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ON SINGULAR VERSIONS OF BIHARI AND WENDROFF-PACHPATTE TYPE INTEGRAL INEQUALITIES AND THEIR APPLICATION

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ABSTRACT. We shall prove some results on singular versions of the Bihari and Wendroff-Pachpatte inequalities and point out to some of their applications.

1. Introduction

We consider the integral inequality of one variable of the form

$$u(t)^r \leqq a(t) + \int_0^t (t-s)^{\beta-1} F_1(s)\omega\big(u(s)\big)\,\mathrm{d}s + \int_0^t F_2(s)\omega\big(u(s)\big)\,\mathrm{d}s\,, \quad t \geqq 0, \tag{1}$$

where $r \geqq 1$, $0 < \beta < 1$, $a(t)$, $F_1(t), F_2(t)$ are continuous, nonnegative functions on $\mathbb{R}^+ = \langle 0, \infty)$, $\omega(u)$ is a continuous, positive, nondecreasing function on $\mathbb{R}^+$, and the integral inequality in two variables of the form

$$\begin{aligned} v(x,y) \leqq a(x) + b(y) &+ \int_0^x \int_0^y (x-s)^{\beta-1}(y-t)^{\beta-1} G_1(s,t)\omega\big(v(s,t)\big)\,\mathrm{d}s\,\mathrm{d}t \\ &+ \int_0^x \int_0^y G_2(s,t)\omega\big(v(s,t)\big)\,\mathrm{d}s\,\mathrm{d}t\,, \end{aligned} \tag{2}$$

where $a(x), b(y)$ are nonnegative C^1-functions, $G_1(x,y), G_2(x,y)$ are continuous, nonnegative functions on $\mathbb{R}^+_2 = \mathbb{R}^+ \times \mathbb{R}^+$ and ω is as above.

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There are many papers concerning the inequality (1) with $\beta = 1$. Some of them are analyzed and quoted in the books [1], [7], [10], [18].

First let us remark that there are five fundamental results concerning the inequality (1):

(I) $r = 1$, $\beta = 1$, $F_1 \equiv 0$, ω is linear and $a(t)$ constant (the Gronwall inequality [6]),

(II) $r = 1$, $\beta = 1$, $f_2 \equiv 0$, ω-nonlinear (the Bihari inequality [2]),

(III) $r = 2$, $\beta = 1$, $F_2 \equiv 0$, ω-nonlinear (the Ou-Iang-Pachpatte inequality [16], [17], [18]),

(IV) $r = 2$, $0 < \beta < 1$ (a weakly singular kernel), $F_2 \equiv 0$, ω-linear (the Henry inequality [8]),

(V) $r \geqq 1$, $0 < \beta < 1$, $F_2 \equiv 0$, ω-nonlinear (Medveď [12]).

The proof of the first result concerning the inequality (2) with $\beta = 1$, $F_2 \equiv 0$, ω-linear can be found in the book [1]. The result is quoted as one of the unpublished Wendroff inequalities. There are many other results concerning this inequality, also for functions in n independent variables (see, e.g., [18]). The paper [12] contains results concerning the case $r \geqq 1$, $0 < \alpha$, $\beta < 1$, $F_2 \equiv 0$, ω-nonlinear for two and also n independent variables.

The aim of the paper is to generalize the results published in [11], [13] on the inequality (1) and (2), respectively, for the case $F_2 \neq 0$, or, $G_2 \neq 0$, respectively and to point out to some of their applications.

2. On the inequality (1)

We assume that the inequality (1) contains also the nonsingular part, i.e., $F_2(t) \neq 0$. We need the following lemma.

Lemma 1. *If $0 < a < 1$, $b > 0$ then there exists a constant $D = D(a,b) > 0$ such that*

$$\int_0^t s^{-a} \mathrm{e}^{bs} \, \mathrm{d}s \leqq D \, \mathrm{e}^{bt}, \qquad t \geqq 0. \tag{3}$$

For the proof see [11]. Now let us apply the desingularization method suggested in [11].

