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BOUNDARY VALUE PROBLEM FOR NONLINEAR ULTRAPARABOLIC EQUATION IN UNBOUNDED WITH RESPECT TO TIME VARIABLE DOMAIN

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Abstract. This article describes the solvability in the generalized Lebesgue spaces of a mixed problem and a problem without initial condition for nonlinear ultraparabolic equation. The result is obtained without restrictions on the behaviour of the solution at infinity. The property of decreasing for the solution as $T \to \infty$ is proved. Some estimates for the solution in the time unbounded variable domain are obtained.

1. Introduction

The equation of ultraparabolic type $u_t + xu_y + u_{xx} = f(x, y, t)$ was firstly introduced in the work of A. N. Kolmogorov in 1934 for description of non-isotropic processes [6]. Later equations of such a type appeared in mechanics, physics, biology and in other branches of science.

In this paper we considered the mixed problem for an ultraparabolic equation in a bounded domain and the problem without initial conditions for this equation. The equation contains power nonlinearities with variable exponents. The solvability is proved in generalized Lebesgue spaces. These spaces were considered in [7]. The property of decreasing of the solution as $T \to \infty$ and estimates for the solution of those problems were founded. The definitions of strong and weak solutions for mixed problem for ultraparabolic equation are introduced. The same definition for hyperbolic operators of the first order was given earlier in [14]. The solvability of the problem without initial conditions is proved without assumption on growth of the initial data and without any restriction on the behaviour of the solution at infinity. These results are similar to those

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one for parabolic equations [1], [3]. Note that problems for ultraparabolic equations were earlier studied in [5]–[6], [10]–[13]and [15]–[17]. The solvability in generalized Lebesgue spaces for equations with first or second derivative on t is considered in [7], [9].

2. Mixed problem in a bounded domain

Let $\Omega \subset \mathbb{R}^n$ and $G \subset \mathbb{R}^l$ be bounded domains with boundaries $\partial\Omega$ and ∂G, respectively, $D = \Omega\times G, T > 0$; $\tau \in (0,T]$; $Q_{0,\tau} = D\times(0,\tau)$; $D_\tau = Q_{0,T}\bigcap\{t=\tau\}$, $\partial\Omega \in C^1$, $\partial D \in C^1$, $\Sigma_{0,T} = \partial\Omega \times G \times (0,T)$, $S_{0,\tau} = \Omega \times \partial G \times (0,\tau)$, ν be an external normal vector to the surface $S_{0,T}$, ∂D be a boundary of the domain D.

Consider functions $a,\ g,\ \lambda,\ p$ that satisfy the following conditions:

(A) $a_i \in L^\infty(Q_{0,T})$, $a_i(x,y,t) \geqslant a_0 > 0$ for almost all $(x,y,t) \in Q_{0,T}$, $i \in \{1,\dots,n\}$;

(G) $g \in L^\infty(Q_{0,T})$, $g(x,y,t) \geqslant g_0 > 0$ for almost all $(x,y,t) \in Q_{0,T}$;

(L) $\lambda_i \in C(0,T;C(\overline{D}))$, $\lambda_{iy_i} \in L^\infty(Q_{0,T})$ for almost all $(x,y,t) \in Q_{0,T}$ and all $i \in \{1,\dots,l\}$;

(P) $p : \Omega \to (1,\infty)$, $p \in L^\infty(\Omega)$, $1 < p_1 = \operatorname*{ess\,inf}_\Omega p(x) \leqslant \operatorname*{ess\,sup}_\Omega p(x) = p_2 < +\infty$.

In the domain $Q_{0,T}$ we consider the problem

$$
\begin{aligned}
u_t + \sum_{i=1}^{l} \lambda_i(x,y,t)u_{y_i} - \sum_{i=1}^{n} \big(a_i(x,y,t)|u_{x_i}|^{p(x)-2}u_{x_i}\big)_{x_i}& \\
+ g(x,y,t)|u|^{p(x)-2}u = f(x,y,t),& \qquad (1)
\end{aligned}
$$

$$
u|_{S^1_{0,T}} = 0; \quad u|_{\Sigma_{0,T}} = 0, \qquad (2)
$$

$$
u(x,y,0) = u_0(x,y). \qquad (3)
$$

Here $S^1_{0,T}$ denotes those part of the surface $S_{0,T}$ on which

$$
\sum_{i=1}^{l} \lambda_i(x,y,t)\cos(\nu,y_i) < 0,
$$

and $S^2_{0,T}$ denotes the part of the surface $S_{0,T}$ on which

$$
\sum_{i=1}^{l} \lambda_i(x,y,t)\cos(\nu,y_i) \geqslant 0.
$$

Let us assume that condition

(S) There exists $\Gamma_1 \subset \mathbb{R}^{l-1}$ such that $\operatorname{mes}\Gamma_1 > 0$ and the surface $S^1_{0,T}$ can be represented as $S^1_{0,T} = \Omega \times \Gamma_1 \times (0,T)$ is fulfilled.

