3 \wedge f_4 \leqslant 2 \wedge f_6 \leqslant 1$.

(3231221521) If $f_4 = 2$ and $l_j = l_{j+1} = 4$ for some $j \in [1, r-1]$, then $\sum_{i=1}^{j-1} l_i + \sum_{i=j+2}^{r+1} l_i = e_n - 3$ and $l_j + l_{j+1} + \sum_{i=r+2}^{p-1} l_i = e_n - 4 \to 2.1$.

(3231221522) $f_4 = 1$.

(32312215221) If $\mathrm{nd}(L) = (3)^2(4)(7)^{2r-4}$, then $l_i^1 := 4 \leqslant l_i - 3$, $i = 4,5$, and $4+4+4+7(r-2) = e_n \to 2.2$.

(32312215222) If $\mathrm{nd}(L) = (4,6)(7)^{2r-4}$, then $l_i^1 := 3 \leqslant l_i - 3$, $i = 3,4$, and $3+3+6+7(r-2) = e_n \to 2.2$.

(3231221523) If $f_4 = 0$, then $\mathrm{nd}(L) = (3)(7)^{\frac{2e_n - 3}{7}}$ and we can use Claim 1.2 with $m := 3$, $i_k := k+1$, $l_{i_k}^1 = 3$, $l_{i_k}^2 = 4$, $k = 1,2,3$, $I^1 := \{1\} \cup [5, r+2]$ and $I^2 := [r+3, 2r]$ (notice that by $S(G_n)$ there are good $G_n$-realisations of the sequences $(3)^4(7)^{\frac{e_n - 12}{7}}$ and $(4)^3(7)^{\frac{e_n - 12}{7}}$ corresponding to the rows 1 and 5 of Table 1).

(3231222) If $\varepsilon = -1$, then $l_i \in [l_1, l_1 + 3]$ for any $i \in [1,p]$.

(32312221) If $l_r = l_1$, then $e_n - 1 = r l_1$ in contradiction with (G1)–(G4).

(32312222) If $l_r = l_1 + 1$, then $l_k \notin [l_1+1, l_1+2]$ for any $k \in [r+1, p]$, $e_n + 1 = (p-r)(l_1+3)$, $l_1 \leqslant 4$ and, because of (G1), $l_1 = 4$. Since $4f_4 + 5f_5 = e_n \geqslant 75$, there is $I \subseteq [1,r]$ such that $\sum_{i \in I} l_i = 20$, and then $\sum_{i \in [1,r+3]-I} l_i = e_n \to 1.1$.

(32312223) If $l_r = l_1 + 2$, then $f_{l_1+3} = 0$, hence $e_n + 1 = (p-r)(l_1+2)$, $l_1 \leqslant 5$ and, because of (G1), $l_1 = 5$ and $e_n - 1 \equiv 5 \pmod 7$.

(323122231) If $f_5 \geqslant 4$, then $\sum_{i=5}^{r+3} l_i = e_n \to 1.1$.

(323122232) If $f_5 \leqslant 3$, then $\mathrm{nd}(L) = (5)(7)^{2r-1}$, $l_i^1 := 3 \leqslant l_i - 3$, $i = 2,3$, and $3 + 3 + 7(r-1) = e_n \to 2.2$.

(32312224) If $l_r = l_1 + 3$, then $e_n + 1 = (p-r)(l_1+3)$, $l_1 \leqslant 4$ and, using (G1), $l_1 = 4$, so that $f_6 = 0$ and $e_n - 1 \equiv 5 \pmod 7$.

(323122241) $\exists j \in [2, r-1]$ $l_j = 5$.

(3231222411) If $f_4 \geqslant 2$, then $\sum_{i=3}^{j-1} l_i + \sum_{i=j+1}^{r+2} l_i = e_n \to 1.1$.

(3231222412) $f_4 = 1$.

(32312224121) If $f_5 \geqslant 4$, then $l_1 + \sum_{i=6}^{r+3} l_i = e_n \to 1.1$.

(32312224122) If $f_5 \leqslant 3$, then $\mathrm{nd}(L) = (4)(5)^3(7)^{2r-5}$, $l_1 + l_2 + \sum_{i=5}^{r+1} l_i = e_n - 4$ and $l_3 + l_4 + \sum_{i=r+2}^{p-1} l_i = e_n - 3 \to 2.1$.

(323122242) $f_5 = 0$.

(3231222421) If $f_4 \geqslant 5$, then $\sum_{i=6}^{r+3} l_i = e_n \to 1.1$.

(3231222422) If $f_4 \leqslant 4$, then $\mathrm{nd}(L) = (4)^3(7)^{\frac{2e_n-12}{7}}$. By $S(G_n)$ there exists a good $G_n$-realisation $(T_1^1, T_2^1, T_3^1, T_4^1, T_5^1)\mathcal{T}^1$ of the sequence $(4)^5(7)^{\frac{e_n-20}{7}}$ such that $v_{i,1} \in V(T_i^1)$, $i = 1,2,3$ and $v_{i,2} \in V(T_{3+i}^1)$, $i = 1,2$ (see the row 4 of Table 1). Therefore, by Proposition 6, there is also a good $G_n$-realisation $(\bar{T}_1^1, \bar{T}_2^1, \bar{T}_3^1, \bar{T}_4^1, \bar{T}_5^1)\bar{\mathcal{T}}^1$ of $(4)^5(7)^{\frac{e_n-20}{7}}$ such that $v_{i,1} \in V(\bar{T}_i^1)$, $i = 1,2$, and $v_{i,2} \in V(\bar{T}_{2+i}^1)$, $i = 1,2,3$. Further, by $S(G_n)$ and Propositions 6 and 16.1, there exist good $G_n$-realisations $(T_1^2, T_2^2)\mathcal{T}^2$ and $(\bar{T}_1^2, \bar{T}_2^2)\bar{\mathcal{T}}^2$ of the sequence $(3)^2(7)^{\frac{e_n-6}{7}}$ such that $v_{i+1,1} \in V(T_i^2)$ and $v_{i,1} \in V(\bar{T}_i^2)$, $i = 1,2$. Since $V(\varphi_n^1(T_i^1)) \cap V(\varphi_n^2(T_{i-1}^2)) \supseteq \{v_{i,1}^1\}$, $i = 2,3$, and $V(\varphi_n^1(\bar{T}_i^1)) \cap V(\varphi_n^2(\bar{T}_i^2)) \supseteq \{v_{i,1}^1\}$, $i = 1,2$, there are trails $T_i \in \varphi_n^1(T_i^1) + \varphi_n^2(T_{i-1}^2)$, $i = 2,3$, and $\bar{T}_i \in \varphi_n^1(\bar{T}_i^1) + \varphi_n^2(\bar{T}_i^2)$, $i = 1,2$. Then both $(\varphi_n^1(T_4^1), \varphi_n^1(T_5^1), \varphi_n^1(T_1^1), T_2, T_3)\varphi_n^1(\mathcal{T}^1)\varphi_n^2(\mathcal{T}^2)$ and $(\varphi_n^1(\bar{T}_3^1), \varphi_n^1(\bar{T}_4^1), \varphi_n^1(\bar{T}_5^1), \bar{T}_1, \bar{T}_2)\varphi_n^1(\bar{\mathcal{T}}^1)\varphi_n^2(\bar{\mathcal{T}}^2)$ are $H_n$-good realisations of $\mathrm{nd}(L)$; the former satisfies "(8,2)-conditions" and the latter one "(7,2)-conditions" of Table 2.

(3232) If $\varepsilon \in [1,2]$, then $l_r = \sum_{i=1}^{r} l_i - \sum_{i=1}^{r-1} l_i \geqslant e_n + \varepsilon - (e_n - 3) = 3 + \varepsilon$. With $l := \min(l_i \colon i \in [1,r], l_i \geqslant 3 + \varepsilon)$ we have $3 + \varepsilon \leqslant l \leqslant l_r$.

(32321) If $l_p \geqslant l+3-\varepsilon$, let $j \in [1,r]$ be such that $l_j = l$. Then $\sum_{i=1}^{j-1} l_i + \sum_{i=j+1}^{r} l_i = e_n + \varepsilon - l \leqslant e_n + \varepsilon - (3+\varepsilon)$ and $l_j + \sum_{i=r+1}^{p-1} l_i = l + (e_n - \varepsilon) - l_p \leqslant e_n - 3 \to 2.1$.

(32322) $\forall i \in [1,p]$ $l_i \in [3, 2+\varepsilon] \cup [l, l+2-\varepsilon]$.

(323221) $\varepsilon = 1$.

(3232211) If $l_r = l+1$, then $e_n - 1 = (p-r)(l+1)$ in contradiction with (G1)–(G4).

(3232212) $l_r = l$.

(32322121) If $l_{p-1} = l$, then $e_n - 1 = (p-r)l + m$ for some $m \in [0,1]$ and $e_n \equiv 1 + m \pmod{l}$ in contradiction with (G1)–(G4).

(32322122) $l_{p-1} = l+1$.

(323221221) If $f_3 = 0$, then $e_n + 1 = rl$, hence from $l \leqslant 6$ and (G1)–(G3) it follows that $l = 4$ and $n = 0$.

(3232212211) If $f_5 \geqslant 3$, then $\sum_{i=5}^{r} l_i + \sum_{i=p-2}^{p} l_i = e_n \to 1.1$.

(3232212212) If $f_5 = 2$, then $\mathrm{nd}(L) = (4)^{35}(5)^2$; for a good $H_0$-realisation of $\mathrm{nd}(L)$ see Graph $H_0$.

(323221222) $f_3 \geqslant 1$.

(3232212221) $l = 4$.

(32322122211) If $f_3 = 1$, then $e_n + 1 = 3 + 4(r-1)$ in contradiction with (G2) and (G3).

(32322122212) If $f_3 \geqslant 2$, then $\sum_{i=3}^{r} l_i + l_p = e_n \to 1.1$.