Let

$$\beta = \frac{1}{1+z}, z > 0,\ \alpha = 1 - \beta, \varepsilon > 0,\ q = \frac{1}{\beta} + \varepsilon = 1 + z + \varepsilon,\ p = \frac{1+z+\varepsilon}{z+\varepsilon}.$$

Then $\frac{1}{p}+\frac{1}{q}=1$ and applying the Hölder inequality and (3) we obtain

$$\begin{aligned} u(t)^r &\leqq a(t) + \left(\int_0^t (t-s)^{-\alpha p}\, \mathrm{e}^{ps}\, \mathrm{d}s\right)^{\frac{1}{p}} \left(\int_0^t F_1(s)^q\, \mathrm{e}^{-qs}\, \omega\big(u(s)\big)^q\, \mathrm{d}s\right)^{\frac{1}{q}} \\ &\quad + \int_0^t F_2(s)\omega\big(u(s)\big)\, \mathrm{d}s \\ &\leqq a(t) + D(\alpha p, p)^{\frac{1}{p}}\, \mathrm{e}^t \left(\int_0^t F_1(s)^q\, \mathrm{e}^{-qs}\, \omega\big(u(s)\big)^q\, \mathrm{d}s\right)^{\frac{1}{q}} \\ &\quad + \int_0^t F_2(s)\omega\big(u(s)\big)\, \mathrm{d}s\,. \end{aligned}$$

The inequality $(A+B+C)^q \leqq 3^{q-1}(A^q+B^q+C^q)$, $A, B, C \geqq 0$ yields

$$u(t)^{rq} \leqq b(t) + K\mathrm{e}^{qt}\int_0^t F_1(s)^q\, \mathrm{e}^{-qs}\, \omega\big(u(s)\big)^q\, \mathrm{d}s + 3^{q-1}\left(\int_0^t F_2(s)\omega\big(u(s)\big)\, \mathrm{d}s\right)^q,$$

where $b(t)=3^{q-1}a(t)^q$, $K=3^{q-1}D(\alpha p,p)^{\frac{q}{p}}$. The Jensen inequality in the form

$$\left(\int_0^t g(s)\, \mathrm{d}s\right)^\delta \leqq t^{\delta-1}\int_0^t g(s)^\delta\, \mathrm{d}s\,, \qquad \delta \geqq 1 \tag{4}$$

enables us to estimate

$$\left(\int_0^t F_2(s)\omega\big(u(s)\big)\, \mathrm{d}s\right)^q \leqq t^{q-1}\int_0^t F_2(s)^q\omega\big(u(s)\big)^q\, \mathrm{d}s$$

and thus we obtain

$$u(t)^{rq} \leqq b(t) + K\mathrm{e}^{qt}\int_0^t F_1(s)^q\, \mathrm{e}^{-qs}\, \omega\big(u(s)\big)^q\, \mathrm{d}s + 3^{q-1}t^{q-1}\int_0^t F_2(s)^q\omega\big(u(s)\big)^q\, \mathrm{d}s\,.$$

Let

$$0<T<\infty,\ \lambda_1 = \max_{0\leqq t\leqq T} b(t),\ \lambda_2 = \max\left\{K\mathrm{e}^{qT}, 3^{q-1}T^{q-1}\right\},$$
$$F(t) = F_1(t)^q\, \mathrm{e}^{-qt} + F_2(t)^q.$$

Then

$$u(t)^{rq} \leqq \lambda_1 + \lambda_2 \int_0^t F(s)\omega\big(u(s)\big)^q \,\mathrm{d}s \qquad t \in \langle 0, T \rangle.$$

Denote by $W(t)$ the right-hand side of this inequality. Then $u(t) \leqq W(t)^{\frac{1}{rq}}$ and since ω is nondecreasing we have $\omega\big(u(t)\big)^q \leqq \omega\left(W(t)^{\frac{1}{rq}}\right)^q$. This yields

$$\frac{\frac{\mathrm{d}}{\mathrm{d}t}W(t)}{\left[\omega\left(W(t)^{\frac{1}{rq}}\right)\right]^q} = \frac{\lambda_2 F(s)\omega\big(u(t)\big)^q}{\omega\left(W(t)^{\frac{1}{rq}}\right)^q} \leqq \lambda_2 F(t).$$

