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# BLOCK MATRICES AND STABILITY THEORY

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Abstract. The relations between matrices and differential equations in the stability theory are studied. Generalizations to block matrices are done. Sufficient conditions for the stability of matrix differential equations by means of Lyapunov matrix equation are obtained. The block versions of the classical Hermite, Routh, Hurwitz and Schwarz matrices are presented.

## 1. Introduction

We consider a homogeneous differential equation of order $m$ with constant real coefficients

$$p\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) z = z^{(m)}(t) + a_1 z^{(m-1)}(t) + \cdots + a_{m-1} z'(t) + a_m z(t) = 0\,, \qquad (1)$$

$z(t) \in \mathbb{R}$ e $t \geq 0$.

The characteristic equation associated to equation (1) is

$$p(x) = x^m + a_1 x^{m-1} + \cdots + a_{m-1} x + a_m = 0\,, \qquad (2)$$

where $p(x)$ is a polynomial of degree $m$ (see Cullen ([5], p. 224).

By means of a reduction to the first order we obtain a linearization

$$y'(t) = Cy(t)\,, \qquad (3)$$

where $y(t) \in \mathbb{R}^m$ and $C \in \mathbb{R}^{m\times m}$ is the companion matrix associated to the polynomial $p(x)$ (see Cullen ([5], p. 228)).

The stability criterion is stated next (([2], p. 76) and ([10], p. 72)).

**Theorem 1.1.** *The equilibrium of the differential equation* (1) *is asymptotically stable if and only if all roots of the polynomial* $p(x)$ *have negative real parts.*

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It is equivalent to:

**Theorem 1.2.** *The equilibrium of the differential equation* (3) *is asymptotically stable if and only if all eigenvalues of the matrix $C$ have negative real parts.*

**Remark 1.1.** The Theorem 1.2 is not restrict to a matrix in a companion form. In fact, it is valid for an equation $y'(t) = Ay(t)$ where $A$ is a general matrix.

The "foundation" of the stability theory is from the nineteenth century and it is due to the famous mathematicians: Hermite [6], Routh [13], Lyapunov [11] and Hurwitz [8] in works with matrices to obtain the stability criterion (see Barnett ([2], p. 76)). We observe that in those years the explicit computation of roots of polynomials was very complicated (see Theorem 1.1). The Schwarz matrix [14] is more recent but we also consider it a classical one. In this paper we study the block versions of this classical matrices and in special of the Lyapunov matrix equation.

## 2. Lyapunov matrix equation

We now extend the Lyapunov matrix equation to a block context where we will obtain sufficient conditions for the asymptotic stability.

We start considering the homogeneous matrix differential equation of order $m$ with constant matrix coefficients

$$P\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) z = z^{(m)}(t) + A_1 z^{(m-1)}(t) + \cdots + A_{m-1} z'(t) + A_m z(t) = 0\,, \tag{4}$$

with $z(t) \in \mathbb{R}^n$, $t \geq 0$ and $A_0 = I_n, A_1, \ldots, A_m \in \mathbb{R}^{n\times n}$ (see Pereira [12]).

Associated to the matrix differential equation (4) is

$$P(X) = X^m + A_1 X^{m-1} + \cdots + A_{m-1} X + A_m = 0\,, \tag{5}$$

with $A_0 = I_n, A_1, \ldots, A_m \in \mathbb{R}^{n\times n}$, $X$ is also a matrix of order $n$, and we call $P(X)$ a matrix polynomial.

In (4) we can also make a reduction to the first order:

$$\left\{\begin{array}{lcl} y_1'(t) & = & z'(t) = y_2(t) \\ y_2'(t) & = & y_3(t) \\ & \vdots & \\ y_{m-1}'(t) & = & y_m(t) \\ y_m'(t) & = & -A_m y_1(t) - A_{m-1} y_2(t) - \cdots - A_1 y_m(t) \end{array}\right. , \tag{6}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_{m-1}(t) \\ y_m(t) \end{bmatrix}' = \begin{bmatrix} 0_n & I_n & 0_n & \cdots & 0_n \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0_n \\ 0_n & \cdots & \cdots & 0_n & I_n \\ -A_m & -A_{m-1} & \cdots & -A_2 & -A_1 \end{bmatrix} \begin{bmatrix} cy_1(t) \\ y_2(t) \\ \vdots \\ y_{m-1}(t) \\ y_m(t) \end{bmatrix}, \tag{7}$$

$$y'(t) = C_b y(t), \tag{8}$$

where $y(t) \in \mathbb{R}^{mn}$ and $C_b \in \mathbb{R}^{mn \times mn}$ is the block companion matrix associated to matrix polynomial $P(X)$ (see Lancaster ([9], p. 510)).