Denote by $\Gamma_2 = \partial D \backslash \Gamma_1$, $\Delta_x = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}$; $\Delta_y = \frac{\partial^2}{\partial y_1^2} + \dots + \frac{\partial^2}{\partial y_l^2}$.
Let us consider the space $L^{p(x)}(D)$ [7] with the norm

$$\|v; L^{p(x)}(D)\| = \inf\Big\{\mu > 0 : \int_D |v|^{p(x)}/\mu^{p(x)}\, \mathrm{d}x\,\mathrm{d}y \leqslant 1\Big\}$$

and introduce spaces

$$V(Q_{0,T}) = \big\{v(x,y,t) :\ v \in L^2(Q_{0,T}); v, v_{x_i} \in L^{p(x)}(Q_{0,T}),\\ i \in \{1,\dots,n\},\quad v|_{\Sigma_{0,T}} = 0\big\}$$

with the norm

$$\|v; V(Q_{0,T})\| = \|v; L^2(Q_{0,T})\| + \sum_{i=1}^{n} \|v_{x_i}; L^{p(x)}(Q_{0,T})\| + \|v; L^{p(x)}(Q_{0,T})\|$$

and

$$W(Q_{0,T}) = \big\{v(x,y,t) :\ v \in W^{1,2}(Q_{0,T}),\ v, v_{x_i} \in L^{p(x)}(Q_{0,T}),\\ i \in \{1,\dots,n\},\quad v|_{\Sigma_{0,T}} = 0,\quad v|_{S^1_{0,T}} = 0\big\}$$

with the norm

$$\|v; W(Q_{0,T})\| = |v; L^2(Q_{0,T})\| + \|v_t; L^2(Q_{0,T})\| + \sum_{i=1}^{l} \|v_{y_i}; L^2(Q_{0,T})\|\\ + \sum_{i=1}^{n} \|v_{x_i}; L^{p(x)}(Q_{0,T})\| + \|v; L^{p(x)}(Q_{0,T})\|;$$

the space $H^2_{\Delta_x}(\Omega)$ [8] with the norm $\left(\int\limits_\Omega \left[u^2 + \sum\limits_{i=1}^{n} u^2_{x_i} + (\Delta_x u)^2\right] \mathrm{d}x\right)^{\frac{1}{2}}$,

$H^1_{1,0}(G) = \big\{v : v \in H^1(G),\, v|_{\Gamma_1} = 0\big\}$, $\quad H^2_{\Delta_x,0}(\Omega) = \big\{v \in H^2_{\Delta_x}(\Omega),\ v|_{\partial\Omega} = 0\big\}$,
the space

$$V^{1,1}(D) = \big\{v :\ v \in W^{1,2}(D),\ \Delta_x v \in L^2(D),\ v_{x_i y_j} \in L^2(D),\ \Delta_x v_{y_j} \in L^2(D),\\ i \in \{1,\dots,n\}, j \in \{1,\dots,l\},\ v|_{\partial\Omega\times G} = 0,\ v|_{\Gamma_1\times\Omega} = 0\big\}$$

with the norm

$$\|v; V^{1,1}(D)\| = \|v; W^{1,2}(D)\| + \|\Delta_x v;\, L^2(D)\| + \|\Delta_x v_{y_j};\, L^2(D)\| + \sum_{i=1}^{n}\sum_{j=1}^{l} \|v_{x_i y_j}; L^2(D)\|.$$

We also consider spaces $V(D),\ W(D)$ that correspond to $V(Q_{0,T}),\ W(Q_{0,T})$:
$V(D) = \{v(x,y):\ v \in L^2(D),\ v,\ v_{x_i} \in L^{p(x)}(D),\ i \in \{1,\dots,n\},\ v|_{\partial\Omega\times G} = 0\}$
$W(D) = \{v(x,y):\ v \in W^{1,2}(D),\ v,\ v_{x_i} \in L^{p(x)}(D),\ i \in \{1,\dots,n\}, v|_{\partial\Omega\times G} = 0,$ $v|_{\Gamma_1\times\Omega} = 0\}$.
The space $C([0,T]; V(D))$ contains functions u that are defined on $[0,T]$ with values $V(D)$, such that

$$\|u(\cdot,\cdot,t); V(D)\| \in ([0,T])$$

and the norm

$$\big\|u; C([0,T];\, V(D))\big\| = \max_{[0,T]} \|u; V(D)\| \text{ is finite.}$$

Remark 1.

1) The spaces $W(Q_{0,T})$, $V(Q_{0,T})$ are Banach spaces.
2) The embedding $C([0,T]; V(D))$ in $V(Q_{0,T})$ is dense and continuous.
3) If $\partial\Omega \subset C^2$, then $H^2_{\Delta_x,0}(\Omega)$ is equal to $H^2_0(\Omega)$.

Denote by

$$L(u,v,f) \equiv u_t v + \sum_{i=1}^{l} \lambda_i(x,y,t) u_{y_i} v + \sum_{i=1}^{n} a_i(x,y,t)|u_{x_i}|^{p(x)-2} u_{x_i} v_{x_i} + g(x,y,t)|u|^{p(x)-2} uv - f(x,y,t)v.$$

Definition 1. Function u is said to be a generalized solution of the problem (1)–(3) if $u \in W(Q_{0,T})$ and it satisfies the equality $\int_{Q_{0,T}} L(u,v,f)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = 0$ for all functions $v \in V(Q_{0,T})$ and the initial condition (3) holds.