If

$$\Lambda(v) = \int_{v_0}^{v} \frac{\mathrm{d}\sigma}{\left[\omega\left(\sigma^{\frac{1}{rq}}\right)\right]^q}, \quad v_0 > 0\,,$$

then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Lambda\big(W(t)\big) &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{v_0}^{W(t)} \frac{\mathrm{d}\sigma}{\left[\omega\left(\sigma^{\frac{1}{rq}}\right)\right]^q} \\
&= \frac{\frac{\mathrm{d}}{\mathrm{d}t}W(t)}{\left[\omega\left(W(t)^{\frac{1}{rq}}\right)\right]^q} \\
&\leqq \lambda_2 F(t).
\end{aligned}$$

This yields

$$\Lambda\big(W(t)\big) - \Lambda\big(W(0)\big) \leqq \lambda_2 \int_0^t F(s)\,\mathrm{d}s$$

and thus we have

$$\Lambda\big(u(t)^{rq}\big) \leqq \Lambda(\lambda_1) + \lambda_2 \int_0^t F(s)\,\mathrm{d}s\,, \qquad 0 \leqq t \leqq T.$$

We have proved the following theorem.

Theorem 1. *Let $0 < T < \infty$, $\varepsilon > 0$, $a(t)$, $F_1(t)$, $F_2(t)$ be nonnegative, continuous functions on the interval $\langle 0, T\rangle$, $0 < \beta < 1$, $r \geqq 1$, ω be a continuous,* *positive, nondecreasing function on* $\mathbb{R}^+ = \langle 0, \infty)$ *and let* $u(t)$ *be a nonnegative, continuous function on* R_+ *satisfying the inequality* (1). *Then*

$$\Lambda\big(u(t)^{rq}\big) \leqq \Lambda(\lambda_1) + \lambda_2 \int_0^t F(s)\,\mathrm{d}s\,, \qquad 0 \leqq t \leqq T,$$

or

$$u(t) \leqq \left\{\Lambda^{-1}\left[\Lambda(\lambda_1) + \lambda_2 \int_0^t F(s)\,\mathrm{d}s\right]\right\}^{\frac{1}{rq}},$$

if

$$\Lambda(\lambda_1) + \lambda_2 \int_0^t F(s)\,\mathrm{d}s \in \mathrm{Dom}\left(\Lambda^{-1}\right),$$

where

$\Lambda(v) = \int\limits_{v_0}^{v} \frac{\mathrm{d}\sigma}{\left[\omega\left(\sigma^{\frac{1}{rq}}\right)\right]^q}$, $q = \frac{1}{\beta} + \varepsilon = 1 + z + \varepsilon$, $\beta = \frac{1}{1+z}$, $z > 0$,
$F(t) = F_1(t)^q\,\mathrm{e}^{-qt} + F_2(t)^q$,
$\lambda_1 = \max\limits_{0 \leqq t \leqq T} b(t)$, $b(t) = 3^{q-1} a(t)^q$,
$\lambda_2 = \max\left\{K\mathrm{e}^{qT}, 3^{q-1}T^{q-1}\right\}$, $K = 3^{q-1}D(\alpha p, p)$, $p = \frac{1+z+\varepsilon}{z+\varepsilon}$, $\alpha = 1 - \beta$, $D(\alpha p, p)$ *is the number from Lemma 1 with* $a = \alpha p$, $b = p$.

3. On an application of Theorem 1

Consider the integral equation

$$z(t) = f(t) + \int_0^t (t-s)^{\beta-1} H_1\big(s, z(s)\big)\,\mathrm{d}s + \int_0^t H_2\big(s, z(s)\big)\,\mathrm{d}s\,, \qquad t \geqq 0 \tag{5}$$

Theorem 2. *Let* $f \in C(\mathbb{R}_+, \mathbb{R}^n)$, $H_1, H_2 \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$, $0 < \beta < 1$ *and* $\varepsilon > 0$. *Assume that*

$$\|H_i(t,z)\| \leqq F_i(t)\omega(\|z\|), \qquad (t,z) \in \mathbb{R}_+ \times \mathbb{R}^n, \qquad i = 1, 2, \tag{6}$$

where $F_1, F_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$ *and* ω *is as in Theorem 1 satisfying the condition*

$$\int_0^\infty \frac{\tau^{q-1}}{\omega(\tau)^q}\mathrm{d}\tau = \infty,$$

where $q = q(\varepsilon) = \frac{1}{\beta} + \varepsilon$. *Then*

$$\lim_{t \to T^-} \sup \|z(t)\| < \infty \tag{7}$$

for any continuous solution $z(t)$ *of the equation* (5) *defined on the interval* $\langle 0, T)$, *with* $0 < T < \infty$.