(3232212222) $l \in [5,6]$.

(32322122221) If $l_{r-1} = 3$, then $e_n + 1 = 3(r-1) + l$ in contradiction with (G1).

(32322122222) If $l_{r-1} = l$, then $\sum_{i=1}^{r-2} l_i + l_{p-1} = (e_n + 1 - 2l) + (l+1) \leqslant e_n - 3$ and $\sum_{i=r-1}^{p-2} l_i = e_n - 3 \to 2.1$.

(323222) If $\varepsilon = 2$, then $e_n - 2 = (p-r)l$ in contradiction with (G1)–(G4).

To conclude the proof of the implication $S(G_n) \Rightarrow S(H_n)$ we have to find good $H_n$-realisations of sequences satisfying "$(t,2)$-conditions" of Table 2, $t \in [1,6]$.

$t = 1$: If $7|2e_n - 12$, then $7|e_n - 6$ and $7|e_n - 13$. By $S(G_n)$ (the row 2 of Table 1) and Proposition 6 there is a good $G_n$-realisation $(T_1^1, T_2^1, T_3^1, T_4^1)\mathcal{T}^1$ of the sequence $(3)^3(4)(7)^{\frac{e_n-13}{7}}$ such that $v_{1,1} \in V(T_4^1)$ and $v_{i,2} \in V(T_i^1)$, $i = 1, 2, 3$. By $S(G_n)$ and Proposition 6 there is a good $G_n$-realisation $(T_1^2, T_2^2)\mathcal{T}^2$ of the sequence $(3)^2(7)^{\frac{e_n-6}{7}}$ with $v_{1,1} \in V(T_1^2)$. As $V\big(\varphi_n^1(T_4^1)\big) \cap V\big(\varphi_n^2(T_1^2)\big) \supseteq \{v_{1,1}^1\}$, there is a trail $T_4 \in \varphi_n^1(T_4^1) + \varphi_n^2(T_1^2)$ and $\big(\varphi_n^1(T_1^1), \varphi_n^1(T_2^1), \varphi_n^1(T_3^1), \varphi_n^2(T_2^2), T_4\big)\varphi_n^1(\mathcal{T}^1)\varphi_n^2(\mathcal{T}^2)$ is an appropriate good $H_n$-realisation of the sequence $(3)^4(7)^{\frac{2e_n-12}{7}}$.

$t = 2$: By $S(G_n)$, Propositions 6 and 16.2, there is a good $G_n$-realisation $(T_1^1, T_2^1, T_3^1, T_4^1)\mathcal{T}^1$ of the sequence $(3)^4(7)^{\frac{e_n-12}{7}}$ such that $v_{i,1} \in V(T_i^1)$, $v_{i,2} \in V(T_{i+2}^1)$, $i = 1, 2$. By $S(G_n)$ and Proposition 16.1 there exists a good $G_n$-realisation $(T_1^2, T_2^2)\mathcal{T}^2$ of the sequence $(4)^3(7)^{\frac{e_n-12}{7}}$ such that $v_{i,1} \in V(T_i^2)$, $i = 1, 2$. As $V\big(\varphi_n^1(T_i^1)\big) \cap V\big(\varphi_n^2(T_i^2)\big) \supseteq \{v_{i,1}^1\}$, there is a trail $T_i \in \varphi_n^1(T_i^1) + \varphi_n^2(T_i^2)$, $i = 1, 2$. Then $\big(\varphi_n^1(T_3^1), \varphi_n^1(T_4^1)\big)\varphi_n^2(\mathcal{T}^2)(T_1, T_2)\varphi_n^1(\mathcal{T}^1)$ is a good $H_n$-realisation of the sequence $(3)^2(4)(7)^{\frac{2e_n-10}{7}}$ having required properties.

$t = 3$: By $S(G_n)$ there are good $G_n$-realisations $(T_1^1, T_2^1)\mathcal{T}^1$ of $(3)^2(7)^{\frac{e_n-6}{7}}$ and $\mathcal{T}^2$ of $(6)(7)^{\frac{e_n-6}{7}}$. By Proposition 16.1 we may suppose without loss of generality that $v_{i,2} \in V(T_i^1)$, $i = 1, 2$. Then $\big(\varphi_n^1(T_1^1),\ \varphi_n^1(T_2^1)\big)\ \varphi_n^2(\mathcal{T}^2)\ \varphi_n^1(\mathcal{T}^1)$ is a necessary good $H_n$-realisation of $(3)^2(6)(7)^{\frac{2e_n-12}{7}}$.

$t = 4$: By $S(G_n)$ and Propositions 6 and 16.1 there exist good $G_n$-realisations $(T_1^1, T_2^1, T_3^1)\mathcal{T}^1$ of $(3)^2(4)(7)^{\frac{e_n-10}{7}}$ and $(T_1^2)\mathcal{T}^2$ of $(3)(7)^{\frac{e_n-3}{7}}$ such that $v_{i,2} \in V(T_i^1)$, $i = 1, 2$, and $v_{1,1} \in V(T_3^1) \cap V(T_1^2)$. There is a trail $T_3 \in \varphi_n^1(T_3^1) + \varphi_n^2(T_1^2)$ and $(\varphi_n^1(T_1^1), \varphi_n^1(T_2^1), T_3)\varphi_n^1(\mathcal{T}^1)\varphi_n^2(\mathcal{T}^2)$ is a good $H_n$-realisation of $(3)^2(7)^{\frac{2e_n-6}{7}}$ we are seeking for.

$t = 5$: By $S(G_n)$ (the row 5 of Table 1) and Proposition 6 there are good $G_n$-realisations $(T_1^j, T_2^j, T_3^j)\mathcal{T}^j$ of the sequence $(4)^3(7)^{\frac{e_n-12}{7}}$ such that $v_{i,1+j} \in V(T_i^j)$, $i = 1, 2, 3$, $j = 1, 2$. Then $\big(\varphi_n^1(T_1^1), \varphi_n^1(T_2^1), \varphi_n^1(T_3^1), \varphi_n^2(T_1^2), \varphi_n^2(T_2^2), \varphi_n^2(T_3^2)\big)\varphi_n^1(\mathcal{T}^1)\varphi_n^2(\mathcal{T}^2)$ is a good $H_n$-realisation of $(4)^6(7)^{\frac{2e_n-24}{7}}$ having required properties.

$t = 6$: By $S(G_n)$ (the row 3 of Table 1) and Proposition 6 there exists a good $G_n$-realisation $\left[\prod_{i=1}^{6}(T_i^1)\right]\mathcal{T}^1$ of the sequence $(4)^6(7)^{\frac{e_n-24}{7}}$ with $v_{i,2} \in V(T_i^1)$, $i = 1, 2, 3$, $v_{i,3} \in V(T_{3+i}^1)$, $i = 1, 2$, and $v_{1,1} \in V(T_6^1)$. By $S(G_n)$ and Proposition 6 there exists also a good $G_n$-realisation $(T_1^2)\mathcal{T}^2$ of the sequence $(3)(7)^{\frac{e_n-3}{7}}$ with $v_{1,1} \in V(T_1^2)$. There is a trail $T_6 \in \varphi_n^1(T_6^1) + \varphi_n^2(T_1^2)$

and $\left[\prod_{i=1}^{5}\left(\varphi_n^1(T_i^1)\right)\right](T_6)\varphi_n^1(\mathcal{T}^1)\varphi_n^2(\mathcal{T}^2)$ is an appropriate good $H_n$-realisation of $(4)^5(7)^{\frac{2e_n-20}{7}}$.

(b) $\big(S(G_n)\wedge S(H_n)\big)\Rightarrow S(G_{n+1})$.

**Claim 3.** A sequence $(l_1,\dots,l_p)\in \mathrm{Sct}(G_{n+1})$ has a good $G_{n+1}$-realisation whenever one of the following conditions is fulfilled:

1. There is $I^1\subseteq[1,p]$ such that $\sum_{i\in I^1} l_i = 2e_n$.

2. There is $m\in[1,p]$ and a decomposition $\{\{i_1,\dots,i_m\},I^1,I^2\}$ of $[1,p]$ such that, for both $j=1,2$, there is a sequence $L^j=(l^j_{i_1},\dots,l^j_{i_m})$ such that $L^jL\langle I^j\rangle\in\mathrm{Sct}(H_n)$ and a good $H_n$-realisation $(T^j_{i_1},\dots,T^j_{i_m})\mathcal{T}^j$ of $L^jL\langle I^j\rangle$ satisfying $l_{i_k}=l^1_{i_k}+l^2_{i_k}$ and $V(T^1_{i_k})\cap V(T^2_{i_k})\cap V^1_{n,2}\neq\emptyset$ for any $k\in[1,m]$.

3. There is a decomposition $\{\{i_1,i_2\},I^1,I^2\}$ of $[1,p]$ and $l^1_{i_k}\in[3,l_{i_k}-3]$, $k=1,2$, such that there is a good $(l^1_{i_1},l^1_{i_2})$-global $H_n$-realisation of the sequence $(l^1_{i_1},l^1_{i_2})L\langle I^1\rangle$.

P r o o f. 1. With $I^2:=[1,p]-I^1$ we have $\sum_{i\in I^2} l_i = e_{n+1}-\sum_{i\in I^1} l_i = 2e_n$. By $S(H_n)$ there exists a good $H_n$-realisation $\mathcal{T}^j$ of $L\langle I^j\rangle$, $j=1,2$. Let $T$ be a trail of length $\neq 4$ of $\mathcal{T}^j$, $j\in[1,2]$, and let $i\in[1,3]$. Since $\psi^j_n(W_{n,i})\subseteq V_{n+1,i}$, from $V(T)\cap W_{n,i}\neq\emptyset$ it follows that $V\big(\psi^j_n(T)\big)\cap V_{n+1,i}\neq\emptyset$. Therefore, $\psi^1_n(\mathcal{T}^1)\psi^2_n(\mathcal{T}^2)$ is a good $G_{n+1}$-realisation of $L\langle I^1\rangle L\langle I^2\rangle\sim L$.