Theorem 1. *Under the hypotheses (A), (G), (L), (S), (P) the problem* (1)–(3) *cannot have more than one generalized solution.*

If, besides, $\{f,\ f_{y_i}\} \subset L^2(Q_{0,T})$, $u_0 \in W(D)$, $f|_{S^1_{0,T}} = 0$*; and* $2 < p_2 < \frac{2(n+l)}{n+l-2}$ *if* $n+l > 2$*, or* $2 < p_2 < +\infty$ *if* $n = l = 1$*, then the generalized solution of the problem* (1)–(3) *exists.*

P r o o f . Firstly, we shall obtain the existence of the generalized solution of the problem (1)–(3). Let $\{\varphi^k\}_{k=1}^{\infty}$ be the orthogonal basis of the space $H^2_{\Delta_x,0}(\Omega)$, which is orthonormal in the space $L^2(\Omega)$, where φ^k are eigenfunctions of the problem $\Delta_x u = \nu u$, $u\big|_{\partial\Omega} = 0$, that correspond to eigenvalues ν_k; $\{\psi^m\}_{m=1}^{\infty}$ be the orthogonal basis of the space $H^1_{1,0}(G)$, orthonormal in the space $L^2(G)$, where ψ^m $(m \geqslant 1)$ are eigenfunctions of the problem

$$\Delta_y u = \mu u, \quad u|_{\Gamma_1} = 0, \quad \frac{\partial u}{\partial v}\bigg|_{\Gamma_2} = 0, \tag{4}$$

that correspond to eigenvalues μ_m. Then $\{\varphi^{k,m}\}_{k,m=1}^{\infty} = \left\{\varphi^k(x)\psi^m(y)\right\}_{k,m=1}^{\infty}$ is the basis of $V^{1,1}(D)$, orthonormal in $L^2(D)$. Admit that the space $V^{1,1}(D)$ is compactly embedded into the space $W(D)$ under the condition

$$p_2 < \frac{2(n+l)}{n+l-2} \quad \text{if} \quad n+l>2, \qquad \text{and} \qquad p_2 < \infty \quad \text{if} \quad n=l=1.$$

Let us consider functions $u^N(x,y,t) = \sum\limits_{k,s=1}^{N} c^N_{ks}(t)\varphi^{k,s}(x,y)$, $N \in \{1,2,\dots\}$, where $c^N_{ks}(t)$ is the solution of the problem

$$\int\limits_{D_\tau} L\left(u^N, \varphi^{k,s}, f\right)\,\mathrm{d}x\,\mathrm{d}y = 0, \tag{5}$$

$$c^N_{ks}(0) = u^N_{0ks}\,, \qquad \{k,\, s\} \subset \{1,\dots,N\}. \tag{6}$$

Here $u^N_0(x,y) = \sum\limits_{k,s=1}^{N} u^N_{0ks}\varphi^{k,s}(x,y)$, $\lim\limits_{N\to\infty} \|u_0 - u^N_0\|_{W(D)} = 0$. Due to Carathéodory theorem [4, p. 54] there exists absolutely continuous solution on $[0,\tau_0]$, $\tau_0 < T$ of the problem (5)–(6). It is easy to prove that $\tau_0 = T$.
Let us use the notation:

$$\lambda^1 = \max_i \operatorname*{ess\,sup}_{Q_{0,T}} |\lambda_{iy_i}(x,y,t)|;$$

$$a^1 = \max_i \operatorname*{ess\,sup}_{Q_{0,T}} |a_{iy_i}(x,y,t)|,$$

$$g^1 = \max_i \operatorname*{ess\,sup}_{Q_{0,T}} |g_{y_i}(x,y,t)|.$$

Multiplying (5) by $c^N_{ks}(t)e^{-\alpha t}$, where the constant $\alpha = 3l\lambda^1 + 2$, summing up on s and k, integrating in t on the interval $[0,\tau]$, we have

$$\int\limits_{Q_{0,\tau}} L(u^N, u^N e^{-\alpha t}, f)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = 0.$$

Taking into account the form of α and estimating every term of the previous equality, it is easy to obtain the estimate

$$\begin{aligned}
&\int_{D_\tau} \left(u^N\right)^2 \,\mathrm{d}x\,\mathrm{d}y + \int_{S^2_{0,\tau}} \sum_{i=1}^{l} \lambda_i(x,y,t) \left(u^N\right)^2 \cos(\nu, y_i)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_{Q_{0,\tau}} \left[\sum_{i=1}^{n} |u^N_{x_i}|^{p(x)} + |u^N|^{p(x)} + |u^N|^2\right] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\qquad \leqslant M_1 \left[\int_{Q_{0,\tau}} |f|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t + \int_{D_0} \left(u_0\right)^2 \mathrm{d}x\,\mathrm{d}y\right], \\
&\qquad\qquad \tau \in (0, T],
\end{aligned} \tag{7}$$

where the constant M_1 does not depend on N. Multiplying (5) by the eigenvalue of the problem (4) $-\mu_m$ and changing term $-\mu_m u^N$ on $-\Delta_y u^N$ according to (4) we have

$$\int_{Q_{0,\tau}} \left[-L\left(u^N, \Delta_y u^N, f\right)\right] e^{-\alpha t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = 0. \tag{8}$$

Let us transform and estimate every term of the equality:

$$\begin{aligned}
\mathcal{I}_1 \quad &\equiv \quad -\int_{Q_{0,\tau}} u^N_t \Delta_y u^N e^{-\alpha t}\,\mathrm{d}x\,dy\,dt = \frac{1}{2}\int_{D_\tau} \sum_{i=1}^{l} \left(u^N_{y_i}\right)^2 e^{-\alpha\tau}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad -\frac{1}{2}\int_{D_0} \sum_{i=1}^{l} \left(u^N_{0y_i}\right)^2 \mathrm{d}x\,\mathrm{d}y + \frac{\alpha}{2}\int_{Q_{0,\tau}} \sum_{i=1}^{l} \left(u^N_{y_i}\right)^2 e^{-\alpha t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t; \\
\mathcal{I}_2 \quad &\equiv \quad -\int_{Q_{0,\tau}} \sum_{i=1}^{l} \lambda_i(x,y,t) u^N_{y_i} \sum_{j=1}^{l} u^N_{y_j y_j} e^{-\alpha t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = -\frac{1}{2}\int_{S_{0,\tau}} \sum_{i,j=1}^{l} \lambda_i(x,y,t) \\
&\quad \times \left[2u^N_{y_i} u^N_{y_j} \cos(\nu, y_j) - |u^N_{y_j}|^2 \cos(\nu, y_i)\right] e^{-\alpha t}\,\mathrm{d}S\,\mathrm{d}t \\
&\quad + \int_{Q_{0,\tau}} \sum_{i,j=1}^{l} \left[\lambda_{iy_j}(x,y,t) u^N_{y_i} u^N_{y_j} - \frac{1}{2}\lambda_{iy_i}(x,y,t)|u^N_{y_j}|^2\right] e^{-\alpha t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\geqslant \quad \mathcal{I}^1_2 - \frac{3}{2} l \lambda^1 \int_{Q_{0,\tau}} \sum_{i=1}^{l} |u^N_{y_i}|^2 e^{-\alpha t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.
\end{aligned}$$