A common fact in the utilization of matrices in the stability theory is the symmetric property. When generalizing to blocks we have a problem: a block symmetric matrix in general is not a symmetric matrix. To surpass this difficulty we introduce the following concepts.

**Definition 2.1.** A matrix $N \in \mathbb{R}^{n\times n}$ is said to be symmetrizable if there exists a positive definite matrix $R = R^T \in \mathbb{R}^{n\times n}$ such that $N^T R = RN$ ([3], p. 67).

**Definition 2.2.** Two matrices $N_1, N_2 \in \mathbb{R}^{n\times n}$ are said to be simultaneously symmetrizable if there exists a positive definite matrix $R = R^T \in \mathbb{R}^{n\times n}$ such that ([1])

$$N_1^T R = RN_1 \text{ and } N_2^T R = RN_2\,. \tag{9}$$

It follows directly from Definition 2.2 that $N_1 + N_2$ and $N_1 - N_2$ are simultaneously symmetrizable.

Furthermore, we will use the following basic fact about matrix equations.

**Lemma 2.1.** *The equation $AX + XB = D$ has a unique solution $X$ for each $D$ if and only if $\sigma(A) \cap \sigma(-B) = \emptyset$, where $A, B$ and $D$ are square matrices* ([7], p. 270) *and* ([9], p. 414).

We are now in condition to present the following result:

**Proposition 2.1.** *Let $y'(t) = C_b y(t)$ be a matrix differential equation, where $y(t) \in \mathbb{R}^{mn}$ and $C_b \in \mathbb{R}^{mn\times mn}$ is the block companion matrix associated to the matrix polynomial $P(X)$, in which $A_0 = I_n$. If*

(i) $\sigma(C_b) \cap \sigma(-C_b) = \emptyset$;

(ii) $A_\alpha A_\beta \in \mathbb{R}^{n\times n}$, *with $0 \le \alpha \le \beta \le m$, are simultaneously symmetrizable, that is $(A_\alpha A_\beta)^T R = RA_\alpha A_\beta$, with $R = R^T$ positive definite,* *then for a symmetric matrix $W$ there exists a unique symmetric solution $V$ of the Lyapunov matrix equation*

$$C_b^T V + V C_b = -W. \tag{10}$$

P r o o f. We define $V = V^T = [V_{ij}] \in \mathbb{R}^{mn\times mn}$ where

$$V_{ij} = V_{ji} = \begin{cases} \sum_{k=0}^{i-1} (-1)^{k+i-1} R A_{m-i-j+k+1} A_{m-k} & \text{if } i+j \text{ is even,} \qquad i \le j\,, \\ 0_n & \text{if } i+j \text{ is odd.} \end{cases} \tag{11}$$

Writing

$$V = \begin{bmatrix} V_{11} & V_{12} & \cdots & V_{1m} \\ V_{12} & V_{22} & \cdots & V_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ V_{1m} & V_{2m} & \cdots & V_{mm} \end{bmatrix} \quad \text{and} \quad C_b = \begin{bmatrix} 0_n & I_n & \cdots & 0_n \\ \vdots & \ddots & \ddots & \vdots \\ 0_n & \cdots & 0_n & I_n \\ -A_m & \cdots & -A_2 & -A_1 \end{bmatrix}$$

we have

$$V C_b = \begin{bmatrix} -V_{1m}A_m & V_{11} - V_{1m}A_{m-1} & \cdots & V_{1(m-1)} - V_{1m}A_1 \\ -V_{2m}A_m & V_{12} - V_{2m}A_{m-1} & \cdots & V_{2(m-1)} - V_{2m}A_1 \\ \vdots & \vdots & \cdots & \vdots \\ -V_{mm}A_m & V_{1m} - V_{mm}A_{m-1} & \cdots & V_{(m-1)m} - V_{mm}A_1 \end{bmatrix} \tag{12}$$

and

$$C_b^T V = \begin{bmatrix} -A_m^T V_{1m} & -A_m^T V_{2m} & \cdots & -A_m^T V_{mm} \\ V_{11} - A_{m-1}^T V_{1m} & V_{12} - A_{m-1}^T V_{2m} & \cdots & V_{1m} - A_{m-1}^T V_{mm} \\ \vdots & \vdots & \cdots & \vdots \\ V_{1(m-1)} - A_1^T V_{1m} & V_{2(m-1)} - A_1^T V_{2m} & \cdots & V_{(m-1)m} - A_1^T V_{mm} \end{bmatrix}. \tag{13}$$