P r o o f. Let $z(t)$ be a continuous solution of the equation (3) defined on the interval $\langle 0, T)$, $0 < T < \infty$. Assume that the (7) does not hold.

From the continuity of the functions f, H_1, H_2, F_1, F_2 and the property (6) it follows that there exist constants λ_1, λ_2 and a continuous function $F(t)$ (see Theorem 1, such that

$$\|z(t)\|^q \leqq \lambda_1 + \lambda_2 \int_0^t F(s)\omega(\|z(s)\|)^q \, \mathrm{d}s$$

and as a consequence of Theorem 1 ($r = 1$) we obtain

$$\Lambda(||z(t)|^q) \leqq \Lambda(\lambda_1) + \lambda_2 \int_0^t F(s) \, \mathrm{d}s \, .$$

This yields

$$\limsup_{t \to T^-} \Lambda(\|z(t)\|^q) = \limsup_{t \to T^-} \int_0^{\|z(t)\|^q} \frac{\mathrm{d}\sigma}{\omega(\sigma^{\frac{1}{q}})^q} = q \int_0^\infty \frac{\tau^{q-1}}{\omega(\tau)^q} \mathrm{d}\tau = \infty.$$

and this is a contradiction. □

Remark. In the papers [3]–[5], [15] and [19] sufficient conditions for the local existence of solutions of integral equations of the form (1) with $H_2 \equiv 0$, $H_1(t, u) = \omega(u)$, $u \geqq 0$ is as above, are proved. Results of that type can be applied in the study of global properties of solutions of parabolic differential equations (see, e.g., [4], [14]).

4. On a singular version of the Wendroff-Pachpatte theorem for the inequality (2)

We assume that the inequality (2) contains also the nonsingular part. We need the following lemma.

Lemma 2. *If* $0 < a < 1$, $0 < b < 1$, $c > 0, d > 0$ *then there exists a constant* $\mathcal{D} = \mathcal{D}(a,b,c,d) > 0$ *such that*

$$I(x,y,a,b,c,d) = \int_0^x \int_0^y (x-s)^{-a}(y-t)^{-b}\, \mathrm{e}^{cs} \cdot \mathrm{e}^{dt}\, \mathrm{d}s\, \mathrm{d}t \leqq \mathcal{D}\mathrm{e}^{cx+dy}, \qquad x, y \geqq 0.$$

P r o o f. Applying Lemma 1 we obtain

$$\begin{aligned}
&\int_0^x \int_0^y (x-s)^{-a}(y-t)^{-b}\, \mathrm{e}^{cs} \cdot \mathrm{e}^{dt}\, \mathrm{d}s\, \mathrm{d}t \\
&= \int_0^x (x-s)^{-a}\, \mathrm{e}^{cs} \left(\int_0^y (y-t)^{-b}\, \mathrm{e}^{dt} \right) \mathrm{d}s \\
&\leqq \int_0^x (x-s)^{-a}\, \mathrm{e}^{cs} \big(D_1(b,d)\, \mathrm{e}^{dy} \big)\, \mathrm{d}s \\
&\leqq \big(D_2(a,c)\, \mathrm{e}^{cx} \big) \big(D_1(b,c)\, \mathrm{e}^{dy} \big) \\
&= \mathcal{D}\mathrm{e}^{cx+dy},
\end{aligned}$$

where $\mathcal{D} = \mathcal{D}(a,b,c,d) = D_1(a,c)D_2(b,d)$ is a positive constant. □

Lemma 3. *Let a function* $h(x,y)$ *be integrable on* $\mathbb{R}_2^+$ *and* $\delta > 1$. *Then*

$$\left(\int_0^x \int_0^y h(x,y)\, \mathrm{d}x\, \mathrm{d}y \right)^{\delta} \leqq x^{\delta-1} y^{\delta-1} \int_0^x \int_0^y h(x,y)^{\delta}\, \mathrm{d}x\, \mathrm{d}y\,.$$

This lemma is a direct consequence of an abstract version of Jensen inequality (see the book [9, Theorem 3, p. 182]).