Hence, $\Gamma_1 \subset C^1$, then some part of the surface Γ_1 can be written as

$$\Psi(y) \equiv \psi(y_1, \dots, y_{s-1}, y_{s+1}, \dots, y_l) - y_s = 0.$$

Then $\cos(\nu, y_i) = \omega(y)\Psi_{y_i}, \big(i \in \{1, \dots, l\}\big)$, where $\omega(y) = \Big(\sum\limits_{i=1}^{l} |\Psi_{y_i}(y)|^2\Big)^{-\frac{1}{2}}$.
That is why $\cos(\nu, y_j) = \frac{\Psi_{y_j}}{\Psi_{y_i}} \cos(\nu, y_i),\ i, j \in \{1, \dots, l\}$.
Hence, $u(x, y_1, \dots, y_{s-1}, \psi(y_1, \dots, y_{s-1}, y_{s+1}, \dots, y_l), y_{s+1}, \dots, y_l, t) = 0$, then $u_{y_i} = u_{y_s}\Psi_{y_i},\ i \in \{1, \dots, l\}$ and $u_{y_j}\cos(\nu, y_i) = u_{y_i}\cos(\nu, y_j)$.
Then

$$\mathcal{I}_2^1|_{S^1_{0,\tau}} = -\frac{1}{2}\int\limits_{S^1_{0,\tau}} \sum_{i,j=1}^{l} \lambda_i(x, y, t)|u^N_{y_j}|^2 \cos(\nu, y_i)e^{-\alpha t}\mathrm{d}S\,\mathrm{d}t\,,$$

and due to $\frac{\partial u^N}{\partial \nu} = \sum\limits_{j=1}^{l} u^N_{y_j}\cos(\nu, y_j) = 0 \quad \text{on} \quad S^2_{0,T}\,,$
the term

$$\mathcal{I}_2^1|_{S^2_{0,\tau}} = -\frac{1}{2}\int\limits_{S^2_{\tau}} \sum_{i,j=1}^{l} \lambda_i(x, y, t)\Big[2u^N_{y_i}\frac{\partial u^N}{\partial \nu} - u^N_{y_i}u^N_{y_j}\cos(\nu, y_j)\Big]e^{-\alpha t}\,\mathrm{d}S\,\mathrm{d}t = 0;$$

$$\begin{aligned}
\mathcal{I}_3 \quad &\equiv \quad -\int\limits_{Q_{0,\tau}} \sum_{i=1}^{n} a_i(x, y, t)|u^N_{x_i}|^{p(x)-2}u^N_{x_i}\Delta_y u^N_{x_i}e^{-\alpha t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\geqslant \quad \frac{a_0(p_1 - 1)}{2}\int\limits_{Q_{0,\tau}} \sum_{k=1}^{l}\sum_{i=1}^{n} |u^N_{x_i}|^{p(x)-2}(u^N_{x_iy_k})^2e^{-\alpha t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\quad -\frac{a^1}{4a_0(p_1 - 1)}\int\limits_{Q_{0,\tau}} \sum_{i=1}^{n} |u^N_{x_i}|^{p(x)}e^{-\alpha t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t; \\
\mathcal{I}_4 \quad &\equiv \quad -\int\limits_{Q_{0,\tau}} g(x, y, t)|u^N|^{p(x)-2}u^N\Delta_y u^N e^{-\alpha t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\geqslant \quad \frac{g_0(p_1 - 1)}{2} \times \int\limits_{Q_{0,\tau}} \sum_{k=1}^{l} |u^N|^{p(x)-2}(u^N_{y_k})^2e^{-\alpha t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\quad -\frac{g^1}{2g_0(p_1 - 1)}\int\limits_{Q_{0,\tau}} |u^N|^{p(x)}e^{-\alpha t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t;
\end{aligned}$$

$$\begin{aligned}\mathcal{I}_5 &\equiv \sum_{i=1}^{l}\int\limits_{Q_{0,\tau}} f(x,y,t)u^N_{y_iy_i}e^{-\alpha t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\ &\geqslant -\frac{1}{2}\int\limits_{Q_{0,\tau}}\sum_{i=1}^{l}\big(|u^N_{y_i}|^2+|f_{y_i}(x,y,t)|^2\big)e^{-\alpha t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,.\end{aligned}$$