So

$C_b^T V + V C_b =$

$$= \begin{bmatrix} -A_m^T V_{1m} - V_{1m}A_m & -A_m^T V_{2m} + V_{11} - V_{1m}A_{m-1} & \cdots \\ V_{11} - A_{m-1}^T V_{1m} - V_{2m}A_m & 2V_{12} - A_{m-1}^T V_{2m} - V_{2m}A_{m-1} & \cdots \\ \vdots & \vdots & \cdots \\ V_{1(m-1)} - A_1^T V_{1m} - V_{mm}A_m & V_{2(m-1)} - A_1^T V_{2m} + V_{1m} - V_{mm}A_{m-1} & \cdots \end{bmatrix}$$

$$\begin{bmatrix} \cdots & -A_m^T V_{mm} + V_{1(m-1)} - V_{1m}A_1 \\ \cdots & V_{1m} - A_{m-1}^T V_{mm} + V_{2(m-1)} - V_{2m}A_1 \\ \cdots & \vdots \\ \cdots & 2V_{(m-1)m} - A_1^T V_{mm} - V_{mm}A_1 \end{bmatrix}.$$

Now, using (11), we consider two cases for $m$:

(1) $m$ is even, so we have

$C_b^T V + V C_b =$

$$= \begin{bmatrix} 0_n & -A_m^T RA_0A_{m-1} + RA_{m-1}A_m & \cdots \\ RA_{m-1}A_m - RA_0A_{m-1}A_m & -A_{m-1}^T RA_0A_{m-1} - RA_0A_{m-1}A_{m-1} & \cdots \\ \vdots & \vdots & \cdots \\ RA_1A_m - RA_0A_1A_m & -A_1^T RA_0A_{m-1} - RA_0A_1A_{m-1} & \cdots \end{bmatrix}$$

$$\begin{bmatrix} \cdots & -A_m^T RA_0A_1 + RA_1A_m \\ \cdots & -A_{m-1}^T RA_0A_1 - RA_0A_{m-1}A_1 \\ \cdots & \vdots \\ \cdots & -A_1^T RA_0A_1 - RA_0A_1A_1 \end{bmatrix}. \tag{14}$$

By hypothesis, $(A_\alpha A_\beta)^T R = RA_\alpha A_\beta, 0 \le \alpha \le \beta \le m$ and $A_0 = I_n$, thus

$$\begin{aligned} RA_\beta A_\alpha &= (RA_0A_\beta)A_\alpha = (A_0A_\beta)^T RA_\alpha = A_\beta^T RA_0A_\alpha \\ &= A_\beta^T (A_0A_\alpha)^T R = A_\beta^T A_\alpha^T R = (A_\alpha A_\beta)^T R = RA_\alpha A_\beta. \end{aligned} \tag{15}$$

Now we consider:

(a) $j$ is odd if $i = 1$, then $u + v$ is odd, so $V_{uv} = 0_n$ and the elements of (14) are $0_n$. If $i = 2, \ldots, m$, the elements of (14) are

$$RA_{m-(i-1)}A_{m-(j-1)} - RA_0A_{m-(j-1)}A_{m-(i-1)}$$

and by (15) we have

$$\begin{aligned} &RA_{m-(i-1)}A_{m-(j-1)} - RA_0A_{m-(j-1)}A_{m-(i-1)} \\ &\qquad = RA_{m-(i-1)}A_{m-(j-1)} - RA_{m-(i-1)}A_{m-(j-1)} = 0_n\,. \end{aligned}$$

(b) $j$ is even, the elements of (14) are

$$-A^T_{m-(i-1)}RA_0A_{m-(j-1)} + (-1)^{i+1}RA_{m-(j-1)}A_{m-(i-1)}.$$

(b1) If $i$ is odd, we have

$$\begin{aligned} &-A^T_{m-(i-1)}RA_0A_{m-(j-1)} + RA_{m-(j-1)}A_{m-(i-1)} \\ &-A^T_{m-(i-1)}(A_0A_{m-(j-1)})^TR + RA_{m-(j-1)}A_{m-(i-1)} \\ = &-(A_{m-(j-1)}A_{m-(i-1)})^TR + RA_{m-(j-1)}A_{m-(i-1)} \\ = &-RA_{m-(j-1)}A_{m-(i-1)} + RA_{m-(j-1)}A_{m-(i-1)} = 0_n. \end{aligned}$$

(b2) If $i$ is even, we have

$$\begin{aligned} &-A^T_{m-(i-1)}RA_0A_{m-(j-1)} - RA_{m-(j-1)}A_{m-(i-1)} \\ = &-2RA_{m-(j-1)}A_{m-(i-1)}. \end{aligned}$$

and hence

$$C_b^TV + VC_b = \begin{bmatrix} 0_n & 0_n & \cdots & 0_n \\ 0_n & -2RA_{m-1}A_{m-1} & \cdots & -2RA_1A_{m-1} \\ \vdots & \vdots & \cdots & \vdots \\ 0_n & -2RA_1A_{m-1} & \cdots & -2RA_1A_1 \end{bmatrix}. \tag{16}$$