2. As $V\big(\psi^1_n(T^1_{i_k})\big)\cap V\big(\psi^2_n(T^2_{i_k})\big)\supseteq\psi^1_n\big(V(T^1_{i_k})\cap V(T^2_{i_k})\cap V^1_{n,2}\big)\neq\emptyset$, there is a trail $T_{i_k}\in\psi^1_n(T^1_{i_k})+\psi^2_n(T^2_{i_k})$ for any $k\in[1,m]$. If $(l_{i_k})_4\neq 0$, there is $j\in[1,2]$ such that $(l^j_{i_k})_4\neq 0$. Clearly, the trail $\psi^j_n(T^j_{i_k})$ is $G_{n+1}$-good (as above), hence, because of $V(T_{i_k})\supseteq V\big(\psi^j_n(T^j_{i_k})\big)$, so is $T_{i_k}$. Thus, $(T_{i_1},\dots,T_{i_m})\psi^1_n(\mathcal{T}^1)\psi^2_n(\mathcal{T}^2)$ is a good $G_{n+1}$-realisation of $(l_{i_1},\dots,l_{i_m})L\langle I^1\rangle L\langle I^2\rangle\sim L$.

3. Let $(T^1_{i_1},T^1_{i_2})\mathcal{T}^1$ be a good $(l^1_{i_1},l^1_{i_2})$-global $H_n$-realisation of $(l^1_{i_1},l^1_{i_2})L\langle I^1\rangle$ such that $\big|E(T^1_{i_k})\big|=l^1_{i_k}$, $k=1,2$; there is $j\in[1,2]$ with $V(T^1_{i_1})\cap(V^j_{n,2}\cup V^j_{n,3})\neq\emptyset$ and $V(T^1_{i_2})\cap(V^{3-j}_{n,2}\cup V^{3-j}_{n,3})\neq\emptyset$. If $l^2_{i_k}:=l_{i_k}-l^1_{i_k}$, $k=1,2$, from $l_{i_1}+l_{i_2}+\sum_{i\in I^1\cup I^2} l_i = 4e_n$ and $l^1_{i_1}+l^1_{i_2}+\sum_{i\in I^1} l_i = 2e_n$, it follows that $L^2:=(l^2_{i_1},l^2_{i_2})L\langle I^2\rangle\in\mathrm{Sct}(H_n)$. Hence, by $S(H_n)$ there is a good $H_n$-realisation $(T^1_{i_2},T^2_{i_2})\mathcal{T}^2$ of $L^2$.

If there is $k\in[1,2]$ such that $V(T^2_{i_1})\cap(V^k_{n,2}\cup V^k_{n,3})\neq\emptyset$ and $V(T^2_{i_2})\cap(V^{3-k}_{n,2}\cup V^{3-k}_{n,3})\neq\emptyset$, by Proposition 19.3 we may suppose without loss of generality that $v^m_{1,2}\in V(T^1_{i_m})\cap V(T^2_{i_m})$, $m=1,2$. Since $V\big(\psi^1_n(T^1_{i_m})\big)\cap V\big(\psi^2_n(T^2_{i_m})\big)\supseteq\{v^{m,1}_{1,2}\}$, there is a trail $T_m \in \psi_n^1(T^1_{i_m}) + \psi_n^2(T^2_{i_m})$, $m = 1,2$, and $(T_1, T_2)\psi_n^1(\mathcal{T}^1)\psi_n^2(\mathcal{T}^2)$ is a good $G_{n+1}$-realisation of $(l_{i_1}, l_{i_2})L\langle I^1\rangle L\langle I^2\rangle \sim L$.

If there is $k \in [1,2]$ such that $V(T^2_{i_1}) \cup V(T^2_{i_2}) \subseteq V(G^k_n)$, by Proposition 19.2,3 we may suppose without loss of generality that $v^1_{1,2} \in V(T^1_{i_1}) \cap V(T^2_{i_1})$, $v^2_{1,3} \in V(T^1_{i_2})$ and $v^1_{1,3} \in V(T^2_{i_2})$. As above, there is a trail $T_1 \in \psi_n^1(T^1_{i_1}) + \psi_n^2(T^2_{i_1})$. Further, since $V\big(\psi_n^1(T^1_{i_2})\big) \cap V\big(\psi_n^2(T^2_{i_2})\big) \supseteq \{v^{2,1}_{1,3}\} = \{v^{1,2}_{1,3}\}$, there exists a trail $T_2 \in \psi_n^1(T^1_{i_2}) + \psi_n^2(T^2_{i_2})$ and $(T_1, T_2)\psi_n^1(\mathcal{T}^1)\psi_n^2(\mathcal{T}^2)$ is a good $G_{n+1}$-realisation of a sequence changeable to $L$. □

**Claim 4.** A sequence $(l_1, \dots, l_p) \in \mathrm{Sct}(G_{n+1})$ has a good $G_{n+1}$-realisation whenever one of the following conditions is fulfilled:

1. There is a decomposition $\big\{\{i_1\}, I^1, I^2\big\}$ of $[1,p]$ such that $\sum_{i\in I^j} l_i \leqslant 2e_n - 3$, $j = 1,2$.

2. There is a decomposition $\big\{\{i_1, i_2\}, I^1, I^2\big\}$ of $[1,p]$ and $l^1_{i_k} \in [3, l_{i_k} - 3]$, $k = 1,2$, such that $\big((l^1_{i_1})_4, (l^1_{i_2})_4\big), \big((l_{i_1} - l^1_{i_1})_4, (l_{i_2} - l^1_{i_2})_4\big) \notin P$ and $l^1_{i_1} + l^1_{i_2} + \sum_{i\in I^1} l_i = 2e_n$.

3. There is a decomposition $\big\{\{i_1, i_2, i_3\}, I^1, I^2, I^3\big\}$ of $[1,p]$ and $l^1_{i_k} \in [3, l_{i_k} - 3]$, $k = 1,2,3$, such that $l^1_{i_1} + l^1_{i_3} + \sum_{i\in I^1} l_i = e_n = l^1_{i_2} + (l_{i_3} - l^1_{i_3}) + \sum_{i\in I^2} l_i$.

4. There is a decomposition $\big\{\{i_1, i_2\}, I^1, I^2, I^3\big\}$ of $[1,p]$ and $l^1_{i_j} \in [3, l_{i_j} - 3]$, $j = 1,2$, such that $l^1_{i_j} + \sum_{i\in I^j} l_i = e_n$, $j = 1,2$.

P r o o f. 1. If $l^j_{i_1} := 2e_n - \sum_{i\in I^j} l_i$, then $l^j_{i_1} \in [3, 2e_n]$, $j = 1,2$, and $l_{i_1} = l^1_{i_1} + l^2_{i_1}$. By $S(H_n)$ there exists a good $H_n$-realisation $(T^j_{i_1})\mathcal{T}^j$ of $(l^j_{i_1})L\langle I^j\rangle$, $j = 1,2$. By Proposition 19.1 we may suppose without loss of generality that $V(T^1_{i_1}) \cap V(T^2_{i_1}) \cap V^1_{n,2} \neq \emptyset$ (notice that a closed trail in $H_n$ necessarily contains a vertex of an eccentric part of $H_n$). Thus, we are done by Claim 3.2.

2. If $l^2_{i_k} := l_{i_k} - l^1_{i_k}$, then $l^2_{i_k} \in [3, l_{i_k} - 3]$, $k = 1,2$. Since we have $L^j := (l^j_{i_1}, l^j_{i_2})L\langle I^2\rangle \in \mathrm{Sct}(H_n)$, by $S(H_n)$ there is a good $H_n$-realisation $\mathcal{T}^j = (\tilde{T}^j_{i_1}, T^j_{i_2})\mathcal{T}^j$ of $L^j$, $j = 1,2$. If there is $j \in [1,2]$ such that $\mathcal{T}^j$ is $(l^j_{i_1}, l^j_{i_2})$-global, we are done by Claim 3.3. So, let $j_1, j_2 \in [1,2]$ be such that $V(T^k_{i_1}) \cup V(T^k_{i_2}) \subseteq V(G^{j_k}_n)$, $k = 1,2$. By Proposition 16.2 there is $m_k \in [2,3]$ such that $\{V(T^k_{i_1}), V(T^k_{i_2})\}$ has a system of distinct representatives in $V^{j_k}_{n,m_k}$, $k = 1,2$. By Proposition 19.1 we may suppose without loss of generality that $v_{k,2} \in V(T^1_{i_k}) \cup V(T^2_{i_k})$, $k = 1,2$. Now, it suffices to use Claim 3.2.

3. Put $l^2_{i_3} := l_{i_3} - l^1_{i_3} \in [3, l_{i_3} - 3]$. By $S(G_n)$ there is a good $G_n$-realisation $(T^1_{i_j}, T^j_{i_3})\mathcal{T}^j$ of $(l^1_{i_j}, l^j_{i_3})L\langle I^j\rangle$, $j = 1,2$; we may suppose without loss of generality

that $v_{1,1} \in V(T^1_{i_3}) \cap V(T^2_{i_3})$. As $V\big(\varphi^1_n(T^1_{i_3})\big) \cap V\big(\varphi^2_n(T^2_{i_3})\big) \supseteq \{v^1_{1,1}\}$, there is a trail $T_3 \in \varphi^1_n(T^1_{i_3}) + \varphi^2_n(T^2_{i_3})$ and $\big(\varphi^1_n(T^1_{i_1}), \varphi^2_n(T^1_{i_2}), T_3\big)\varphi^1_n(\mathcal{T}^1)\varphi^2_n(\mathcal{T}^2)$ is a good $(l^1_{i_1}, l^1_{i_2})$-global $H_n$-realisation of the sequence $(l^1_{i_1}, l^1_{i_2}, l^1_{i_3})L\langle I^1\rangle L\langle I^2\rangle \sim (l^1_{i_1}, l^1_{i_2})L\big\langle\{i_3\}\cup I^1\cup I^2\big\rangle$. Now, using the decomposition $\big\{\{i_1, i_2\}, \{i_3\}\cup I^1\cup I^2, I^3\big\}$ of $[1,p]$ we are done by Claim 3.3 (and by Lemma 1).