According to estimations of integrals $\mathcal{I}_1 - \mathcal{I}_5$ and to the form of α from (8) we obtain inequality

$$\begin{aligned}&\sum_{k=1}^{l}\left[\int\limits_{D_\tau}\big(u^N_{y_k}\big)^2\mathrm{d}x\,\mathrm{d}y+\int\limits_{Q_{0,\tau}}\left(\sum_{i=1}^{n}|u^N_{x_i}|^{p(x)-2}\big(u^N_{x_iy_k}\big)^2+|u^N|^{p(x)-2}\,\big(u^N_{y_k}\big)^2\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\right]\\ &\leqslant M_2\left[\int\limits_{Q_{0,T}}\left(\sum_{k=1}^{l}|f_{y_k}|^2+|f|^2\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t+\int\limits_{D}\left[\sum_{k=1}^{l}(u_{0y_k})^2+(u_0)^2\right]\mathrm{d}x\,\mathrm{d}y\right],\\ &\hspace{22em}\tau\in(0,T],\end{aligned}\tag{9}$$

where the constant M_2 does not depend in N. Multiplying (5) by $c^N_{kst}(t)$, summing up in s and k and integrating on t from 0 to τ we have equality

$$\int\limits_{Q_{0,\tau}} L\big(u^N,u^N_t,f\big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t=0.\tag{10}$$

Applying (7) and (9) to (10) we get

$$\begin{aligned}&\int\limits_{Q_{0,\tau}}\big(u^N_t\big)^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t+\int\limits_{D_\tau}\left[\sum_{i=1}^{l}|u^N_{x_i}|^{p(x)}+|u^N|^{p(x)}\right]\mathrm{d}x\,\mathrm{d}y\\ &\leqslant M_3\left[\int\limits_{Q_{0,T}}\left(\sum_{i=1}^{l}|f_{y_i}|^2+|f|^2\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\right.\\ &\quad\left.+\int\limits_{D_\tau}\left[\sum_{k=1}^{l}\big(u^N_{0y_k}\big)^2+\big(u^N_0\big)^2+\big(u^N_0\big)^{p(x)}+\sum_{i=1}^{n}\big(u^N_{0x_i}\big)^{p(x)}\right]\mathrm{d}x\,\mathrm{d}y,\right],\\ &\hspace{22em}\tau\in(0,T],\end{aligned}\tag{11}$$

where the constant M_3 does not depend on N. Then from estimates (7), (9), (11) we obtain such convergence of some subsequence of sequence $\{u^N:\ N\geqslant 1\}$ as

$N \to \infty$ (we shall preserve the same notation for it) that:

$$\begin{aligned}
&u^N \to u \, * \text{- weakly in } L^\infty((0,T);L^2(D));\\
&u^N_{x_i} \to u_{x_i} \text{ weakly in } L^{p(x)}(Q_{0,T}),\\
&u^N \to u \text{ weakly in } L^{p(x)}(Q_{0,T});\\
&u^N_{y_k} \to u_{y_k} \, * \text{- weakly in } L^\infty((0,T);L^2(D));\\
&u^N_t \to u_t \text{ weakly in } L^2((0,T);L^2(D)).
\end{aligned} \tag{12}$$

$\big(k \in \{1,\dots,l\},\ i \in \{1,\dots,n\}\big)$. Because of

$$\begin{aligned}
&\int\limits_{Q_{0,\tau}} \Bigg[\bigg(\sum_{i=1}^n a_i(x,y,t)|u^N_{x_i}|^{p(x)-2}u^N_{x_i}\bigg)^{p'(x)}\\
&\quad + \Big(g(x,y,t)|u^N|^{p(x)-2}u^N\Big)^{p'(x)}\Bigg] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\
&\leqslant M_4 \int\limits_{Q_{0,\tau}} \Bigg[\sum_{i=1}^n |u^N_{x_i}|^{p(x)} + |u^N|^{p(x)}\Bigg] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\
&\leqslant M_5,
\end{aligned}$$

the term
$\sum\limits_{i=1}^n a_i(x,y,t)|u^N_{x_i}|^{p(x)-2}u^N_{x_i} + g(x,y,t)|u^N|^{p(x)-2}u^N \to \chi$ weakly in $L^{p'(x)}(Q_{0,T})$, $i \in \{1,...,n\}$.

From (12) it follows that $u^N \to u$ weakly in $W^{1,2}\big(G;L^2((0;T)\times\Omega)\big)$ as $N \to \infty$. Then $u \in C\big(G;L^2((0,T)\times\Omega)\big)$, $u|_{S^1_{0,T}} \in L^2\big((0,T)\times\Omega\big)$, $u|_{S^1_{0,T}} = 0$. By analogous way we prove that $u \in C\big([0,T];L^2(D)\big)$ and $u(x,y,0) = u_0(x,y)$. Taking into account convergences (12) and using method of monotonicity from [15, p. 171] we prove that u is the solution to the problem (1)–(3).

Let us prove the uniqueness of solution of the problem (1)–(3). Assume that there exist two solutions u_1, u_2 of the problem (1)–(3). Their difference $u = u_1 - u_2$ satisfies the equality $\int_{Q_{0,T}} \big[L(u_1,v,f) - L(u_2,v,f)\big]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = 0$ for all functions $v \in V(Q_{0,T})$ and conditions (2), (3) hold with $u_0(x,y) \equiv 0$. Choosing $v = u$ in this equality and noting that $\int_{Q_{0,\tau}} \sum_{i=1}^n a_i(x,y,t)\big[|u_{1x_i}|^{p(x)-2}u_{1x_i} - |u_{2x_i}|^{p(x)-2}u_{2x_i}\big](u_{1x_i} - u_{2x_i})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \geqslant 0$, we obtain that $\int_{Q_{0,T}} u^2\,\mathrm{d}x\,\mathrm{d}y \leqslant 0$. Hence $u \equiv 0$ almost everywhere in $Q_{0,T}$, therefore $u_1 = u_2$. $\square$

Using the method from [2] we prove.