(2) $m$ is odd, so

$$\begin{aligned} &C_b^TV + VC_b \\ = &\begin{bmatrix} A_m^TRA_0A_m - RA_0A_mA_m & RA_{m-1}A_m - RA_0A_mA_{m-1} & \cdots & \cdots & -A_m^TRA_0A_1 - RA_0A_mA_1 \\ -A_{m-1}^TRA_0A_m + RA_{m-1}A_m & 0_n & \cdots & \cdots & -A_{m-1}^TRA_0A_1 + RA_1A_{m-1} \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ -A_1^TRA_0A_m - RA_0A_1A_m & RA_1A_{m-1} - RA_0A_1A_{m-1} & \cdots & \cdots & -A_1^TRA_0A_1 - RA_0A_1A_1 \end{bmatrix}. \end{aligned} \tag{17}$$

Considering

(1) $j$ is even, the elements of (17) are

$$RA_{m-(i-1)}A_{m-(j-1)} - RA_0A_{m-(j-1)}A_{m-(i-1)}$$

and by (15) we obtain

$$\begin{aligned} &RA_{m-(i-1)}A_{m-(j-1)} - RA_0A_{m-(j-1)}A_{m-(i-1)} \\ =\,&RA_{m-(i-1)}A_{m-(j-1)} - RA_{m-(i-1)}A_{m-(j-1)} = 0_n. \end{aligned}$$

(2) $j$ is odd, the elements of (17) are

$$-A^T_{m-(i-1)}RA_0A_{m-(j-1)} + (-1)^i RA_{m-(j-1)}A_{m-(i-1)}.$$

(b1) If $i$ is even, we have

$$\begin{aligned} &-A^T_{m-(i-1)}RA_0A_{m-(j-1)} + RA_{m-(j-1)}A_{m-(i-1)} \\ =&-A^T_{m-(i-1)}(A_0A_{m-(j-1)})^T R + RA_{m-(j-1)}A_{m-(i-1)} \\ =&-(A_{m-(j-1)}A_{m-(i-1)})^T R + RA_{m-(j-1)}A_{m-(i-1)} \\ =&-RA_{m-(j-1)}A_{m-(i-1)} + RA_{m-(j-1)}A_{m-(i-1)} = 0_n. \end{aligned}$$

(b2) If $i$ is odd, we have

$$\begin{aligned} &-A^T_{m-(i-1)}RA_0A_{m-(j-1)} - RA_{m-(j-1)}A_{m-(i-1)} \\ =&-2RA_{m-(j-1)}A_{m-(i-1)}. \end{aligned}$$

and hence

$$C_b^TV + VC_b = \begin{bmatrix} -2RA_mA_m & 0_n & \cdots & -2RA_1A_m \\ 0_n & 0_n & \cdots & 0_n \\ \vdots & \vdots & \cdots & \vdots \\ -2RA_1A_m & 0_n & \cdots & -2RA_1A_1 \end{bmatrix} \tag{18}$$

Then, by (16) and (18), we get

$$C_b^TV + VC_b = -W, \quad \text{with} \quad W = W^T = [W_{ij} \in \mathbb{R}^{mn\times mn}$$

where

$$W_{ij} = W_{ji} = \begin{cases} 2RA_{m+1-j}A_{m+1-i} & \text{if } m+i \quad \text{and} \quad m+j \ \text{ are even,} \quad i \le j\,, \\ 0_n & \text{if } m+i \quad \text{or} \quad m+j \ \text{ is odd,} \end{cases} \tag{19}$$

$(i, j = 1, 2, \dots, m)$. Thus $V$ is solution of (10).

Now, by hypothesis of $\sigma(C_b) \cap \sigma(-C_b) = \emptyset$, the solution $V$ is unique from Lemma 2.1. □

**Corollary 2.1.** *If the Lyapunov matrix equation $C_b^TV + VC_b = -W$ has a symmetric solution $V$ and furthermore $V$ is positive definite for a $W$ positive semi-definite such that*

$$\begin{bmatrix} W^{\frac{1}{2}} \\ W^{\frac{1}{2}} C_b \\ \vdots \\ W^{\frac{1}{2}} C_b^{m-1} \end{bmatrix} \tag{20}$$

*is of full rank. Then all eigenvalues of $C_b$ have negative real parts.*

**Corollary 2.2.** *The equilibrium of the matrix differential equation $y'(t) = C_b y(t)$ is asymptotically stable.*

**Remark 2.1.** For an equation $A^T V + V A = -W$, where $A$ is a general matrix and $V$ is positive definite, the condition that matrix (19) is of full rank implies that $A$ has all eigenvalues with negative real parts, we use this fact in Corollary 2.1 (see Lancaster ([9], p. 449)).