4. By $S(G_n)$ there is a good $G_n$-realisation $(T^1_{i_j})\mathcal{T}^j$ of $(l^1_{i_j})L\langle I^j\rangle$, $j = 1,2$. Since $\big(\varphi^1_n(T^1_{i_1}), \varphi^2_n(T^1_{i_2})\big)\varphi^1_n(\mathcal{T}^1)\varphi^2_n(\mathcal{T}^2)$ is a good $(l^1_{i_1}, l^1_{i_2})$-global $H_n$-realisation of $(l^1_{i_1}, l^1_{i_2})L\langle I^1\rangle L\langle I^2\rangle \sim (l^1_{i_1}, l^1_{i_2})L\langle I^1 \cup I^2\rangle$, Claim 3.3 with the decomposition $\big\{\{i_1, i_2\}, I^1 \cup I^2, I^3\big\}$ of $[1,p]$ can be used. $\square$

Consider $L \in \mathrm{Sct}(G_{n+1})$, assume that $\mathrm{nd}(L) = (l_1, \ldots, l_p)$ and let $q \in [1,p]$ be defined by the inequalities $\sum\limits_{i=1}^{q-1} l_i \leqslant 2e_n - 3$ and $\sum\limits_{i=1}^{q-1} l_i + l_p > 2e_n - 3$.

Instead of (G1)–(G4) we are going to use the following assertions:

$$\forall m \in \{3,5,6\}\ 2e_n \equiv 0 \pmod m, \tag{H1}$$
$$n = 0 \Rightarrow 2e_n \equiv 2 \pmod 4, \tag{H2}$$
$$n \in [1,\infty) \Rightarrow 2e_n \equiv 0 \pmod 4, \tag{H3}$$
$$\exists m \in \{3,5,6\}\ 2e_n \equiv m \pmod 7, \tag{H4}$$
$$n = 0 \Rightarrow 2e_n \equiv 6 \pmod 8, \tag{H5}$$
$$n \in [1,\infty) \Rightarrow 2e_n \equiv 0 \pmod 8. \tag{H6}$$

(1) $\sum\limits_{i=1}^{q-1} l_i + l_p = 2e_n \to 3.1$.

(2) If $\sum\limits_{i=1}^{q-1} l_i + l_p \geqslant 2e_n + 3$, then $\sum\limits_{i=q}^{p-1} l_i \leqslant 2e_n - 3 \to 4.1$.

(3) $\exists \delta \in \{-2,-1,1,2\}$ $\sum\limits_{i=1}^{q-1} l_i + l_p = 2e_n + \delta$.

(31) $l_{p-1} \geqslant 9$.

(311) $\delta = -2$.

(3111) If $l_q \leqslant 4$, then $\sum\limits_{i=2}^{q-1} l_i + l_p \leqslant 2e_n - 2 - 3$ and $l_1 + \sum\limits_{i=q}^{p-2} l_i = l_1 + e_{n+1} - (2e_n - 2) - l_{p-1} \leqslant 4 + 2e_n + 2 - 9 \to 4.1$.

(3112) If $l_q \in [5, l_p - 1]$, then $\sum\limits_{i=1}^{q} l_i \leqslant 2e_n - 3$ and $\sum\limits_{i=q+1}^{p-1} l_i = 2e_n + 2 - l_q \leqslant 2e_n - 3 \to 4.1$.

(3113) If $l := l_p$ and $l_i = l$ for any $i \in [q,p]$, we obtain $2e_n + 2 = (p-q)l$ and $2e_n - 2 \geqslant l|2e_n + 2$, so that $l \leqslant e_n + 1$ and $p - q \geqslant 2$. Using (H1)–(H3) we see that $3 \nmid l$, $4 \nmid l$ (because of (H3), $4|l$ implies $n = 0$, $l|152$ and $l \leqslant 8$, while we have $l \geqslant 9$) and $5 \nmid l$, hence $l \geqslant 11$.

(31131) If $l_1 \leqslant l - 5$, then $\sum\limits_{i=2}^{q-1} l_i + l_p \leqslant 2e_n - 2 - 3$ and $l_1 + \sum\limits_{i=q}^{p-2} l_i = 2e_n + 2 - (l - l_1) \leqslant 2e_n - 3 \to 4.1$.

(31132) If $l_j = l-2$ for some $j \in [1, q-1]$, then $\sum\limits_{i=1}^{j-1} l_i + \sum\limits_{i=j+1}^{q+1} l_i = 2e_n \to 3.1$.

(31133) $\forall i \in [1, q-1]$ $l_i \in \{l-4, l-3, l-1, l\}$.

(311331) If $f_{l-1} \geqslant 2$ and $l_j = l_{j+1} = l - 1$ for some $j \in [1, q-1]$, then $\sum\limits_{i=1}^{j-1} l_i + \sum\limits_{i=j+2}^{q+2} l_i = 2e_n \to 3.1$.

(311332) If $f_{l-4} + f_{l-3} \geqslant 2$, then $l-4 \leqslant l_1 \leqslant l_2 \leqslant l-3$, $\sum\limits_{i=3}^{q+1} l_i = 2e_n - 2 + l - (l_1 + l_2) \leqslant 2e_n + 6 - l \leqslant 2e_n - 3$ and $l_1 + l_2 + \sum\limits_{i=q+2}^{p-1} l_i = 2e_n + 2 - (l - l_1) - (l - l_2) \leqslant 2e_n - 4 \to 4.1$.

(311333) If $f_{l-4} + f_{l-3} + f_{l-1} \leqslant 2$, then $2e_n - 2 \equiv -4f_{l-4} - 3f_{l-3} - f_{l-1} \pmod{l}$, $2e_n + 2 \equiv 0 \pmod{l}$, $-4f_{l-4} - 3f_{l-3} - f_{l-1} \equiv -4 \pmod{l}$, $\mathrm{nd}(L) = (l_1, l_2)(l)^{p-2}$, $p = 2q \geqslant 4$ and $(l_1, l_2) \in \{(l-3, l-1), (l-4, l)\}$. Since $pl - 4 = 4e_n$, we have $pl \equiv 0 \pmod 4$.

(3113331) If $q$ is even, $q = 2r$, then $3 \leqslant l_1^1 := \lfloor \frac{l-2}{2} \rfloor \leqslant l - 7 \leqslant l_1 - 3$, $3 \leqslant l_2^1 := \lceil \frac{l-2}{2} \rceil \leqslant l - 4 \leqslant l_2 - 3$, $3 \leqslant l_3^1 := \lceil \frac{l}{2} \rceil \leqslant l - 3 = l_3 - 3$ and $\lfloor \frac{l-2}{2} \rfloor + \lceil \frac{l}{2} \rceil + (r-1)l = e_n = \lceil \frac{l-2}{2} \rceil + (l - \lceil \frac{l}{2} \rceil) + (r-1)l \to 4.3$ (with $i_1 := 1$, $i_2 := 2$, $i_3 := 3$, $I^1 := [4, r+2]$, $I^2 := [r+3, 2r+1]$ and $I^3 := [2r+2, 4r]$).

(3113332) If $q$ is odd, $q = 2r+1$, then $pl = 2(2r+1)l \equiv 0 \pmod 4$ implies that $l$ is even, $l = 2m \geq 12$, $3 \leqslant l_1^1 := m - 1 \leqslant 2m - 7 \leqslant l_1 - 3$, $3 \leqslant l_2^1 := m - 1 \leqslant 2m - 7 \leqslant l_2 - 3$ and $m - 1 + r \cdot 2m = e_n \to 4.4$ (with $i_1 := 1$, $i_2 := 2$, $I^1 := [3, r+2]$, $I^2 := [r+3, 2r+2]$ and $I^3 := [2r+3, 4r+2]$).

(312) If $\delta = -1$, then $3 \leqslant l_{p-1}^1 := 6 \leqslant l_{p-1} - 3$, $3 \leqslant l_p^1 := l_p - 5 \leqslant l_p - 3$, $\big((6)_4, (l_p - 5)_4\big), \big((l_{p-1} - 6)_4, (5)_4\big) \notin P$ and $6 + (l_p - 5) + \sum\limits_{i=1}^{q-1} l_i = 2e_n \to 4.2$.

(313) If $\delta = 1$, then $3 \leqslant l_{p-1}^1 := 5 \leqslant l_{p-1} - 3$, $3 \leqslant l_p^1 := l_p - 6 \leqslant l_p - 3$, $\big((5)_4, (l_p - 6)_4\big), \big((l_{p-1} - 5)_4, (6)_4\big) \notin P$ and $5 + (l_p - 6) + \sum\limits_{i=1}^{q-1} l_i = 2e_n \to 4.2$.

(314) $\delta = 2$.

(3141) If there is $j \in [1, q-1]$ such that $l_j \in [5, l_{p-1}-1]$, then $\sum_{i=1}^{j-1} l_i + \sum_{i=j+1}^{q-1} l_i + l_p \leqslant 2e_n + 2 - 5$ and $l_j + \sum_{i=q}^{p-2} l_i = 2e_n - 2 - (l_{p-1} - l_j) \leqslant 2e_n - 3 \to 4.1$.

(3142) $\forall i \in [1, q-1]$ $l_i \in \{3, 4, l_{p-1}\}$.

(31421) If $f_3 + f_4 \geqslant 2$, then $\sum_{i=3}^{q-1} l_i + l_p \leqslant (2e_n + 2) - 2 \cdot 3$ and $l_1 + l_2 + \sum_{i=q}^{p-2} l_i \leqslant 2 \cdot 4 + (2e_n - 2) - 9 \to 4.1$.