Now we shall apply the desingularization method proposed in the papers [11], [13], a version of Jensen inequality, and the result by B. G. Pachpatte [18, Theorem 5.2.1] on nonlinear integral inequality of the Bihari type for functions in two variables.

Let $\beta = \frac{1}{z+1}$, $z > 0$, $\alpha = 1 - \beta$, $q = q(\varepsilon) = \frac{1}{\beta} = 1 + z + \varepsilon$, $p = p(\varepsilon) = \frac{1+z+\varepsilon}{z+\varepsilon}$, where $\varepsilon > 0$. Obviously, $\frac{1}{p} + \frac{1}{q} = 1$. Applying the Hölder inequality and Lemma 2 we obtain for $0 \leqq x \leqq T_1 < \infty$, $0 \leqq y \leqq T_2 < \infty$:

$$
\begin{aligned}
v(x,y) \leqq & a(x) + b(y) \\
& + \int_0^x \int_0^y (x-s)^{-\alpha} \mathrm{e}^{\varepsilon s} (y-t)^{-\alpha} \mathrm{e}^{\varepsilon t} \cdot \mathrm{e}^{-\varepsilon(t+s)} G_1(s,t) \omega\big(v(s,t)\big) \,\mathrm{d}s\,\mathrm{d}t \\
& + \int_0^x \int_0^y G_2(s,t) \omega\big(v(s,t)\big) \,\mathrm{d}s\,\mathrm{d}t \\
& \leqq a(x) + b(y) + \left(\int_0^x \int_0^y (x-s)^{-\alpha p} \mathrm{e}^{\varepsilon s p} (y-t)^{-\alpha p} \mathrm{e}^{\varepsilon p t} \,\mathrm{d}s\,\mathrm{d}t \right)^{\frac{1}{p}} \\
& \times \left(\int_0^x \int_0^y \mathrm{e}^{-\varepsilon q(t+s)} G_1(s,t)^q \omega\big(v(s,t)\big)^q \,\mathrm{d}s\,\mathrm{d}t \right)^{\frac{1}{q}} \qquad (8) \\
& \leqq a(x) + b(y) + I(T_1, T_2, \alpha p, \alpha p, p, p)^{\frac{1}{p}} \cdot \mathrm{e}^{\varepsilon(x+y)} \\
& \times \left(\int_0^x \int_0^y \mathrm{e}^{-\varepsilon q(t+s)} G_1(s,t)^q \omega\big(v(s,t)\big)^q \,\mathrm{d}s\,\mathrm{d}t \right)^{\frac{1}{q}} \\
& + \int_0^x \int_0^y G_2(s,t) \omega\big(v(s,t)\big) \,\mathrm{d}s\,\mathrm{d}t \,,
\end{aligned}
$$

where $I = I(T_1, T_2, \alpha p, \alpha p, p, p)$ is the value of the function from Lemma 2. Now assume that the couple (ω, η) , where $\eta \colon \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, positive, nondecreasing function, satisfies the (r,q) condition

$$
\mathrm{e}^{-rt} \omega(v)^q \leqq R(t) \eta(\mathrm{e}^{-rt} v^q), \qquad t \geqq 0, \quad v \geqq 0, \tag{9}
$$

where $r = \varepsilon q$. This inequality yields

$$
\begin{aligned}
v(x,y)^q \leqq & 3^{q-1} \big(a(x) + b(y)\big)^q \\
& + 3^{q-1} I^{\frac{q}{p}} \mathrm{e}^{\varepsilon q(x+y)} \int_0^x \int_0^y \mathrm{e}^{-\varepsilon q(t+s)} G_1(s,t)^q \omega\big(v(s,t)\big)^q \,\mathrm{d}s\,\mathrm{d}t \\
& + 3^{q-1} \left(\int_0^x \int_0^y G_2(s,t) \eta\big(v(s,t)\big) \,\mathrm{d}s\,\mathrm{d}t \right)^q
\end{aligned}
$$

