Tatra Mt. Math. Publ. **35** (2007), 107–113

TATRA MOUNTAINS
Mathematical Publications

THE LEBESGUE AND DE LA VALLÉE POUSSIN'S THEOREMS ON DERIVATION OF AN INTEGRAL

Omar Dzagnidze

ABSTRACT. Theorem 1 proven in the present paper is a generalization of classical theorems on derivatives of an indefinite integral and of the definite integral with a parameter.

1. Introduction

We mean the following Lebesgue and Ch. J. de la Vallée Poussin's theorems.

Theorem L. *Let ψ be a summable function on a closed interval $[a,b]$, and suppose that*

$$\Psi(x)=\int_a^x \psi(t)\,dt\,.$$

Then for almost all $x\in[a,b]$ the equality

$$\Psi'(x)=\psi(x) \tag{1}$$

is fulfilled.

Theorem VP. *Let a function $f(x,y)$ be summable with respect to x on a closed interval $[a,b]$ for every fixed y from a closed interval $[c,d]$. Consider a finite on $[c,d]$ function, a definite integral with parameter y,*

$$\Phi(y)=\int_a^b f(x,y)\,dx\,. \tag{2}$$

Suppose that the following conditions are fulfilled:

(A) $f(x,y)$ is a function, absolutely continuous with respect to y on $[c,d]$ for every fixed $x\in[a,b]$;

2000 Mathematics Subject Classification: 26B05.
Keywords: derivative, strong partial derivative, strong gradient, differentiability.

(B) partial derivative f'_y with respect to y is a summable function on a closed rectangle $Q = \{(x, y \in \mathbb{R}^2 \colon a \leqslant x \leqslant b,\ c \leqslant y \leqslant d\}$.

Then

$$\Phi'(y) = \int_a^b f'_y(x, y)\, dx \tag{3}$$

for almost all $y \in [c, d]$.

In the present note we prove one theorem which is a generalization of Theorems L and VP under the assumptions (A) and (B) of Theorem VP.

To prove that generalized theorem we shall need some statements and definitions.

1. A use will be made of the following Tolstov's theorem on partial derivatives of an indefinite double integral

$$\Omega(x, y) = \int_a^x \int_c^y \omega(t, \tau)\ dt\, d\tau\,, \tag{4}$$

where the function ω is assumed to be summable on the rectangle Q.

Theorem T. ([1, § 7]) *To the function ω there correspond measurable sets $e_1 \subset [a, b]$ and $e_2 \subset [c, d]$, respectively, with $|e_1| = b - a$ and $|e_2| = d - c$, such that the function Ω defined by equality (4) possesses the following properties:*

(1) the function $\Omega'_x(x, y)$ is finite at every point (x, y) with $x \in e_1$ and $c \leqslant y \leqslant d$, and

$$\Omega'_x(x, y) = \int_c^y \omega(x, \tau)\, d\tau\,;$$

(2) the function $\Omega'_y(x, y)$ is finite at each point (x, y) with $a \leqslant x \leqslant b$ and $y \in e_2$, and

$$\Omega'_y(x, y) = \int_a^x \omega(t, y)\, dt\,;$$

(3) there exists a measurable set $E \subset Q$ with $|E| = |Q|$ such that at every point $(x, y) \in E$ there exist equalities

$$\Omega''_{x,y}(x, y) = \omega(x, y) = \Omega''_{y,x}(x, y)\,,$$

with finite terms.[1]

[1] Here we have introduced the symbols $\Omega''_{x,y}(x, y) = \partial y \partial x \Omega$ and $\Omega''_{y,x}(y, x) = \partial x \partial y \Omega$.

Regarding Theorem T it should be said that using Theorem L for the function Ω, we can establish the existence of the measurable set $e \subset [a,b]$ with $|e| = b-a$, such that the equality

$$\Omega'_x(x,y) = \int\limits_c^y \omega(x,\tau)\,d\tau$$

is realized for all $x \in e$ and fixed $y \in [c,d]$.

Thus the set e depends on the variable $y \in [c,d]$. Just this very undesirable phenomenon is eliminated by means of Theorem T.

2. Here we shall need the notion of a strong gradient introduced on the base of the notion of a strong partial derivative. It is included in the following. Let a finite function of two variables $F(x,y)$ be defined in the neighborhood of the point $(x_0,y_0) \in \mathbb{R}^2$. A strong partial derivative of the function F at the point (x_0,y_0) with respect to the variable x, denoted by $F'_{[x]}(x_0,y_0)$, is called the limit

$$F'_{[x]}(x_0,y_0) = \lim_{(h,k)\to(0,0)} \frac{F(x_0+h,y_0+k)-F(x_0,y_0+k)}{h}$$

which is finite or infinite of fixed sign.