**Remark 2.2.** In Corollary 2.1 if we consider $W$ positive definite (instead of positive semi-definite) we can discard the condition that matrix (19) is of full rank and this is in the original work of Lyapunov [11].

## 3. The classical matrices in block form

We present now the block versions of Hurwitz, Routh, Schwarz and Hermite matrices showing the connection among them.

**Definition 3.1.** Let $P(X) = A_0 X^m + A_1 X^{m-1} + \cdots + A_{m-1} X + A_m$, be a matrix polynomial. The matrix

$$H_b = \left[H_{b(ij)}\right] \in \mathbb{R}^{mn \times mn},$$

where

$$H_{b(ij)} = A_{2j-i}$$

in which $A_r = 0_n$ if $r < 0$ or $r > m$ $(i, j = 1, 2, \ldots, m)$.
That is

$$H_b = \begin{bmatrix} A_1 & A_3 & A_5 & \cdots & \cdots & \cdots & A_{2m-1} \\ A_0 & A_2 & A_4 & \cdots & \cdots & \cdots & A_{2m-2} \\ 0_n & A_1 & A_3 & \cdots & \cdots & \cdots & A_{2m-3} \\ 0_n & A_0 & A_2 & \ddots & \cdots & \cdots & A_{2m-4} \\ 0_n & 0_n & A_1 & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \cdots & \ddots & \vdots \\ 0_n & 0_n & 0_n & \cdots & \cdots & \cdots & A_m \end{bmatrix},$$

which we call the block Hurwitz matrix ([4]).

**Definition 3.2.** If we perform the block elimination of the block Hurwitz matrix, $H_b$, we obtain the matrix

$$\begin{bmatrix} C_{21} & C_{22} & C_{23} & \cdots & \cdots & \cdots \\ 0_n & C_{31} & C_{32} & C_{33} & \cdots & \cdots \\ 0_n & 0_n & C_{41} & C_{42} & \cdots & \cdots \\ 0_n & 0_n & 0_n & C_{51} & C_{52} & \cdots \\ \vdots & & & \ddots & \ddots & \ddots \\ 0_n & 0_n & 0_n & \cdots & 0_n & C_{(m+1)1} \end{bmatrix} \in \mathbb{R}^{mn\times mn}, \tag{21}$$

which we call the block Routh matrix.

**Definition 3.3.** Let $S_b = \left[S_{b(ij)}\right] \in \mathbb{R}^{mn\times mn}$,
in which

$$S_{b(ij)} = \begin{cases} I_n & \text{if} \quad j-i=1, \\ -S_k & \text{if} \quad i-j=1, \qquad (k=2,\ldots,m) \\ -S_1 & \text{if} \quad j=i=m, \\ 0_n & \text{otherwise} \end{cases}.$$

That is

$$S_b = \begin{bmatrix} 0_n & I_n & 0_n & \cdots & 0_n \\ -S_m & 0_n & I_n & \ddots & \vdots \\ 0_n & \ddots & \ddots & \ddots & 0_n \\ \vdots & \ddots & -S_3 & 0_n & I_n \\ 0_n & \cdots & 0_n & -S_2 & -S_1 \end{bmatrix}, \tag{22}$$

which we call the block Schwarz matrix ([4]).

We can obtain a matrix in a block Schwarz form from the block companion matrix by a similarity transformation. This will be useful to verify the stability of the matrix differential equation.

For that we consider the matrix

$$T=\begin{bmatrix}
I_n & \cdot & \cdot & 0_n & 0_n & 0_n & 0_n & 0_n & 0_n & 0_n \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
C^{-1}_{(m-1)1}C_{(m-1)2} & \cdot & \cdot & I_n & 0_n & 0_n & 0_n & 0_n & 0_n & 0_n \\
0_n & \cdot & \cdot & 0_n & I_n & 0_n & 0_n & 0_n & 0_n & 0_n \\
C^{-1}_{(m-3)1}C_{(m-1)3} & \cdot & \cdot & C_{61}^{-1}C_{62} & 0_n & I_n & 0_n & 0_n & 0_n & 0_n \\
0_n & \cdot & \cdot & 0_n & C_{51}^{-1}C_{52} & 0_n & I_n & 0_n & 0_n & 0_n \\
C^{-1}_{(m-5)1}C_{(m-5)4} & \cdot & \cdot & C_{41}^{-1}C_{43} & 0_n & C_{41}^{-1}C_{42} & 0_n & I_n & 0_n & 0_n \\
0_n & \cdot & \cdot & 0_n & C_{31}^{-1}C_{33} & 0_n & C_{31}^{-1}C_{32} & 0_n & I_n & 0_n \\
\cdot & \cdot & \cdot & C_{21}^{-1}C_{24} & 0_n & C_{21}^{-1}C_{23} & 0_n & C_{21}^{-1}C_{22} & 0_n & I_n
\end{bmatrix}, \quad (23)$$

where $C_{ij}$ are the elements of the block Routh matrix, with $C_{i1}\in\mathbb{R}^{n\times n}$, $(i=2,\cdots,m)$, being nonsingular. This appeared in [15] but the authors did not show how the elements of $S_b$ can be computed. We will pursue this next. The elements of the block Schwarz are

$$S_1=C_{21} \text{ and } S_i=C^{-1}_{(i-1)1}C_{(i+1)1}, \quad \text{with} \quad C_{11}=I_n \text{ and } i=2,\dots m. \tag{24}$$

Hence $T^{-1}S_bT=C_b$.