$$\begin{aligned}&\leqq 3^{q-1}\big(a(x)+b(y)\big)^q\\&\quad+3^{q-1}I^{\frac{q}{p}}\,\mathrm{e}^{\varepsilon q(x+y)}\int\limits_0^x\int\limits_0^y G_1(s,t)^q R(s+t)\omega\big(\mathrm{e}^{-\varepsilon qt}\,v(s,t)^q\big)^q\,\mathrm{d}s\,\mathrm{d}t\\&\quad+3^{q-1}\left(\int\limits_0^x\int\limits_0^y G_2(s,t)\eta\big(v(s,t)\big)\,\mathrm{d}s\,\mathrm{d}t\right)^q\end{aligned}$$

Now applying Lemma 3 we obtain

$$\begin{aligned}&\left(\int\limits_0^x\int\limits_0^y G_2(s,t)\omega\big(v(s,t)\big)\,\mathrm{d}s\,\mathrm{d}t\right)^q\\&\qquad\leqq x^{q-1}y^{q-1}\int\limits_0^x\int\limits_0^y G_2(s,t)^q\,\mathrm{e}^{q(s+t)}\cdot\mathrm{e}^{-q(s+t)}\,\omega\big(v(s,t)\big)^q\,\mathrm{d}s\,\mathrm{d}t\\&\qquad\leqq x^{q-1}y^{q-1}\int\limits_0^x\int\limits_0^y G_2(s,t)\,\mathrm{e}^{q(s+t)}\,R(s+t)\eta\big(\mathrm{e}^{-q(s+t)}\,v(s,t)^q\big)\,\mathrm{d}s\,\mathrm{d}t\,.\end{aligned}$$

Therefore if $w(x,y)=\mathrm{e}^{\varepsilon(t+s)}\,v(x,y)$ then from the inequality (8) we obtain

$$w(x,y)\leqq c(x)+d(y)+L\int\limits_0^x\int\limits_0^y G(s,t)\eta\big(w(s,t)\big)\,\mathrm{d}s\,\mathrm{d}t\,,$$

where $c(x)=6^{q-1}a(x,y)^q$, $d(y)=6^{q-1}b(y)^q$ (we have used there the inequality $(A+B)^q\leqq 2^{q-1}(A^q+B^q)$, $A,B\geqq 0$), $L=3^{q-1}\big(I^{\frac{q}{p}}\,\mathrm{e}^{q(T_1+T_2)}+T_1^{q-1}\cdot T_2^{q-1}\big)$,

$$G(s,t)=\mathrm{e}^{-q(t+s)}\,G_1(s,t)^q+\mathrm{e}^{q(s+t)}\,R(s+t)G_2(s,t)^q. \tag{10}$$

If $a(x),b(y)$ are positive, continuous functions, $a'(x)$, $b'(y)$ are nonnegative, continuous functions and $\omega(u)$ is continuously differentiable, nondecreasing function, then the the same properties have also the functions $c(x,)$ $d(y)$ and η, respectively. Now applying the result by B.G. Pachpatte [18, Theorem 5.2.1] we obtain the inequality

$$\Omega_\eta\big(w(x,y)\big)\leqq\Omega\big(c(0)+d(y)\big)+\int\limits_0^x\frac{c'(s)}{\eta\big(c(s)+d(0)\big)}\,\mathrm{d}s\,+L\int\limits_0^x\int\limits_0^y G(s,t)\,\mathrm{d}s\,\mathrm{d}t\,,$$

where $\Omega_\eta(w)=\int\limits_{w_0}^{w}\frac{\mathrm{d}\sigma}{\eta(\sigma)}$, $\quad w\geqq w_0>0$.