Theorem 2. *Assume that Ω is any bounded domain of $\mathbb{R}^n$. Let u be the generalized solution of the problem* (1), (2), (3) *with $f \equiv 0$ and $\sum\limits_{i=1}^{l} \lambda_{iy_i}(x,y,t) \leqslant 0$ for all $(x,y,t) \in Q_{0,T}$. Then for all $t > 0$*

$$\int\limits_D |u(x,y,t)|^2 \,\mathrm{d}x\,\mathrm{d}y \leqslant \max\left\{ \left[\frac{C}{t}\right]^{\frac{2}{p_1-2}} ; \left[\frac{C}{t}\right]^{\frac{2}{p_2-2}} \right\},$$

where the positive constant C depends only on p, l, n.

P r o o f. Take $f \equiv 0$, $p_1 > 2$. We denote by $g(t) = \int_D |u(x,y,t)|^2 \,\mathrm{d}x\,\mathrm{d}y$. Taking into account that $\sum_{i=1}^{l} \lambda_{iy_i}(x,y,t)u^2 \leqslant 0$, the proof of Theorem 1 with $\alpha = 0$ implies the inequality

$$\frac{1}{2} g'(t) + a_0 \int\limits_D \sum_{i=1}^{n} |u_{x_i}|^{p(x)} \,\mathrm{d}x\,\mathrm{d}y \leqslant 0$$

and, besides, for further use, we can assume that

$$2\int\limits_{t_1}^{t_2}\int\limits_D \sum_{i=1}^{n} |u|^{p(x)} \,dx\,\mathrm{d}y\,\mathrm{d}t \leqslant g(t_1) - g(t_2) \leqslant g(t_1) \quad \text{for all} \quad t_2 > t_1 \geqslant 0.$$

From the embedding theorem

$$\begin{aligned} g(t) = \int\limits_D |u|^2 \,\mathrm{d}x\,\mathrm{d}y \ &\leqslant \|u; L^{p(x)}(D)\|^2 \\ &\leqslant \max\left\{ \left(\int\limits_D |u|^{p(x)} \,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{2}{p_2}} ; \left(\int\limits_D |u|^{p(x)} \,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{2}{p_1}} \right\} \leqslant C_1 \big[g'(t)\big]^{\frac{2}{p_*}}, \end{aligned}$$

where p_* is founded from the condition

$$\left(\int\limits_D |u|^{p(x)} \,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{2}{p_*}} = \max\left\{ \left(\int\limits_D |u|^{p(x)} \,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{2}{p_2}} ; \left(\int\limits_D |u|^{p(x)} \,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{2}{p_1}} \right\}.$$

Therefore $g(t) \leqslant C_2 |g'(t)|^{\frac{2}{p_*}}$. Noting that $g(t)$ decreases, we get

$$C_3 \big[g(t)\big]^{\frac{p_*}{2}} \leqslant |g'(t)|, \quad C_3 \big[g(t)\big]^{\frac{p_*}{2}} \leqslant -g'(t), \quad g'(t) \leqslant -C_3 [g(t)]^{\frac{p_*}{2}}.$$

Solving this differential inequality, we obtain

$$\big[g(t)\big]^{\frac{2-p_*}{2}} \leqslant C_4 t + \big[g(0)\big]^{\frac{2-p_*}{2}}.$$

Hence,

$$g(t) \leqslant \left[C_4 t + [g(0)]^{\frac{2-p_*}{2}}\right]^{\frac{2}{2-p_*}} \leqslant [C_4 t]^{\frac{2}{2-p_*}} = \left[\frac{C_5}{t}\right]^{\frac{2}{p_*-2}}.$$

□

3. Problem without initial conditions

Now, let the domain be unbounded with respect to the time variable. Denote it by $Q_{-\infty,T} = \Omega \times G \times (-\infty; T]$. Consider spaces $W_{\mathrm{loc}}(\overline{Q}_{-\infty,T}) = \{v : \ v \in W(Q_{t_1,T}) \text{ for every } t_1 < T\}$, $V_{\mathrm{loc}}(\overline{Q}_{-\infty,T}) = \big\{v : \ v \in V(Q_{t_1,T}) \text{ for every } t_1 < T\big\}$.

Definition 2. Function $u \in W_{\mathrm{loc}}(\overline{Q}_{-\infty,T}) \cap C\big((-\infty; T]; L^2(D)\big)$ is said to be a strong solution of the problem (1)–(2) (the problem without initial conditions) if it satisfies

$$\int\limits_{Q_{t_1,t_2}} L(u, v, f)\, dx\, \mathrm{d}y\, \mathrm{d}t = 0$$

for $t_1,\ t_2 \in (-\infty, T],\ t_1 < t_2$ and for all functions $v \in V_{\mathrm{loc}}(\overline{Q}_{-\infty,T})$.

Definition 3. Function u is said to be a weak solution the problem (1), (2), if u is the limit in the space $V_{\mathrm{loc}}(\overline{Q}_{-\infty,T}) \cap C\big((-\infty; T];\ L^2(D)\big)$ of the sequence of functions $\{u^k\}_{k=1}^{\infty}$ such that for all $k \in \mathbb{N}$ function u^k is the strong solution of the equation

$$\begin{aligned} u_t &+ \sum_{i=1}^{l} \lambda_i(x,y,t) u_{y_i} \\ &- \sum_{i=1}^{n} \Big(a_i(x,y,t)|u_{x_i}|^{p(x)-2} u_{x_i}\Big)_{x_i} \\ &+ g(x,y,t)|u|^{p(x)-2} u \\ =& f^k(x,y,t), \end{aligned}$$

with conditions (2), where the sequence $\{f^k\}_{k=1}^{\infty}$ is convergent to the function f in the space $L^{p'(x)}_{\mathrm{loc}}\big((-\infty, T];\ L^{p'(x)}(D)\big)$.