Now we consider the Lyapunov matrix equation for the block companion matrix $C_b^TV+VC_b=-W$
and so

$$\begin{aligned}C_b^TV+VC_b=-W \Leftrightarrow{}& T^TS_b^T\left(T^{-1}\right)^TV+VT^{-1}S_bT\\ ={}&-W\Leftrightarrow S_b^T\left(T^{-1}\right)^TVT^{-1}+\left(T^{-1}\right)^TVT^{-1}S_b=-\left(T^{-1}\right)^TWT^{-1}.\end{aligned}$$

Considering
$$M = \left(T^{-1}\right)^T VT^{-1} \text{ e } Q = \left(T^{-1}\right)^T WT^{-1},$$
we have $S_b^T M + MS_b = -Q$.

So we have an equivalent Lyapunov matrix equation with the block Schwarz matrix. We are now in condition to obtain a stability criterion in terms of the block Schwarz matrix.

**Proposition 3.1.** *Let $S_b \in \mathbb{R}^{mn\times mn}$ be a block Schwarz matrix.*

*If $S_1, S_1S_2, S_1S_2S_3, \ldots, S_1S_2S_3\cdots S_{m-1}S_m \in \mathbb{R}^{n\times n}$, are simultaneously symmetrizable and positive definite, then for a symmetric and positive semi-definite matrix $Q$ there exists a unique solution $M$ symmetric and positive definite of the Lyapunov matrix equation $S_b^T M + MS_b = -Q$.*

Proof. By hypothesis $S_1, S_1S_2, S_1S_2S_3, \ldots, S_1S_2\cdots S_m$ are simultaneously symmetrizable. Hence, there exists a positive definite matrix $R = R^T \in \mathbb{R}^{n\times n}$ such that
$$\begin{aligned}&S_1^T R = RS_1, \quad (S_1S_2)^T R = R(S_1S_2), \\ &(S_1S_2S_3)^T R = R(S_1S_2S_3), \ \ldots, \ (S_1S_2\cdots S_m)^T R = R(S_1S_2\cdots S_m).\end{aligned} \tag{25}$$

We define
$$M = \begin{bmatrix} RS_1S_2\cdots S_m & 0_n & \cdots & \cdots & 0_n \\ 0_n & RS_1S_2\cdots S_{m-1} & 0_n & \cdots & 0_n \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_n & \cdots & 0_n & RS_1S_2 & 0_n \\ 0_n & \cdots & \cdots & 0_n & RS_1 \end{bmatrix} \in \mathbb{R}^{mn\times mn}$$
a block diagonal matrix in which the block elements in diagonal are positive definite. So, $M$ is symmetric and positive definite.

Then we have

$S_b^T M =$
$$= \begin{bmatrix} 0_n & -S_m^T RS_1S_2\cdots S_{m-1} & 0_n & \cdots & 0_n \\ RS_1S_2\cdots S_m & 0_n & -S_{m-1}^T RS_1S_2\cdots S_{m-2} & \ddots & \vdots \\ 0_n & RS_1S_2\cdots S_{m-1} & \ddots & \ddots & 0_n \\ \vdots & \ddots & \ddots & 0_n & -S_2^T RS_1 \\ 0_n & \cdots & 0_n & RS_1S_2 & -S_1^T RS_1 \end{bmatrix}.$$

and

$$MS_b = \begin{bmatrix} 0_n & RS_1S_2\cdots S_m & 0_n & \cdots & 0_n \\ -RS_1S_2\cdots S_m & 0_n & RS_1S_2\cdots S_{m-1} & \ddots & \vdots \\ 0_n & -RS_1S_2\cdots S_{m-1} & \ddots & \ddots & 0_n \\ \vdots & \ddots & \ddots & 0_n & RS_1S_2 \\ 0_n & \cdots & 0_n & -RS_1S_2 & -RS_1^2 \end{bmatrix}.$$