(31422) If $f_3 + f_4 \leqslant 1$, then $l_i = l_{p-1} =: l$ for any $i \in [2, p-1]$, $2e_n + 2 = \sum_{i=1}^{q-1} l_i + l_p \equiv j + l_p \pmod{l}$ for some $j \in \{0, 3, 4\}$ and $2e_n - 2 \equiv 0 \pmod{l}$, so that $j + l_p \equiv 4 \pmod{l}$ and, using (H1)–(H3) $3 \nmid l$, $4 \nmid l$ (as $l \geqslant 9$), $5 \nmid l$ and $l \geqslant 11$.

(314221) If $l_p = l + 2$, then $\sum_{i=1}^{q} l_i = 2e_n \to 3.1$.

(314222) If $l_p \geqslant l + 5$, then $\sum_{i=1}^{q} l_i = (2e_n + 2) - (l_p - l) \leqslant 2e_n - 3$ and $\sum_{i=q+1}^{p-1} l_i = (2e_n - 2) - l \leqslant 2e_n - 11 \to 4.1$.

(314223) If $l_p = l + k$ for some $k \in \{0, 1, 3, 4\}$, then $j + k \equiv 4 \pmod{l}$. Consequently, from $l \geqslant 11$ it follows that $j + k = 4$ and $(j, k) \in \{(0, 4), (3, 1), (4, 0)\}$.

(3142231) If $(j, k) = (0, 4)$, then $p = 2q$, $\mathrm{nd}(L) = (l)^{2q-1}(l+4)$ and $2ql + 4 = 4e_n$, so that $2ql \equiv 0 \pmod{4}$.

(31422311) If $q$ is even, $q = 2r$, then $3 \leqslant l_p^1 := \lfloor \frac{l+2}{2} \rfloor \leqslant l + 1 = l_p - 3$, $3 \leqslant l_1^1 := \lceil \frac{l+2}{2} \rceil \leqslant l - 3 = l_1 - 3$, $3 \leqslant l_2^1 := \lceil \frac{l}{2} \rceil \leqslant l - 3 = l_2 - 3$ and $\lfloor \frac{l+2}{2} \rfloor + \lceil \frac{l}{2} \rceil + (r-1)l = e_n = \lceil \frac{l+2}{2} \rceil + (l - \lceil \frac{l}{2} \rceil) + (r-1)l \to 4.3$.

(31422312) If $q$ is odd, $q = 2r + 1$, then $l$ must be even, $l = 2m$, $3 \leqslant l_p^1 := m + 1 \leqslant 2m + 1 = l_p - 3$, $3 \leqslant l_1^1 := m + 1 \leqslant 2m - 3 = l_1 - 3$ and $m + 1 + r \cdot 2m = e_n \to 4.4$.

(3142232) If $(j, k) \in \{(3, 1), (4, 0)\}$, then $p = 2q + 1$, $\mathrm{nd}(L) = (j)(l)^{2q-1}(l + 4 - j)$ and $2ql \equiv 0 \pmod{4}$.

(31422321) If $q$ is even, $q = 2r$, then $3 \leqslant l_p^1 := \lfloor \frac{l+1-j}{2} \rfloor \leqslant l - 3 \leqslant l_p - 3$, $3 \leqslant l_2^1 := \lceil \frac{l+3-j}{2} \rceil \leqslant l - 3 = l_2 - 3$, $3 \leqslant l_3^1 := \lceil \frac{l+1-j}{2} \rceil \leqslant l - 3 = l_3 - 3$ and $\lfloor \frac{l+1-j}{2} \rfloor + \lceil \frac{l+1-j}{2} \rceil + j + (r-1)l = e_n = \lceil \frac{l+3-j}{2} \rceil + (l - \lceil \frac{l+1-j}{2} \rceil) + (r-1)l \to 4.3$.

(31422322) If $q$ is odd, $q = 2r+1$, then $l$ is even, $l = 2m \geq 12$, $3 \leqslant l_p^1 := m+1-j \leqslant 2m+4-j = l_p$, $3 \leqslant l_2^1 := m+1 \leqslant 2m-3 = l_2 - 3$ and $(m+1-j)+j+r\cdot 2m = e_n = m+1+r\cdot 2m \to 4.4$.

(32) If $l_{p-1} \leqslant 8$, let $r \in [q,p]$ be defined by the inequalities $\sum\limits_{i=1}^{r-1} l_i \leqslant 2e_n - 3$ and $\sum\limits_{i=1}^{r} l_i > 2e_n - 3$.

(321) $\sum\limits_{i=1}^{r} l_i = 2e_n \to 3.1$.

(322) If $\sum\limits_{i=1}^{r} l_i \geqslant 2e_n + 3$, then $\sum\limits_{i=r+1}^{p} l_i \leqslant 2e_n - 3 \to 4.1$.

(323) $\exists \varepsilon \in \{-2,-1,1,2\}$ $\sum\limits_{i=1}^{r} l_i = 2e_n + \varepsilon$.

(3231) $\varepsilon \in [-2,-1]$.

(32311) If $l_p \geqslant l_1 + 3 - \varepsilon$, then $\sum\limits_{i=2}^{r} l_i = 2e_n + \varepsilon - l_1 \leqslant 2e_n - 4$ and $l_1 + \sum\limits_{i=r+1}^{p-1} l_i = l_1 + (2e_n - \varepsilon) - l_p \leqslant 2e_n - 3 \to 4.1$.

(32312) $l_p \leqslant l_1 + 2 - \varepsilon$

(323121) If there are $j \in [1,r]$ and $k \in [r+1,p]$ such that $l_k = l_j - \varepsilon$, then $\sum\limits_{i=1}^{j-1} l_i + \sum\limits_{i=j+1}^{r} l_i + l_k = 2e_n \to 3.1$.

(323122) $\forall j \in [1,r]$ $\forall k \in [r+1,p]$ $l_k \neq l_j - \varepsilon$

(3231221) $\varepsilon = -2$

(32312211) If $l_r = l_1$, then $2e_n - 2 \equiv 0 \pmod{l_1}$ and from (H1)–(H6) it follows that $n = 0$, $l_1 = 4$ and $r = 37$.

(323122111) If $f_5 \geqslant 2$, then $35\cdot 4 + 2\cdot 5 = 2e_0 \to 3.1$.

(323122112) If $f_7 \geqslant 2$, then $34\cdot 4 + 2\cdot 7 = 2e_0 \to 3.1$.

(323122113) If $f_5 + f_7 \leqslant 2$, then $2e_0 + 2 = \sum\limits_{i=38}^{p} l_i \equiv 5f_5 + 7f_7 \pmod 2$ (recall that $l_p \leqslant 8$), and so $f_5 = f_7 \leqslant 1$.

(3231221131) If $l_{p-1} \geqslant 7$, then $3 =: l_{p-1}^1 \leqslant l_{p-1} - 3$, $3 =: l_p^1 \leqslant l_p - 3$, $\big((3)_4,(3)_4\big)$, $\big((l_{p-1}-3)_4,(l_p-3)_4\big) \notin P$ (here we use the inequality $f_7 \leqslant 1$) and $3+3+36\cdot 4 = 2e_0 \to 4.2$.

(3231221132) If $l_{p-1} \leqslant 6$, then $(l_{p-1}, l_p) \in \big\{(4,4),(4,8),(5,7)\big\}$.

(32312211321) If $\mathrm{nd}(L) \in \big\{(4)^{75},(4)^{73}(8)\big\}$, we can use the fact that the graph $G_1 = K_{10,10,10}$ is an edge-disjoint union of graphs $K^i \cong K_{10,10}$, $i =$

$1,2,3$. By [7], the graph $K_{10,10}$ is ADCT, there is a $K^i$-realisation $\mathcal{T}^i$ of $(4)^{25}$, $i=1,2,3$, and a $K^3$-realisation $\bar{\mathcal{T}}^3$ of $(4)^{23}(8)$. Then $\mathcal{T}^1\mathcal{T}^2\mathcal{T}^3$ is a good $H_1$-realisation of $(4)^{75}$ and $\mathcal{T}^1\mathcal{T}^2\bar{\mathcal{T}}^3$ is a good $H_1$-realisation of $(4)^{73}(8)$.

(32312211322) If $\mathrm{nd}(L)=(4)^{72}(5,7)$, consider again the above $H_1$-realisation of $(4)^{75}$. By Lemma 1 and Proposition 6 we may suppose without loss of generality that $\mathcal{T}^1\mathcal{T}^2\mathcal{T}^3=(T_1,T_2,T_3)\mathcal{T}$, where $V(T_i)=\{v_{1,i},\ v_{2,i},\ v_{1,i+1},\ v_{2,i+1}\}$, $i=1,2$, and $V(T_3)=\{v_{1,1},\ v_{2,1},\ v_{1,3},\ v_{2,3}\}$. Now $\mathcal{T}\big((v_{1,1},\ v_{1,2},\ v_{2,1},\ v_{2,2},\ v_{1,3},\ v_{1,1}),\ (v_{1,1},\ v_{2,2},\ v_{2,3},\ v_{2,1},\ v_{1,3},\ v_{1,2},\ v_{2,3},\ v_{1,1})\big)$ is a good $G_1$-realisation of $\mathrm{nd}(L)$.

(32312212) If $l_r=l_1+1$, then $f_{l_1+2}=f_{l_1+3}=0$.

(323122121) If $l_{p-1}=l_1+1$, then $2e_n+2\le(p-r)(l_1+1)+3$, $p-r\ge\frac{2e_n-1}{l_1+1}\ge\lfloor\frac{149}{8}\rfloor=18$ and $l_{r+1}=l_{r+2}=l_1+1$.

(3231221211) If $l_2=l_1$, then $\sum\limits_{i=3}^{r+2}l_i=2e_n\to 3.1$.