A strong partial derivative of the function F at the point (x_0,y_0) with respect to the variable y,

$$F'_{[y]}(x_0,y_0) = \lim_{(h,k)\to(0,0)} \frac{F(x_0+h,y_0+k)-F(x_0+h,y_0)}{k},$$

is defined analogously.

If there exist finite or infinite $F'_{[x]}(x_0,y_0)$ and $F'_{[y]}(x_0,y_0)$, then the strong gradient of the function F at the point (x_0,y_0), in short strgrad $F(x_0,y_0)$, is defined by the following equality ([2], [3])

$$\operatorname{strgrad} F(x_0,y_0) = \big(F'_{[x]}(x_0,y_0), F'_{[y]}(x_0,y_0)\big)\,.$$

Note that the finiteness of strgrad $F(x_0,y_0)$ means that one of its components.

We have the following statements ([2], [3]):

(a) the finiteness of strgrad $F(x_0,y_0)$ implies the differentiability of the function F at the point (x_0,y_0), and not vice versa;

(b) the continuity of the gradient

$$\operatorname{grad} F(x,y) = \big(F'_x(x,y), F'_y(x,y)\big)$$

at the point (x_0,y_0) implies the finiteness of strgrad $F(x_0,y_0)$, and not vice versa.

2. Main Result

Thus we have the following generalization of Theorem L and Theorem VP.

Theorem 1. *Let assumptions* (A) *and* (B) *of Theorem VP be fulfilled. Then the function, an indefinite integral with parameter* y,

$$F(x,y)=\int_a^x f(t,y)\,dt \tag{5}$$

possesses the following properties:

(*i*) *there exists* $e_1\subset[a,b]$ *such that* $|e_1|=b-a$, $F'_{[x]}(x_0,y_0)$ *is finite for* $(x_0,y_0)\in e_1\times[c,d]$, *and*

$$F'_{[x]}(x_0,y_0)=f(x_0,y_0)\,; \tag{6}$$

(*ii*) *there exists* $e_2\subset[s,d]$ *such that* $|e_2|=d-c$, $F'_{[y]}(x_0,y_0)$ *is finite for* $(x_0,y_0)\in[a,b]\times e_2$, *and*

$$F'_{[y]}(x_0,y_0)=\int_a^x f'_y(t,y_0)\,dt\,; \tag{7}$$

(*iii*) *the* strgrad $F(x_0,y_0)$ *is finite at almost all points* $(x_0,y_0)\in Q$, *in particular, there exists the total differential* $dF(x_0,y_0)$.

P r o o f. (i) Taking condition (A) of Theorem VP into account, we have the relation

$$\begin{aligned}
&\frac{F(x_0+h,y_0+k)-F(x_0,y_0+k)}{h}\\
&=\frac{1}{h}\int_{x_0}^{x_0+h}\big[f(t,y_0+k)-f(t,y_0)\big]\,dt+\frac{1}{h}\int_{x_0}^{x_0+h}f(t,y_0)\,dt\\
&=\frac{1}{h}\int_{x_0}^{x_0+h}\bigg(\int_{y_0}^{y_0+k}f'_\tau(t,\tau)\,d\tau\bigg)\,dt+\frac{1}{h}\int_{x_0}^{x_0+h}f(t,y_0)\,dt\\
&\equiv A_{h,k}(x_0,y_0)+B_h(x_0,y_0)\,.
\end{aligned}$$

By the assumption (B) of Theorem VP, the function f'_τ is summable on Q. Therefore

$$A_{h,k}(x_0,y_0)=\frac{1}{h}\int_{x_0}^{x_0+h}\int_{y_0}^{y_0+k}f'_\tau(t,\tau)\,dt\,d\tau\,. \tag{8}$$

The right-hand side of equality (8) tends to zero for almost all $x_0 \in [a,b]$ and for all $y_0 \in [c,d]$ as $(h,k) \to (0,0)$ ([2], statement (1) of Theorem 6.9 or statement (1) of Theorem 9.5).

Further, the finiteness of integral (2) for all $y \in [c,d]$ implies that of the indefinite integral with parameter y (see equality (5)),

$$\int_a^x f(t,y)\,dt = F(x,y)$$

for all $(x,y) \in Q$.