Theorem 3. *Let the assumptions of* Theorem 1 *fulfilled for every bounded subdomain of the domain $Q_{-\infty,T}$, conditions (A), (G), (P), (L) be true in the entire domain $Q_{-\infty,T}$. Then the problem* (1), (2) *has at least one weak solution.*

Proof. Let us examine the sequence of functions $\{f^k\}_{k=1}^{\infty}$ that is convergent to f in the space $L^{p'(x)}_{\mathrm{loc}}\big((-\infty,T];\ L^{p'(x)}(D)\big)$ and such that f^k, $f^k_{y_i}$, $f^k_t \in L^2_{\mathrm{loc}}\big((-\infty,T];\ L^2(D)\big)$ and $f^k|_{S^1_T}=0$ for all natural k. Let us introduce functions

$$F^k(x,y,t)=\begin{cases} f^k(x,y,t)\xi(t), & (x,y,t)\in Q_{T-k,T}\ ,\\ 0, & (x,y,t)\in Q_{T-k}\ ,\end{cases}$$

where $\xi\in C^1(\mathbb{R}),\ 0\leqslant\xi(t)\leqslant 1$, $\xi(t)=1$ as $t\geqslant T-k+1$, $\xi(t)=0$ as $t\leqslant T-k$. Remark that $F^k\to f$ in the space $L^{p'(x)}_{\mathrm{loc}}\Big((-\infty,T];\ L^{p'(x)}(D)\Big)$ as $k\to\infty$.

Consider in the domain $Q_{T-k,T}$ the problem for the equation (1) with the boundary conditions (2) and with the initial condition $u(x,y,T-k)=0$. Theorem 1 gives us conditions of existence and uniqueness of the strong solution u^k of this problem. Extend every function u^k by zero in the entire domain $Q_{-\infty,T-k}$ and use the same denotation for it. Obviously, the function u^k satisfies the equality

$$\int\limits_{Q_{t_1,t_2}} L\left(u^k,v,F^k\right)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t=0$$

for all $v\in V_{\mathrm{loc}}(\overline{Q}_{-\infty,T})$, $t_1<t_2\leqslant T$. Consider the function $\theta_1(t)$, $t\in\mathbb{R}$ [3, p. 24] such that

$$\theta_1\in C^1(\mathbb{R}),\quad \theta_1'(t)\geqslant 0\quad\text{on}\quad\mathbb{R},\quad 0\leqslant\theta_1(t)\leqslant 1;$$

$$\begin{aligned}
&\theta_1(t)=0 && \text{if}\quad t\in(-\infty;-1],\\
&\theta_1(t)=\exp\left\{\frac{-1}{(t+1)}\right\} && \text{if}\quad t\in\left(-1;\frac{-1}{2}\right],\\
&\theta_1(t)\geqslant\exp\{-2\} && \text{if}\quad t\in\left(\frac{-1}{2};0\right),\\
&\theta_1(t)=1 && \text{if}\quad t\in[0;+\infty).
\end{aligned}$$

Note that $\sup\limits_{\mathbb{R}}\theta_1'(t)\theta_1^{-\varkappa}(t)\leqslant C$, where $0<\varkappa<1$, C is a constant which depends only on $\varkappa$.

Denote by $\theta(t)=\theta_1\left(\frac{t-t_1}{\delta}\right)$, $\delta>0$, $u^{k,m}=u^k-u^m$. Let us prove that the sequence $\{u^k\}_{k=1}^{\infty}$ is fundamental in the spaces $C\big((-\infty,T];\ L^2(D)\big)$, $V_{\mathrm{loc}}(\overline{Q}_T)$. We write the previous equality for the function u^m, u^k, subtract these equalities and choose $v=u^{k,m}\theta(t)e^{-\alpha t}$ in the obtained equality, $t_1-\delta$ and ζ instead of t_1,

t_2 respectively, where ζ is arbitrary number from the interval $[t_1, t_2]$. We get

$$\int\limits_{Q_{t_1-\delta,\zeta}} \Big[L\left(u^k, u^{k,m}\theta(t), F^k(x,y,t) - F^m(x,y,t)\right) \\ -L\left(u^m, u^{k,m}\theta(t), F^k(x,y,t) - F^m(x,y,t)\right)\Big]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = 0. \quad (13)$$

Choose $T-k < t_1-\delta$, $T-m < t_1-\delta$. Then $F^k - F^m \equiv 0$ as $t \in [t_1-\delta, t_2]$. Transforming and estimating every term of the equality (13), similarly to the estimates $\mathcal{I}_1 - \mathcal{I}_5$ and using estimate

$$\frac{1}{2}\int\limits_{Q_{t_1-\delta,\zeta}} \left(u^{k,m}\right)^2 \theta'(t) e^{-\alpha t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ \leqslant \frac{\delta_1}{p_1}\int\limits_{Q_{t_1-\delta,\zeta}} |u^{k,m}|^{p(x)}\theta(t)e^{-\alpha t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t + C_1[\delta_1\delta]^{\frac{-2}{p_1-2}},$$

where $\delta_1 > 0$, we obtain inequality

$$\frac{1}{2}\int\limits_{D_\zeta} \left(u^{k,m}\right)^2\,\mathrm{d}x\,\mathrm{d}y \\ +\int\limits_{S_\tau^2}\sum_{i=1}^{l}\lambda_i(x,y,t)\cos(\nu, y_i)\left(u^{k,m}\right)^2 e^{-\alpha t}\theta(t)\,\mathrm{d}S\,\mathrm{d}t \\ +\int\limits_{Q_{t_1-\delta,\zeta}}\Bigg[\left(\frac{\alpha}{2} - \frac{\lambda^1 l}{2}\right)\left(u^{k,m}\right)^2 + \left(\frac{-\delta_1}{p_1} + g_0\right)|u^{k,m}|^{p(x)} \\ + a_0\sum_{i=1}^{n}|u^{k,m}_{x_i}|^{p(x)}\Bigg]\theta(t)e^{-\alpha t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ \leqslant C_1[\delta\delta_1]^{-2/(p_1-2)}. \quad (14)$$