By (25), we obtain

$S_b^T M =$

$$= \begin{bmatrix} 0_n & -RS_1S_2\cdots S_{m-1}S_m & 0_n & \cdots & 0_n \\ RS_1S_2\cdots S_m & 0_n & -RS_1S_2\cdots S_{m-2}S_{m-1} & \ddots & \vdots \\ 0_n & RS_1S_2\cdots S_{m-1} & \ddots & \ddots & 0_n \\ \vdots & \ddots & \ddots & 0_n & -RS_1S_2 \\ 0_n & \cdots & 0_n & RS_1S_2 & -RS_1^2 \end{bmatrix}.$$

Thus,

$$S_b^T M + MS_b = \begin{bmatrix} 0_n & 0_n & \cdots & \cdots & 0_n \\ 0_n & 0_n & 0_n & \cdots & 0_n \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_n & \cdots & 0_n & 0_n & 0_n \\ 0_n & \cdots & \cdots & 0_n & -2RS_1^2 \end{bmatrix} = -Q.$$

Since $RS_1^2 = RS_1S_1 = S_1^T RS_1 = S_1^T S_1^T R = (S_1S_1)^T R = (RS_1S_1)^T = (RS_1^2)^T$and $RS_1$ is positive definite then $Q \in \mathbb{R}^{mn\times mn}$ is a symmetric and positive semi-definite. Hence $M$ is a solution.

Now we prove that $M$ is unique. For that we consider the matrix

$$\begin{bmatrix} Q^{\frac{1}{2}} \\ Q^{\frac{1}{2}} S_b \\ \vdots \\ Q^{\frac{1}{2}} S_b^{m-1} \end{bmatrix} \tag{26}$$

and considering

$$S_{b(m(m-1))} = -S_2, \quad S^2_{b(m(m-2))} = S_2S_3,$$
$$S^3_{b(m(m-3))} = -S_2S_3S_4,\ \ldots,\ S^{m-1}_{b(m1)} = (-1)^{m+1} S_2S_3S_3\cdots S_m.$$

We have that the last block row of:

$$
\begin{aligned}
&Q^{\frac{1}{2}} && \text{is} && \Big[\ \underbrace{0_n \quad \cdots \quad 0_n}_{m-1} \quad B \ \Big],\\
&Q^{\frac{1}{2}}S_b && \text{is} && \Big[\ \underbrace{0_n \quad \cdots \quad 0_n}_{m-2} \quad -BS_2 \quad -BS_1 \ \Big],\\
&Q^{\frac{1}{2}}S_b^2 && \text{is} && \Big[\ \underbrace{0_n \quad \cdots \quad 0_n}_{m-3} \quad BS_2S_3 \quad \cdots \ \Big],\\
&Q^{\frac{1}{2}}S_b^3 && \text{is} && \Big[\ \underbrace{0_n \quad \cdots \quad 0_n}_{m-4} \quad -BS_2S_3S_4 \quad \cdots \ \Big],\\
&\vdots && && \\
&Q^{\frac{1}{2}}S_b^{m-1} && \text{is} && \big[\ (-1)^{m+1}BS_2S_3S_4\cdots S_m \quad \cdots \ \big].
\end{aligned}
$$

Making a suitable permutation of block rows in the matrix (26) we obtain a rank equivalent matrix

$$
\begin{bmatrix}
(-1)^{m+1}BS_2S_3S_4\cdots S_m & \cdots & & \cdots & \cdots & & \\
0_n & (-1)^m BS_2S_3S_4\cdots S_{m-1} & \cdots & & & & \\
\vdots & \ddots & \ddots & & & & \\
0_n & & 0_n & -BS_2S_3S_4 & & & \\
 & & & 0_n & BS_2S_3 & \cdots & \\
\vdots & & & & 0_n & -BS_2 & -BS_1 \\
0_n & \cdots & & & \cdots & 0_n & B \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}. \tag{27}
$$

Since $S_1$, $S_1S_2$, $S_1S_2S_3$, $\ldots, S_1S_2\cdots S_m$ are positive definite then the matrix (27) has rank $mn$ and it is of full rank. Hence, the matrix (26) is of full rank.

By Remark 2.1 we have that all eigenvalues of the matrix $S_b$ have negative real parts[1] and so $\sigma(S_b) \cap \sigma(-S_b) = \emptyset$. Thus $M$ is a unique solution. $\square$

**Corollary 3.1.** *The equilibrium of the matrix differential equation $y'(t) = C_b y(t)$ is asymptotically stable.*

We now present the block version of the Hermite matrix.

**Definition 3.4.** Let $P(X) = A_0X^m + A_1X^{m-1} + \cdots + A_{m-1}X + A_m$, be a matrix polynomial. The matrix $H_b^e = \left[H_{b(ij)}^e\right] \in \mathbb{R}^{mn\times mn}$, in which

$$H_{b(ij)}^e = H_{b(ji)}^e = \begin{cases} \displaystyle\sum_{k=0}^{i-1} (-1)^{k+i-1} A_k A_{i+j-k-1} & \text{if } i+j \text{ is even}, \quad i \le j \\ 0_n & \text{if } i+j \text{ is odd} \end{cases} \tag{28}$$

$(i, j = 1, 2, \ldots, m)$ is said to be the block Hermite matrix of $P(X)$.

