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CONNECTED G_δ FUNCTIONS OF ARBITRARILY HIGH BOREL CLASS

Piotr Szuca

ABSTRACT. The class of real functions with connected G_δ graph has some nice properties. For example, it is known that Sharkovskiĭ's ordering of periods holds in this family of functions. We modify the example of function constructed by [J. Jastrzębski: *An answer to a question of R. G. Gibson and F. Roush*, Real Anal. Exchange **15** (1989–1990), 340–341] to show that there are connected G_δ functions of arbitrary high Borel class.

1. Introduction

Our notation and terminology conforms to that used in the most recent real analysis literature. We use a symbol $\overline{A}$ to denote the closure of the set A. All considered functions maps the set of reals $\mathbb{R}$ to $\mathbb{R}$ (or the interval $\mathbb{I} = [0,1]$ to $\mathbb{I}$). For given functions f and g we use the notation "$f \circ g$" to denote the composition of g and f. We use symbols C, Conn, D and B_1 to denote the class of all continuous functions, all functions with connected graph (we do not distinguish the function f from the graph of f), all Darboux functions (i.e., all functions with intermediate value property), and all Baire 1 functions (pointwise limits of continuous functions), respectively. We say that a function f is G_δ if its graph is a G_δ subset of the plane. For properties of these and the other Darboux-like functions see, e.g., [3]. In particular, it is known that

$$\mathrm{Conn} \subsetneq \mathrm{D} \quad \text{and} \quad \mathrm{C} \subsetneq \mathrm{Conn} \cap \mathrm{B}_1 = \mathrm{D} \cap \mathrm{B}_1 \subsetneq \mathrm{Conn} \cap \mathrm{G}_\delta .$$

In papers [9] and [10] we have shown that the class of all functions with connected G_δ graph (i.e., functions from the class $\mathrm{Conn} \cap \mathrm{G}_\delta$) is similar to the class of continuous functions from the combinatorial dynamics point of view. For example, the following variant of Sharkovskiĭ's theorem holds (for the class of continuous functions it was proved by Sharkovskiĭ in [8].)

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Theorem 1. ([9] *and* [10]) *If* $f\colon \mathbb{R}\to\mathbb{R}$ *is a composition of connected* G_δ *functions and* f *has a point of the prime order* P *then it also has a point of prime order* Q *for every* $Q \lhd P$, *where the symbol "*$\lhd$*" denotes the following Sharkovskiĭ's ordering:*

$$\begin{gathered}3 \rhd 5 \rhd 7 \rhd \cdots 2\cdot 3 \rhd 2\cdot 5 \rhd 2\cdot 7 \rhd \cdots\\ 2^2\cdot 3 \rhd 2^2\cdot 5 \rhd 2^2\cdot 7 \rhd \cdots\\ \cdots \rhd 2^4 \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1\,.\end{gathered}$$

In [9] we have shown an example of connected G_δ function f such that $f\circ f \notin \mathrm{G}_\delta$. It is well-known that every G_δ function is Borel, and there are G_δ functions of arbitrarily high Borel class, i.e., for every $\alpha < \omega_1$ there is a G_δ function $f\colon \mathbb{I}\to\mathbb{I}$ such that f is not of the Borel class α (for a simple argument see, e.g., [2 p. 581].) It seems interesting to find out if the above statement is true in the narrower class of functions with connected G_δ graph. Our main result is the following

Theorem 2. *For every* $\alpha < \omega_1$ *there is a connected* G_δ *function* $f\colon \mathbb{I}\to\mathbb{I}$ *such that* f *does not belong to the Borel class* α.

Recall that we do not know any characterization of functions which are of the form

$$f_1 \circ f_2 \circ \cdots \circ f_n\,,$$

where all f_i's are connected G_δ (see, e.g., section "Problems" in [9] and the question about a characterization of the composition of two derivatives, or Darboux Baire 1 functions in [4].) An author does not even know whether there exists a connected Baire class 2 function which is not a composition of two connected G_δ functions.

Proof of Theorem 2

The function constructed in Theorem 2 is continuous on the complement of ternary Cantor set. The first example of G_δ function of such type has been found in [1, Example 2] (but see also [6]).