(3231221212) If $l_2=l_1+1$, then $2e_n-2=r(l_1+1)-1$ and $2e_n\equiv 1 \pmod{l_1+1}$ in contradiction with (H1)–(H6).

(323122122) If $l_{p-1}=l_1+4$, then $l_1\leqslant 4$.

(3231221221) If $l_1=3$, there is $I\subseteq[1,r]$ such that $\sum\limits_{i\in I}l_i=12$ and $\sum\limits_{i\in[1,r]-I}l_i+l_{p-1}+l_p=2e_n\to 3.1$.

(3231221222) If $l_1=4$, then $4f_4+5f_5=2e_n-2\equiv 0 \pmod 2$, hence $f_5\geqslant 2$, $\sum\limits_{i=1}^{r-2}l_i+l_p=2e_n-4$ and $\sum\limits_{i=r-1}^{p-2}l_i=2e_n-4\to 4.1$.

(32312213) If $l_r=l_1+2$, then $l_j=l_1+3$ for any $j\in[r+1,p]$, $l_i\in\{l_1,l_1+2\}$ for any $i\in[1,r]$, $2e_n+2\equiv 0 \pmod{l_1+3}$ and, since $l_1\leqslant 5$, from (H1) it follows that $l_1\in[4,5]$. We have also $2e_n+2=(p-r)(l_1+3)$, hence $l_{r+1}=l_{r+2}=l_1+3$.

(323122131) If $l_1=4$, there is $I\subseteq[1,r]$ such that $\sum\limits_{i\in I}l_i=12$, hence $\sum\limits_{i\in[1,r+2]-I}l_i=2e_n\to 3.1$.

(323122132) If $l_1=5$, then $n=0$ and $5f_5+7f_7=2e_0-2=148$.

(3231221321) If $f_5\geqslant 2$, then $\sum\limits_{i=3}^{r+1}l_i=2e_0-4=l_1+l_2+\sum\limits_{i=r+2}^{p-1}l_i\to 4.1$.

(3231221322) If $f_7\geqslant 2$, then $\sum\limits_{i=1}^{r-2}l_i+l_{r+1}+l_{r+2}=2e_0\to 3.1$.

(32312214) $l_r=l_1+3$.

(323122141) If $l_{p-1} = l_1 + 3$, then $l_i \in \{l_1, l_1 + 2, l_1 + 3\}$ for any $i \in [1, r]$, there is $m \in [0, 1]$ such that $2e_n + 2 \equiv m \pmod{l_1 + 3}$, $l_1 \in [4, 5]$ and $l_{r+1} = l_{r+2} = l_1 + 3$.

(3231221411) $l_1 = 4$.

(32312214111) If $f_4 \geqslant 3$ or $f_6 \geqslant 2$, there is $I \subseteq [1, r]$ such that $\sum_{i \in I} l_i = 12$ and $\sum_{i \in [1, r+2] - I} l_i = 2e_n \to 3.1$.

(32312214112) $f_4 \leqslant 2 \wedge f_6 \leqslant 1$.

(323122141121) If $l_p = 7$, then $\mathrm{nd}(L) = (4, 6)(7)^{2r-2}$, $14r - 4 = 4e_n$, hence $r$ is even, $r = 2s$ and $7(s-1)+6 = e_n$. By $S(G_n)$ (see the row 2 of Table 1) and Proposition 6 there are good $G_n$-realisations $(T_1^1, T_2^1, T_3^1, T_4^1)\mathcal{T}^1$ of $(3)^3(4)(7)^{s-2}$ and $(T_1^2, T_2^2, T_3^2)\mathcal{T}^2$ of $(3, 4, 6)(7)^{s-2}$ such that $v_{i,2} \in V(T_i^1)$, $i = 1, 2, 3$, and $v_{1,1} \in V(T_4^1) \cap V(T_1^2)$. As $V\big(\varphi_n^1(T_4^1)\big) \cap V\big(\varphi_n^2(T_1^2)\big) \supseteq \{v_{1,1}^1\}$, there is $T \in \varphi_n^1(T_4^1) + \varphi_n^2(T_1^2)$ and $\left[\prod_{i=1}^{3} \big(\varphi_n^1(T_i^1)\big)\right] \big(\varphi_n^2(T_2^2), \varphi_n^2(T_3^2), T\big)\varphi_n^1(\mathcal{T}^1)\varphi_n^2(\mathcal{T}^2)$ is a good $H_n$-realisation of $(3)^3(4, 6)(7)^{2s-3}$ with $v_{i,2}^1 \in V\big(\varphi_n^1(T_i^1)\big)$, $i = 1, 2, 3$. By $S(H_n)$ (the row 7 of Table 2) there is a good $H_n$-realisation $(T_1^3, T_2^3, T_3^3)\mathcal{T}^3$ of $(4)^3(7)^{2s-2}$ such that $v_{i,2}^1 \in V(T_i^3)$, $i = 1, 2, 3$. Since $V\big(\psi_n^1\big(\varphi_n^1(T_i^1)\big)\big) \cap V\big(\psi_n^2(T_i^3)\big) \supseteq \{v_{i,2}^{1,1}\}$, there is a trail $T_i \in \psi_n^1\big(\varphi_n^1(T_i^1)\big) + \psi_n^2(T_i^3)$, $i = 1, 2, 3$, and $\big(\psi_n^1\big(\varphi_n^2(T_2^2)\big)\big), \psi_n^1\big(\varphi_n^2(T_3^2)\big), T_1, T_2, T_3, \big(\psi_n^1(T)\big)\psi_n^1\big(\varphi_n^1(\mathcal{T}^1)\varphi_n^2(\mathcal{T}^2)\big)\psi_n^2(\mathcal{T}^3)$ is a good $G_{n+1}$-realisation of $\mathrm{nd}(L)$.

(323122141122) If $l_p = 8$, then $\mathrm{nd}(L) = (4)(7)^{2r-2}(8)$, $14r-2 = 4e_n$, hence $r$ is odd, $r = 2s+1$, $3 =: l_p^1 \leqslant l_p - 3$, $3 =: l_2^1 \leqslant l_2 - 3$, $\big((3)_4, (3)_4\big), \big((5)_4, (4)_4\big) \notin P$ and $3 + 3 + 2s \cdot 7 = 2e_n \to 4.2$.

(3231221412) If $l_1 = 5$, then $n = 0$, $l_p = 8$, $2e_0 - 2 = 148 \equiv 5f_5 + 7f_7 \pmod 2$ and $f_5 + f_7 \geqslant 2$.

(32312214121) If $f_5 \geqslant 2$, then $\sum_{i=3}^{r+1} l_i = 2e_0 - 4 = l_1 + l_2 + \sum_{i=r+2}^{p-1} l_i \to 4.1$.

(32312214122) If $f_7 \geqslant 2$, and $l_j = l_{j+1} = 7$ for some $j \in [2, r-2]$, then $\sum_{i=1}^{j-1} l_i + \sum_{i=j+2}^{r+2} l_i = 2e_0 \to 3.1$.

(32312214123) If $f_5 = f_7 = 1$, then $\mathrm{nd}(L) = (5, 7)(8)^{36}$. By $S(G_0)$ and Propositions 6 and 16.1 there are good $G_0$-realisations $(T_1^1, T_2^1, T_3^1)\mathcal{T}^1$ of $(3)(4)^2(8)^8$ and $(T_1^2, T_2^2, T_3^2)\mathcal{T}^2$ of $(4)^2(3)(8)^8$ such that $v_{i,1} \in V(T_i^1) \cap V(T_i^2)$, $i = 1, 2$. As $V\big(\varphi_n^1(T_i^1)\big) \cap V\big(\varphi_n^2(T_i^2)\big) \supseteq \{v_{i,1}^1\}$, there is a trail $T_i \in \varphi_n^1(T_i^1) + \varphi_n^2(T_i^2)$, $i = 1, 2$, $\big(\varphi_n^2(T_3^2), \varphi_n^1(T_3^1), T_1, T_2\big)\varphi_n^1(\mathcal{T}^1)\varphi_n^2(\mathcal{T}^2)$ is a good $(3, 4)$-global

$H_0$-realisation of $(3,4,7)(8)^{17}$. Since $3 =: l_3^1 \leqslant l_3 - 3$ and $3 \leqslant l_4^1 := 4 \leqslant l_4 - 3$, we are done by Claim 3.3.

(323122142) If $l_{p-1} = l_1 + 4$, then $l_1 \leqslant 4$ and $f_{l_1+2} = 0$.

(3231221421) If $l_1 = 3$, there is $I \subseteq [1,r]$ such that $\sum_{i\in I} l_i = 12$ and $\sum_{i\in[1,r]-I} l_i + l_{p-1} + l_p = 2e_n \to 3.1$.

(3231221422) If $l_1 = 4$, then $\sum_{i=2}^{r-1} l_i + l_p = 2e_n - 5$ and $l_1 + \sum_{i=r}^{p-2} l_i = 2e_n - 3 \to 4.1$.

(32312215) If $l_r = l_1 + 4$, then $2e_n + 2 \equiv 0 \pmod{l_1 + 4}$, $2e_n - 2 \equiv l_1 \pmod{l_1 + 4}$, $f_{l_1+2} = 0$ and $l_1 \leqslant 4$.

(323122151) $l_1 = 3$.

(3231221511) If $f_3 \geqslant 4$, $f_4 \geqslant 3$ or $f_6 \geqslant 2$, there is $I \subseteq [1,r]$ such that $\sum_{i\in I} l_i = 12$ and $\sum_{i\in[1,r+2]-I} l_i = 2e_n \to 3.1$.

(3231221512) $f_3 \leqslant 3 \wedge f_4 \leqslant 2 \wedge f_6 \leqslant 1$.

(32312215121) If $f_4 = 2$ and $l_j = l_{j+1} = 4$ for some $j \in [2, r-2]$, then $\sum_{i=1}^{j-1} l_i + \sum_{i=j+2}^{r+1} l_i = 2e_n - 3$ and $l_j + l_{j+1} + \sum_{i=r+2}^{p-1} l_i = 2e_n - 4 \to 4.1$.