Example 3.1. For $m = 5$, we have

$$H_b^e = \begin{bmatrix} A_0A_1 & 0_n & A_0A_3 & 0_n & A_0A_5 \\ 0_n & A_1A_2 - A_0A_3 & 0_n & A_1A_4 - A_0A_5 & 0_n \\ A_0A_3 & 0_n & A_0A_5 - A_1A_4 + A_2A_3 & 0_n & A_2A_5 \\ 0_n & A_1A_4 - A_0A_5 & 0_n & A_3A_4 - A_2A_5 & 0_n \\ A_0A_5 & 0_n & A_2A_5 & 0_n & A_4A_5 \end{bmatrix}.$$

**Remark 3.1.** If the block elements of $H_b^e$ are simultaneously symmetrizable then the matrix $V$, in the Proposition 2.1, can be obtained by $V = JDH_b^eJ^{-1}$, where

$$J = \begin{bmatrix} 0_n & \cdots & 0_n & I_n \\ \vdots & ⋰ & I_n & 0_n \\ 0_n & ⋰ & ⋰ & \vdots \\ I_n & 0_n & \cdots & 0_n \end{bmatrix} \in \mathbb{R}^{mn\times mn}, \qquad D = \begin{bmatrix} R & & \\ & \ddots & \\ & & R \end{bmatrix} \in \mathbb{R}^{mn\times mn}$$

and $R = R^T \in \mathbb{R}^{n\times n}$ is positive definite.

**Proposition 3.2.** *Let $H_b^e$ be a block Hermite matrix of $P(X)$. If $H_b^e$ is positive definite and $A_\alpha A_\beta \in \mathbb{R}^{n\times n}$ are simultaneously symmetrizable, $0 \le \alpha \le \beta \le m$, and $A_0$ is nonsingular, then the equilibrium of the matrix differential equation $y'(t) = C_b y(t)$ is asymptotically stable.*

---

[1] At this point we have the proof that the equilibrium of an equivalent matrix differential equation $b'(t) = S_b b(t)$ is asymptotically stable.

Proof. For the sake of simplicity, we suppose $A_0 = I_n$. Besides $A_\alpha A_\beta \in \mathbb{R}^{n\times n}$ $(0 \le \alpha \le \beta \le m)$ are simultaneously symmetrizable. So, it follows from Remark 3.1 that $V = JDH_b^e J^{-1}$, where $D$ and $H_b^e$ are positive definite. Hence, $V$ is positive definite.

Next, using the matrix (23), we have

$$M = \left(T^T\right)^{-1} VT^{-1} = \begin{bmatrix} RS_1S_2\cdots S_m & 0_n & \cdots & \cdots & 0_n \\ 0_n & RS_1S_2\cdots S_{m-1} & 0_n & \cdots & 0_n \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_n & \cdots & 0_n & RS_1S_2 & 0_n \\ 0_n & \cdots & \cdots & 0_n & RS_1 \end{bmatrix},$$

which is symmetric and since $V$ and $T$ are positive definite, $M$ is positive definite.

We have $M = M^T$ and so

$$\begin{aligned} RS_1 &= (RS_1)^T = S_1^T R \\ RS_1S_2 &= (RS_1S_2)^T = (S_1S_2)^T R \\ RS_1S_2S_3 &= (RS_1S_2S_3)^T = (S_1S_2S_3)^T R \\ &\vdots \\ RS_1S_2\cdots S_m &= (RS_1S_2\cdots S_m)^T = (S_1S_2\cdots S_m)^T R. \end{aligned} \tag{29}$$

By (29) we have that $S_1$, $S_1S_2$, $S_1S_2S_3, \dots, S_1S_2\cdots S_m$, are simultaneously symmetrizable. Furthermore, $M$ is positive definite and so $S_1$, $S_1S_2$, $S_1S_2S_3$, …, $S_1S_2\cdots S_m$, are positive definite.

By Proposition 3.1 and from Corollary 3.1 we have that the equilibrium of the matrix differential equation $y'(t) = C_b y(t)$ is asymptotically stable. □

## 4. Final remarks

In this work we dealt with matrices having general blocks. If we restrict the matrices to commuting blocks we can extend the theory and investigate:

– Systems of first order linear differential equations with matrix constant coefficients that can be written in a matrix form $y'(t) = A_b y(t)$. So we can define a characteristic equation if $A_b$ has commuting blocks.

– The Classical (scalar) Hermite, Routh and Hurwitz criteria are based in the minors of the matrices. In the block context we can only define a block minor if a matrix has commuting blocks.

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