Lemma 3. *Let* D *be a closed nowhere dense subset of* $\mathbb{I}$, $D = \mathbb{I}\setminus \bigcup_{i\in\omega}(a_i,b_i)$, *where all intervals* (a_i,b_i) *are pairwise disjoint. Let* $D_0 = \mathbb{I}\setminus\bigcup_{i\in\omega}[a_i,b_i]$. *Suppose that* $f\colon\mathbb{I}\to\mathbb{I}$ *is such that:*

1. (1) f *is connected on every interval* $[a_i,b_i]$,
2. (2) *for every* $x\in D_0$ *and a subsequence* $a_{p_n}\to x$ *there exists a sequence* $(x_n)_n$ *such that* $x_n\in[a_{p_n},b_{p_n}]$ *for each* n *and* $\langle x_n, f(x_n)\rangle \to \langle x, f(x)\rangle$.

Then f is connected.

Proof. The proof presented here is a modification of that from the paper [5].

Suppose that f is not connected. Then there exist non-empty sets $A, B \subset f$ such that $A \cup B = f$ and there is no $\langle x, f(x)\rangle \in (\overline{A} \cap B) \cup (A \cap \overline{B})$. Let

$$A' = \{x \in \mathbb{I}\colon \langle x, f(x)\rangle \in A\} \text{ and } B' = \{x \in \mathbb{I}\colon \langle x, f(x)\rangle \in B\}.$$

Without lost of generality we can assume that $1 \in B'$. Let $s = \sup A'$. Since $f \restriction [a_i, b_i]$ is connected for each $i \in \omega$, then either $[a_i, b_i] \subset A'$ or $[a_i, b_i] \subset B'$. Thus $s \in D$. There are three possible cases:

(1) $s \in A' \cap D$;

(2) $s \in B' \cap (D_0 \cup \{a_n\colon n \in \omega\})$;

(3) $s \in B' \cap \{b_n\colon n \in \omega\}$.

In the first case $s < 1$ and $s \neq a_i$ for each i. By assertion (2) there exists a subsequence $(a_{p_n}, b_{p_n})_n$ of intervals such that $a_{p_n} \searrow s$, and $f(s)$ is a limit point of $f([a_{p_n}, b_{p_n}])$. Since $(a_{p_n}, b_{p_n}) \subset B'$ for every $a_{p_n} > s$, $\langle s, f(s)\rangle \in A \cap \overline{B}$—a contradiction.

In the second case there exists a sequence $(x_n)_n \subset A'$ such that $x_n \nearrow s$. We may assume that $(x_n)_n \subset \bigcup_{i\in\omega} [a_i, b_i]$. Indeed, for every $x \in A' \cap D_0$ there exists a sequence of intervals $[a_{p_n}, b_{p_n}]$ with $\lim_{n\to\infty} a_{p_n} = \lim_{n\to\infty} b_{p_n} = x$. By assertion (2) there exists a sequence (y_n) such that $y_n \in [a_{p_n}, b_{p_n}]$ for each n and $\langle y_n, f(y_n)\rangle \to \langle x, f(x)\rangle$. Since almost all elements of $(y_n)_n$ belong to A' (in the other case $\langle x, f(x)\rangle \in A \cap \overline{B}$), for each n and every open neighbourhood of x_n there is $x'_n \in A' \cap \bigcup_{i\in\omega} [a_i, b_i]$.

Let q_n be a unique integer such that $x_n \in [a_{q_n}, b_{q_n}]$. Then $[a_{q_n}, b_{q_n}] \subset A'$, and from assertion (2) it follows that $\langle s, f(s)\rangle \in \overline{A}$. This is a contradiction with $\langle s, f(s)\rangle \in B$.

In the last case $s = b_N$ for an $N \in \omega$. By the assertion (1) $[a_N, b_N] \subset B'$, which contradicts with $s = \sup A'$. □

Proof of Theorem 2. Fix $\alpha < \omega_1$. In the sequel we will assume that $\alpha \geqslant 2$. Let $C = \mathbb{I} \setminus \bigcup_{i\in\omega} (a_i, b_i)$, where C is a ternary Cantor set and all intervals (a_i, b_i) are pairwise disjoint. Let $C_0 = \mathbb{I} \Big\backslash \bigcup_{i\in\omega} [a_i, b_i]$.