(32312215122) $f_4 = 1$.

(323122151221) If $\mathrm{nd}(L) = (3)^2(4)(7)^{\frac{4e_n-10}{7}}$, by $S(H_n)$ (the rows 1 and 7 of Table 2) there are good $H_n$-realisations $(T_1^1, T_2^1, T_3^1)\mathcal{T}^1$ of $(3)^4(7)^{\frac{2e_n-12}{7}}$ and $(T_1^2, T_2^2, T_3^2)\mathcal{T}^2$ of $(4)^3(7)^{\frac{2e_n-12}{7}}$ such that $V(T_i^1) \cap V(T_i^2) \cap V_{n,2}^1 \supseteq \{v_{i,2}^1\}$, $i = 1,2,3 \to 3.2$ (with $i_1 := 4$, $i_2 := 5$, $I^1 := [1,2] \cup [6, r+2]$, $I^2 := \{3\} \cup [r+3, 2r-1]$ and $l_{i_k}^1 := 3$, $l_{i_k}^2 := 4$, $k = 1,2$).

(323122151222) If $\mathrm{nd}(L) = (4,6)(7)^{\frac{4e_n-10}{7}}$, by $S(H_n)$ (the rows 4 and 7 of Table 2) there are good $H_n$-realisations $(T_1^1, T_2^1)\mathcal{T}^1$ of $(3)^2(6)(7)^{\frac{2e_n-12}{7}}$ and $(T_1^2, T_2^2)\mathcal{T}^2$ of $(4)^3(7)^{\frac{2e_n-12}{7}}$ such that $V(T_i^1) \cap V(T_i^2) \cap V_{n,2}^1 \supseteq \{v_{i,2}^1\}$, $i = 1,2 \to 3.2$.

(32312215123) If $f_4 = 0$, then $\mathrm{nd}(L) = (3)(7)^{\frac{4e_n-3}{7}}$ and we can proceed analogously as in (323122151221) (with $i_1 := 2$, $i_2 := 3$, $i_3 := 4$, $I^1 := \{1\} \cup [5, r+2]$, $I^2 := [r+3, 2r]$ and $l_{i_k}^1 := 3$, $l_{i_k}^2 := 4$, $k = 1,2,3$).

(3231222) If $\varepsilon = -1$, then $l_p \leqslant l_1 + 3$.

(32312221) If $l_r = l_1$, then $2e_n - 1 \equiv 0 \pmod{l_1}$ in contradiction with (H1)–(H6).

(32312222) If $l_r = l_1 + 1$, then $l_j = l_1 + 3$ for any $j \in [r+1, p]$, $2e_n + 1 \equiv 0 \pmod{l_1 + 3}$, $l_1 \leqslant 5$ and, because of (H1)–(H6), $l_1 = 4$. Since $4f_4 + 5f_5 = 2e_n - 1 \geqslant 149$, there is $I \subseteq [1, r]$ such that $\sum_{i \in I} l_i = 20$ and $\sum_{i \in [1, r+3] - I} l_i = 2e_n \to 3.1$.

(32312223) If $l_r = l_1 + 2$, then $l_j = l_1 + 2$ for any $j \in [r+1, p]$, hence $2e_n + 1 \equiv 0 \pmod{l_1 + 2}$, $l_1 \leqslant 6$ and, because of (H1)–(H6), $l_1 = 5$ and $2e_n - 1 \equiv 5 \pmod 7$. Moreover, $l_i \in \{5, 7\}$ for any $i \in [1, r]$.

(323122231) If $f_5 \geqslant 4$, then $\sum_{i=5}^{r+3} l_i = 2e_n \to 3.1$.

(323122232) If $f_5 \leqslant 3$, then $\mathrm{nd}(L) = (5)(7)^{\frac{4e_n - 5}{7}}$, $l_i^1 := 3 \leqslant l_i - 3$, $i = 2, 3$, by $S(G_n)$ there is a good $G_n$-realisation $\mathcal{T}^1$ of $(3)(7)^{\frac{e_n - 3}{7}}$ and $\varphi_n^1(\mathcal{T}^1)\varphi_n^2(\mathcal{T}^1)$ is a good $(3,3)$-global $H_n$-realisation of $(3)^2(7)^{\frac{2e_n - 6}{7}} \to 3.3$.

(32312224) If $l_r = l_1 + 3$, then $2e_n + 1 \equiv 0 \pmod{l_1 + 3}$, $l_1 \leqslant 5$, from (H1)–(H6) it follows that $l_1 = 4$, hence $f_6 = 0$ and $2e_n - 1 \equiv 5 \pmod 7$.

(323122241) $\exists j \in [2, r-1]\ l_j = 5$.

(3231222411) If $f_4 \geqslant 2$, then $\sum_{i=3}^{j-1} l_i + \sum_{i=j+1}^{r+2} l_i = 2e_n \to 3.1$.

(3231222412) $f_4 = 1$.

(32312224121) If $f_5 \geqslant 4$, then $l_1 + \sum_{i=6}^{r+3} l_i = 2e_n \to 3.1$.

(32312224122) If $f_5 \leqslant 3$, then $\mathrm{nd}(L) = (4)(5)^3(7)^{\frac{4e_n - 19}{7}}$, $l_1 + l_2 + \sum_{i=5}^{r+1} l_i = 2e_n - 4$ and $l_3 + l_4 + \sum_{i=r+2}^{p-1} l_i = 2e_n - 3 \to 4.1$.

(323122242) $f_5 = 0$.

(3231222421) If $f_4 \geqslant 5$, then $\sum_{i=6}^{r+3} l_i = 2e_n \to 3.1$.

(3231222422) If $f_4 \leqslant 4$, then $\mathrm{nd}(L) = (4)^3(7)^{\frac{4e_n - 12}{7}}$. By $S(H_n)$ (the rows 4 and 6 of Table 2) and Proposition 19.1 there are good $H_n$-realisations $(T_1^1, T_2^1)\mathcal{T}^1$ of $(3)^2(7)^{\frac{2e_n - 6}{7}}$ and $[\prod_{i=1}^{5}(T_i)]\mathcal{T}^2$ of $(4)^5(7)^{\frac{2e_n - 20}{7}}$ such that $v_{i,3}^2 \in V(T_i^1)$, $i = 1, 2$, $v_{i,2}^1 \in V(T_i^2)$, $i = 1, 2, 3$, and $v_{i,3}^1 \in V(T_{3+i}^2)$, $i = 1, 2$. As $V(\psi_n^1(T_i^1)) \cap V(\psi_n^2(T_{3+i}^2)) \supseteq \{v_{i,3}^{2,1}\} = \{v_{i,3}^{1,2}\}$, there is a trail $T_i \in \psi_n^1(T_i^1) + \psi_n^2(T_{3+i}^2)$, $i = 1, 2$, and $\psi_n^2((T_1^2, T_2^2, T_3^2)\mathcal{T}^2)(T_1, T_2)\psi_n^1(\mathcal{T}^1)$ is a good $G_{n+1}$-realisation of $\mathrm{nd}(L)$ with $v_{i,2}^{1,2} \in \psi_n^2(T_i^2)$, $i = 1, 2, 3$. Since $p_{(5 \cdot 2^n + 1)3}(v_{i,2}^{1,2}) = 2$, $i = 1, 2, 3$, by

Proposition 6 there is a good $G_{n+1}$-realisation $(\bar{T}_1, \bar{T}_2, \bar{T}_3)\bar{\mathcal{T}}$ of $\mathrm{nd}(L)$ such that $v_{i,1} \in V(\bar{T}_i)$, $i = 1, 2, 3$ (satisfying the conditions of the row 5 of Table 1).

(3232) If $\varepsilon \in [1, 2]$, then $l_r = \sum_{i=1}^{r} l_i - \sum_{i=1}^{r-1} l_i \geqslant 2e_n + \varepsilon - (2e_n - 3) = 3 + \varepsilon$. With $l := \min(l_i \colon i \in [1, r], l_i \geqslant 3 + \varepsilon)$ we have $3 + \varepsilon \leqslant l \leqslant l_r$.

(32321) If $l_p \geqslant l + 3 - \varepsilon$, let $j \in [1, r]$ be such that $l_j = l$. Then $\sum_{i=1}^{j-1} l_i + \sum_{i=j+1}^{r} l_i = 2e_n + \varepsilon - l \leqslant 2e_n + \varepsilon - (3 + \varepsilon)$ and $l_j + \sum_{i=r+1}^{p-1} l_i = l + (2e_n - \varepsilon) - l_p \leqslant 2e_n - 3 \to 4.1$.

(32322) $\forall i \in [1, p]$ $l_i \in [3, 2 + \varepsilon] \cup [l, l + 2 - \varepsilon]$.

(323221) $\varepsilon = 1$.

(3232211) If $l_r = l + 1$, then $2e_n - 1 \equiv 0 \pmod{l + 1}$ in contradiction with $l \leqslant 7$ and (H1)–(H6).

(3232212) $l_r = l$.

(32322121) If $l_{p-1} = l$, then $2e_n - 1 \equiv k \pmod{l}$ for some $k \in [0, 1]$ and then from (H1)–(H6) it follows that $l = 4$, $n = 0$, $2e_0 + 1 \equiv 3 \pmod 4$ and $f_3 \geqslant 1$.

(323221211) If $f_3 \geqslant 3$, then $\sum_{i=4}^{r+2} l_i = 2e_0 \to 3.1$.

(323221212) If $f_3 \leqslant 2$, then $\mathrm{nd}(L) = (3)(4)^{73}(5)$. If $\mathcal{T}$ is the sequence of closed trails from (32312211322), then $\big((v_{1,1}, v_{1,2}, v_{1,3}, v_{1,1}), (v_{1,3}, v_{2,1}, v_{2,3}, v_{2,2}, v_{1,3})\big)$ $\mathcal{T}$ $\big((v_{1,1}, v_{2,2}, v_{2,1}, v_{1,2}, v_{2,3}, v_{1,1})\big)$ is a good $G_1$-realisation of $\mathrm{nd}(L)$.