Choose $\delta_1 = (g_0-1)p_1$. Fix $\varepsilon > 0$. Let us choose δ in such way that the right-hand side of inequality (14) is less than ε. Then for fixed t_1, $t_1 < T$ the sequence $\{u^k\}_{k=1}^{\infty}$ is fundamental in the spaces $C\big([t_1, T]; L^2(D)\big)$ and $V(Q_{t_1,T})$. From here we get convergence (using estimates between functionals and norms of functions in the generalized Lebesgue spaces) $u^k \to u$ in $C\big((-\infty, T]; L^2(D)\big) \cap V_{\mathrm{loc}}(\overline{Q}_T)$ as $k \to \infty$. Since F^k converges to f in $L^{p'(x)}_{\mathrm{loc}}\big((-\infty; T]; L^{p'(x)}(D)\big)$, according to Definition 3 function u is a weak solution of the problem (1)–(2). $\square$

Theorem 4. *Let the conditions (A), (G), (L), (P) be fulfilled in the domain $Q_{-\infty,T}$. Then the problem* (1)–(2) *can not have more than one weak solution.*

P r o o f. Let there exist two weak solutions u_1 and u_2 of the problem (1)–(2). According to Definition 3 there exist sequences $\{u_l^k\}_{k=1}^{\infty}, l = 1, 2$, that converge to u_l in the space $V_{\mathrm{loc}}(\overline{Q}_{-\infty,T}) \cap C\big((-\infty, T]; L^2(D)\big)$ as $k \to \infty$. Then their difference $u^k = u_1^k - u_2^k$ satisfies equality

$$\int\limits_{Q_{t_1-\delta,\zeta}} \Big[L\big(u_1^k, u^k, F_1^k(x, y, t)\big) - L\big(u_2^k, u^k, F_2^k(x, y, t)\big)\Big]\theta(t)e^{-\alpha t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = 0.$$

Similarly as from (13) we obtain equality

$$\int\limits_{D_\zeta}(u^k)^2 e^{-\zeta\alpha}\,\mathrm{d}x\,\mathrm{d}y + \int\limits_{Q_{t_1-\delta,\zeta}}\left[(u^k)^2 + |u^k|^{p(x)} + \sum_{i=1}^{n}|u^k|^{p(x)}\right]\theta(t)e^{-\alpha t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$
$$\leqslant C_1[\delta\delta_1]^{\frac{-2}{(p_1-2)}} + C_1\int\limits_{Q_{t_1-\delta,\zeta}}|f_1^k - f_2^k|^{p(x)}\theta(t)e^{-\alpha t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.$$

Fix $t_0 \in (-\infty, T)$ and $t_1 < t_0$. From the triangle inequality we get

$$\|f_2^k - f_1^k\|_{L^{p'(x)}(Q_{t_1-\delta,\zeta})} \leqslant \|f_2^k - f\|_{L^{p'(x)}(Q_{t_1-\delta,\zeta})} + \|f_1^k - f\|_{L^{p'(x)}(Q_{t_1-\delta,\zeta})}.$$

Fix $\varepsilon > 0$ and choose δ from the condition:

$$C_1[\delta\delta_1]^{\frac{-2}{(p_1-2)}} < \frac{\varepsilon^2}{8}.$$

Since sequences $\{f_l^k\}_{k=l}^{\infty}$, $l \in \{1, 2\}$ are convergent to function f in the space

$$L^{p'(x)}_{\mathrm{loc}}\big((-\infty, T);\ L^{p'(x)}(D)\big),$$

there exists such a natural number $k_0(\varepsilon)$ that for all

$$k > k_0\ \|f_l^k - f\|_{L^{p'(x)}(Q_{t_1-\delta,\zeta})} < \frac{1}{2}\left(\frac{\varepsilon}{8}\right)^{\frac{2}{p_1'}},\quad l = 1, 2.$$

From (15) we shall get

$$\|u_2^k - u_1^k\|_{C\big([t_0,T];L^2(D)\big)} < \varepsilon.$$

From

$$\begin{aligned}
&\|u_2 - u_1\|_{C\left([t_0,T];L^2(D)\right)} \\
&\leqslant \|u_2^k - u_2\|_{C\left([t_0,T];L^2(D)\right)} \\
&\quad + \|u_2^k - u_1^k\|_{C\left([t_0,T];L^2(D)\right)} \\
&\quad + \|u_1^k - u_1\|_{C\left([t_0,T];L^2(D)\right)}
\end{aligned}$$

and

$$\|u_l^k - u_l\|_{C\left([t_0,T];L^2(D)\right)} \to 0, \quad l = 1, 2,$$

as $k \to \infty$, we deduce existence of such natural $k_1 \geqslant k_0$ that for all $k > k_1$ norm

$$\|u_2 - u_1\|_{C\left([t_0,T];L^2(D)\right)} \leqslant 3\varepsilon.$$

Since t_0 and ε are arbitrary numbers, $u_2 = u_1$ almost everywhere in $Q_{-\infty,T}$. □

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