The assumptions (A) and (B) of Theorem VP result in the equality

$$\int_a^x f(t,y)\,dt = \int_a^x \int_c^y f'_\tau(t,\tau)\,dt\,d\tau + \int_a^x f(t,c)\,dt \tag{9}$$

for all $(x,y) \in Q$.

According to statement (1) of Theorem T, we have

$$\frac{\partial}{\partial x} \int_a^x \int_c^y f'_\tau(t,\tau)\,dt\,d\tau = \int_c^y cf'_\tau(x,\tau)\,d\tau = f(x,y) - f(x,c) \tag{10}$$

for all (x,y) with $x \in e_1$ and $y \in [c,d]$, where $|e_1| = b-a$.

Besides, by Theorem L we have the equality

$$\left(\int_a^x f(t,c)\,dt\right)' = f(x,c) \tag{11}$$

for all x from the set $e \subset [a,b]$ with $|e| = b-a$, and the set e depends only on the constant c.

Now, from equalities (9)–(11) we obtain

$$\left(\int_a^x f(t,y)\,dt\right)'_x = f(x,y)$$

for all (x,y) with $x \in e \cap e_1$ and $y \in [c,d]$.

This means that the expression $B_h(x_0,y_0)$ for almost all $x_0 \in [a,b]$ and for all $y_0 \in [c,d]$ tends to $f(x_0,y_0)$ as $h \to 0$. Thus statement (i) is established.

(ii) Now,

$$\begin{aligned}
&\frac{F(x_0+h,y_0+k)-F(x_0+h,y_0)}{k}\\
&=\frac{1}{k}\int_a^{x_0+h}\left[f(t,y_0+k)-f(t,y_0)\right]dt=\frac{1}{k}\int_a^{x_0+h}\left(\int_{y_0}^{y_0+k}f'_\tau(t,\tau)\,d\tau\right)dt\\
&=\frac{1}{k}\int_{y_0}^{y_0+k}\int_a^{x_0}f'_\tau(t,\tau)\,dt\,d\tau+\frac{1}{k}\int_{y_0}^{y_0+k}\int_{x_0}^{x_0+h}f'_\tau(t,\tau)\,dt\,d\tau\\
&\equiv C_k(x_0,y_0)+D_{h,k}(x_0,y_0)\,.
\end{aligned}$$

By statement (2) of Theorem T we can get the equality

$$\lim_{k\to 0}C_k(x_0,y_0)=\int_a^{x_0}f'_\tau(t,y_0)\,dt$$

for all $x_0\in[a,b]$ and for almost all $y_0\in[c,d]$ (analogous reasoning has been given for $B_h(x_0,y_0)$).

Next, the equality

$$\lim_{(h,k)\to(0,0)}D_{h,k}(x_0,y_0)=0$$

for all $x_0\in[a,b]$ and for almost all $y_0\in[c,d]$ is similar to the equality established above for $A_{h,k}(x_0,y_0)$.

Consequently, statement (ii) holds.

(iii) The finiteness of strgrad $F(x_0,y_0)$ for almost all $(x_0,y_0)\in Q$ follows from statements (i) and (ii). According to the above-mentioned statement (a), at the same points (x_0,y_0) there exists total differential $dF(x_0,y_0)$.

Finally, if in equality (5) we put $x=b$, then equality (7) for $x_0=b$ takes the form of equality (3) because the derivative of the function of one variable is, in fact, its strong partial derivative with respect to the same variable, if we consider this function as the function of two variables, constant with respect to the second variable. Thus equality (7) is the generalization of equality (3).

Equality (1) is obtained analogously from equality (6) if the function f in equality (5) is assumed to be independent of the variable y. $\square$

REFERENCES

TOLSTOV, G. P.: *On the second mixed derivative*, Mat. Sb. **24** (1949), 27–51. (In Russian)

[2] DZAGNIDZE, O. P.: *On the differentiability of functions of two variables and of indefinite double integrals*, Proc. A. Razmadze Math. Inst. **106** (1993), 7–48.

[3] DZAGNIDZE, O. P.: *A necessary and sufficient condition for differentiability of functions of several variables*, Proc. A. Razmadze Math. Inst. **123** (2000), 23–29.

Received October 10, 2005

A. Razmadze Mathematical Institute
Georgian Academy of Sciences
Aleksidze St. 1
G–0193 Tbilisi
GEORGIA
E-mail: odzagni@rmi.acnet.ge