(32322122) $l_{p-1} = l + 1$.

(323221221) If $f_3 = 0$, then $2e_n + 1 = rl$, hence from $l \leqslant 7$ and (H1)–(H4) it follows that $l = 7$, $\sum_{i=1}^{r-2} l_i + l_p = 2e_n - 5$ and $\sum_{i=r-1}^{p-2} l_i = 2e_n - 3 \to 4.1$.

(323221222) $f_3 \geqslant 1$.

(3232212221) $l = 4$.

(32322122211) If $f_3 = 1$, then $2e_n + 1 \equiv 3 \pmod 4$, hence from (H3) it follows that $n = 0$ and $2e_0 - 1 \equiv 1 \pmod 4$. Further, $2e_0 - 1 \equiv f_5 \pmod 4$, $f_5 \geqslant 5$ (as $l_{p-1} = l_p = 5$) and $\sum_{i=1}^{r-4} l_i + \sum_{i=p-2}^{p} l_i = 2e_0 \to 3.1$.

(32322122212) If $f_3 \geqslant 2$, then $\sum_{i=3}^{r} l_i + l_p = 2e_n \to 3.1$.

(3232212222) $l \in [5,7]$.

(32322122221) If $l_{r-1} = 3$, then $2e_n + 1 \equiv l \pmod 3$, hence, by (H1), $l = 7$ and $\sum_{i=4}^{r} l_i + l_p = 2e_n \to 3.1$.

(32322122222) If $l_{r-1} = l$, then $\sum_{i=1}^{r-2} l_i + l_{p-1} = (2e_n + 1 - 2l) + (l+1) \leqslant 2e_n - 3$ and $\sum_{i=r-1}^{p-2} l_i = 2e_n - 3 \to 4.1$.

(323222) If $\varepsilon = 2$, then $2e_n - 2 \equiv 0 \pmod l$ in contradiction with (H1) and (H4).

Now, it remains to be proved that there are good $G_{n+1}$-realisations of four sequences from $\mathrm{Sct}(G_{n+1})$ according to the row $t$ of Table 1, $t \in [1,4]$.

$t = 1$: By $S(H_n)$ (the row 4 of Table 2) and Proposition 19.1 there are good $H_n$-realisations $(T_1^1, T_2^1)\mathcal{T}^1$ and $(T_1^2)\mathcal{T}^2$ of $(3)^2(7)^{\frac{2e_n-6}{7}}$ such that $v_{i,2}^1 \in V(T_i^1)$, $i = 1,2$, and $v_{3,2}^1 \in V(T_1^2)$. If $\bar{T}_i := \psi_n^1(T_i^1)$, $i = 1,2$, and $\bar{T}_3 := \psi_n^2(T_1^2)$, then $(\bar{T}_1, \bar{T}_2, \bar{T}_3)\psi_n^1(\mathcal{T}^1)\psi_n^2(\mathcal{T}^2)$ is a good $G_{n+1}$-realisation of $(4)^3(7)^{\frac{4e_n-12}{7}}$ with $v_{i,2}^{1,1} \in V(\bar{T}_i)$, $i = 1,2,3$. As $p_{(5\cdot 2^{n+1})3}(v_{i,2}^{1,1}) = 2$, $i = 1,2,3$, it suffices to use Proposition 6.

$t = 2$: By $S(H_n)$ (the row 2 of Table 2) and Proposition 19.1 there are good $H_n$-realisations $(T_1^1, T_2^1)\mathcal{T}^1$ of $(3)^2(4)(7)^{\frac{2e_n-10}{7}}$ and $(T_1^2)\mathcal{T}^2$ of $(3)(7)^{\frac{2e_n-3}{7}}$ such that $v_{i,2}^1 \in V(T_i^1)$, $i = 1,2$, and $v_{3,2}^1 \in V(T_1^2)$. If $\bar{T}_i := \psi_n^1(T_i^1)$, $i = 1,2$, and $\bar{T}_3 := \psi_n^2(T_1^2)$, then $(\bar{T}_1, \bar{T}_2, \bar{T}_3)\psi_n^1(\mathcal{T}^1)\psi_n^2(\mathcal{T}^2)$ is a good $G_{n+1}$-realisation of $(3)^3(4)(7)^{\frac{4e_n-13}{7}}$ with $v_{i,2}^{1,1} \in V(\bar{T}_i)$, $i = 1,2,3$. As for $t = 1$, employ Proposition 6.

$t = 3$: By $S(H_n)$ (the rows 7 and 8 of Table 2) and Proposition 19.1 there are good $H_n$-realisations $(T_1^j, T_2^j, T_3^j)\mathcal{T}^j$ of $(4)^3(7)^{\frac{2e_n-12}{7}}$, $j = 1,2$, such that $v_{i,3}^1 \in V(T_i^1)$, $i = 1,2,3$, $v_{i,2}^1 \in V(T_i^2)$, $i = 1,2$, and $v_{1,1}^1 \in V(T_3^2)$. If $\bar{T}_i^j := \psi_n^j(T_i^1)$, $i = 1,2,3$, $j = 1,2$, then $\left[\prod_{i=1}^{3}(\bar{T}_i^1)\right]\left[\prod_{i=1}^{3}(\bar{T}_i^2)\right]\psi_n^1(\mathcal{T}^1)\psi_n^2(\mathcal{T}^2)$ is a good $G_{n+1}$-realisation of $(4)^6(7)^{\frac{4e_n-24}{7}}$ with $v_{i,3}^{1,1} \in V(\bar{T}_i^1)$, $i = 1,2,3$, $v_{i,2}^{1,2} \in V(\bar{T}_i^2)$, $i = 1,2$, and $v_{1,1}^{1,2} \in V(\bar{T}_3^2)$. Since $p_{(5\cdot 2^{n+1})3}(v_{i,3}^{1,1}) = 3$, $i = 1,2,3$, $p_{(5\cdot 2^{n})3}(v_{i,2}^{1,2}) = 2$, $i = 1,2$, and $p_{(5\cdot 2^{n+1})3}(v_{1,1}^{1,2}) = 1$, we are done again due to Proposition 6.

$t = 4$: By $S(H_n)$ (the row 5 of Table 2) and Proposition 19.1 there are good $H_n$-realisations $\left[\prod_{i=1}^{6}(T_i^1)\right]\mathcal{T}^1$ of $(4)^6(7)^{\frac{2e_n-24}{7}}$ and $(T_1^2)\mathcal{T}^2$ of $(3)(7)^{\frac{2e_n-3}{7}}$ such that $v_{i,2}^1 \in V(T_i^1)$, $v_{i,3}^2 \in V(T_{3+i}^1)$, $i = 1,2,3$, and $v_{3,3}^1 \in V(T_1^2)$. As

$V(\psi_n^1(T_6^1)) \cap V(\psi_n^2(T_1^2)) \supseteq \{v_{3,3}^{2,1}\} = \{v_{3,3}^{1,2}\}$, there is a trail $\bar{T} \in \psi_n^1(T_6^1) + \psi_n^2(T_1^2)$ and $\left[\prod_{i=1}^{5}(\psi_n^1(T_i^1))\right](\bar{T})\psi_n^1(\mathcal{T}^1)\psi_n^2(\mathcal{T}^2)$ is a good $G_{n+1}$-realisation of $(4)^5(7)^{\frac{4e_n-20}{7}}$ with $v_{i,2}^{1,1} \in V(\bar{T}_i)$, $i = 1,2,3$, and $v_{i,3}^{2,1} \in V(\bar{T}_{3+i})$, $i = 1,2$. Because of $p_{(5\cdot 2^{n+1})3}(v_{i,2}^{1,1}) = 2$, $i = 1,2,3$, and $p_{(5\cdot 2^n)3}(v_{i,3}^{2,1}) = 3$, $i = 1,2$, we are done with help of Proposition 6. $\square$

## REFERENCES


[1] BALISTER, P. N.: *Packing circuits into $K_N$*, Combin. Probab. Comput. **6** (2001), 463–499.

[2] BALISTER, P. N.: *Packing closed trails into dense graphs*, J. Combin. Theory Ser. B **88** (2003), 107–118.

[3] BILLINGTON, E. J.—CAVENAGH, N. J.: *Decompositions of complete multipartite graphs into cycles of even length*, Graphs Combin. **16** (2000), 49–65.

[4] BILLINGTON, E. J.—CAVENAGH, N. J.: *On decomposing complete tripartite graphs into 5-cycles*, Australas. J. Combin. **22** (2000), 41–62.

[5] CAVENAGH, N. J.: *Further decompositions of complete tripartite graphs into 5-cycles*, Discrete Math. **256** (2002), 55–81.

[6] COCKAYNE, J. E.—HARTNELL, B. L.: *Edge partitions of complete multipartite graphs into equal length circuits*, J. Combin. Theory Ser. B **23** (1977), 174–183.

[7] HORŇÁK, M.—WOŹNIAK, M.: *Decomposition of complete bipartite even graphs into closed trails*, Czechoslovak Math. J. **53** (2003), 127–134.

[8] SOTTEAU, D.: *Decomposition of $K_{m,n}$ $(K_{m,n}^*)$ into cycles (circuits) of length $2k$*, J. Combin. Theory Ser. B **30** (1981), 75–81.


Received September 2, 2004

*Mirko Horňák*
*Institute of Mathematics*
*P. J. Šafárik University*
*SK–040 01 Košice*
*SLOVAK REPUBLIC*
*E-mail*: mirko.hornak@upjs.sk

*Zuzana Kocková*
*Credium, a.s.*
*Vladislavova 17*
*CZ–110 00 Praha 1*
*CZECH REPUBLIC*
*E-mail*: kockova@credium.cz