Theorem 3. *Let $0 < \beta < 1$, $\varepsilon > 0$, $0 < T_1 < \infty$, $0 < T_2 < \infty$ be constants, $G_1(x,y)$, $G_2(x,y)$ be continuous, nonnegative functions on $D(T_1,T_2) = \langle 0,T_1\rangle \times \langle 0,T_2\rangle$, $a(x)$, $b(y)$ be a C^1-functions with $a'(x) \geqq 0$, $b'(y) \geqq 0$, ω be as in Theorem 1 and moreover we assume that it is continuously differentiable. Let $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous, nondecreasing positive function such that the couple $(\omega,\, \eta)$ satisfies the (r,q) condition*

$$\mathrm{e}^{-rt}\,\omega(v)^q \leqq R(t)\eta(\mathrm{e}^{-rt}\,v^q), \qquad t \geqq 0, \quad v \geqq 0,$$

with $r = \varepsilon q$, where $R(t)$ is a continuous function. Assume that $u(t)$ is a continuous, nonnegative function satisfying the inequality (2) on $D(T_1,T_2)$. Then

$$\Omega_\eta\big(\mathrm{e}^{-\varepsilon q(x+y)}\,v(x,y)^q\big) \leqq \Omega_\eta\big(c(0)+d(y)\big) + \int_0^x \frac{c'(s)}{\eta\big(c(s)+d(0)\big)}\,\mathrm{d}s + L\int_0^x\int_0^y G(s,t)\,\mathrm{d}s\,\mathrm{d}t\,,$$

or

$$v(x,y) \leqq \mathrm{e}^{\varepsilon q(x+y)}\,\Omega_\eta^{-1}\Bigg(\Omega_\eta\big(c(0)+d(y)\big) + \int_0^x \frac{c'(s)}{\eta\big(c(s)+d(0)\big)}\,\mathrm{d}s + L\int_0^x\int_0^y G(s,t)\,\mathrm{d}s\,\mathrm{d}t)\Bigg)^{\frac{1}{q}},$$

for $0 \leqq x \leqq \tau_1 \leqq T_1$, $0 \leqq y \leqq \tau_2 \leqq T_2$, where τ_1, τ_2 are such that the argument of the function Ω_η^{-1} belongs to $\mathrm{Dom}(\Omega_\eta^{-1})$,

$$\Omega_\eta(w) = \int_{w_0}^{w} \frac{\mathrm{d}\sigma}{\eta(\sigma)}, \qquad w \geqq w_0 > 0,$$

$q = q(\varepsilon) = \frac{1}{\beta} + \varepsilon$, $\beta = 1-\alpha, c(x,y) = 6^{q-1}a(x,y)^q$, $d(x,y) = 6^{q-1}b(x,y)$, $L = 3^{q-1}\big(I^{\frac{q}{p}}\,\mathrm{e}^{q(T_1+T_2)} + T_1^{q-1}\cdot T_2^{q-1}\big)$, $G(s,t)$ is defined by (10), $I = I(T_1,T_2,\alpha p, \alpha p, p, p)$, $p = \frac{q}{q-1}$, $I(x,y,a,b,c,d)$ is the function from Lemma 2.

5. On an application of Theorem 3

One can prove the following theorem applying Theorem 3 in the same way as we have used Theorem 1 in the proof of Theorem 2.

Theorem 4. *Let $0 < \beta < 1, f(x,y), G(x,y)$ be continuous, nonnegative functions on $\mathbb{R}_2^+ = \langle 0,\infty) \times \langle 0,\infty), f$ bounded, ω, η be as in Theorem 3, $H(x,y,v)$ be a nonnegative, continuous function on $\mathbb{R}_2^+$ with*

$$|H(x,y,v)| \leqq G(x,y)\omega(|v|), \qquad \text{for all} \quad (x,y,v) \in \mathbb{R}_2^+ \times \mathbb{R}$$
$$|f(x,y)| \leqq a(x) + b(y),$$

where the functions $a(x)$, $b(y)$ are as in Theorem 3. Assume that

$$\int_0^\infty \frac{\mathrm{d}\tau}{\left[\eta(\tau)\right]} = \infty,$$

where $q = q(\varepsilon) = \frac{1}{\beta} + \varepsilon$, $\varepsilon > 0$. Let $v(x,y)$ be a nonnegative continuous function on the set $M = \{(x,y) \in R^2 : 0 \leqq x < P < \infty,\ 0 \leqq y < Q < \infty$ satisfying the inequality

$$v(x,y) = f(x,y) + \int_0^x \int_0^y (x-s)^{\beta-1}(y-t)^{\beta-1} H\big(s,t,v(s,t)\big)\,\mathrm{d}s\,\mathrm{d}t\,.$$

Then

$$\limsup_{x\to P^-, y \to Q^-} |v(x,y)| < \infty.$$

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