First, we define a G_δ function $f_1\colon \bigcup_{i\in\omega} [a_i, b_i] \to \mathbb{I}$ such that if we extend it to any function $f\colon \mathbb{I} \to \mathbb{I}$ then f fulfills the assumption of Lemma 3.

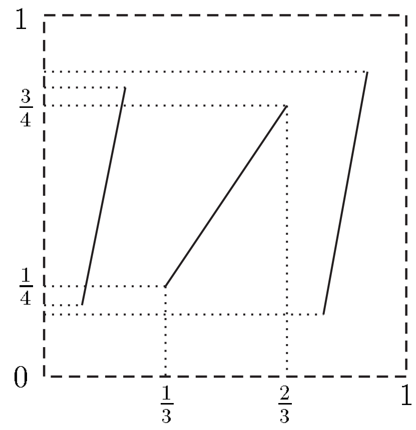

FIGURE 1. Illustration of the function f_1 constructed in the proof of Theorem 2.

Let f_1 be such that $f_1(a_i) = \frac{1}{i+4}$, $f_1(b_i) = 1 - \frac{1}{i+4}$ and f_1 is linear on every interval $[a_i, b_i]$ (see Fig. 1). We claim that the graph of f_1 is a G_δ subset of the plane. Following the argument from [2, Example 2.1], $f_1 = P \cup Q$ where

$$P = f_1 \restriction \bigcup_{i\in\omega} \{a_i, b_i\} = \bigcup_{i\in\omega} \left\{ \left\langle a_i, \frac{1}{i+4} \right\rangle, \left\langle b_i, 1 - \frac{1}{i+4} \right\rangle \right\}$$

and

$$Q = f_1 \restriction \bigcup_{i\in\omega} (a_i, b_i) = \bigcup_{i\in\omega} f_1 \restriction (a_i, b_i) =$$

$$= \bigcap_{n\in\omega} \bigcup_{i\in\omega} \bigcup_{x\in(a_i,b_i)} \{x\} \times \left(f_1(x) - \frac{1}{n+1}, f_1(x) + \frac{1}{n+1} \right) \in \mathrm{G}_\delta .$$

Since P is a discrete space, $P \in G_\delta$. Thus $f_1 \in G_\delta$.

Since $C_0 \in G_\delta$ and $\alpha \geqslant 2$, it is sufficient to find a function $f_2 \colon C_0 \to \mathbb{I}$ such that f_2 is G_δ and f_2 is not of the Borel class α.

Let $A \subset C_0$ be a Borel set which does not belong to the Borel class α. According to Alexandroff theorem ([7, Th. 3.11, p. 17]), C_0 is a Polish space. By the Lusin-Suslin theorem ([7, Th. 13.7, p. 83]), there are one-to-one continuous functions $\tilde{g_0} \colon A' \to C_0$ and $\tilde{g_1} \colon B' \to C_0$ such that (the symbol $\mathbb{Q}$ denotes the set of rationals):

(1) A' is a closed subset of $\left(0, \frac{1}{2}\right) \setminus \mathbb{Q}$,
(2) B' is a closed subset of $\left(\frac{1}{2}, 1\right) \setminus \mathbb{Q}$,
(3) $\tilde{g_0}(A') = A$,
(4) $\tilde{g_1}(B') = C_0 \setminus A$.

Set $f_2 = \tilde{g_0}^{-1} \cup \tilde{g_1}^{-1}$.

Since the graph of continuous function $\tilde{g_0}$ ($\tilde{g_1}$, respectively) is a closed subset of G_δ set $A' \times \mathbb{I}$ ($B' \times \mathbb{I}$, resp.), $f_2 \in \mathrm{G}_\delta$. By definition of the function f_2, $f_2^{-1}\big((0, \frac{1}{2})\big) = A$, and therefore f_2 is not of the Borel class α. Moreover, $f = f_1 \cup f_2$ fulfills the assertion of Lemma 3. It follows that f is a connected G_δ function which does not belong to the Borel class α. $\square$

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Institute of Mathematics
University of Gdańsk
ul. Wita Stwosza 57
PL–80-952 Gdańsk
POLAND

E-mail: pszuca@radix.com.pl