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THE SUBCHROMATIC INDEX OF GRAPHS

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ABSTRACT. In an edge coloring of a graph each color class forms a subgraph without a path of length two (a matching). An edge subcoloring extends this concept: each color class in an edge subcoloring forms a subgraph without path of length three. While every graph with maximum degree at most two is edge 2-subcolorable, we point out in this paper that recognizing edge 2-subcolorable graphs with maximum degree three is NP-complete, even when restricted to triangle-free graphs. As by-products, we obtain NP-completeness results for the star index and the subchromatic number for several classes of graphs. It is also proved that recognizing edge 3-subcolorable graphs is NP-complete.

Moreover, edge subcolorings and subchromatic index of various basic graph classes are studied. In particular, we show that every unicyclic graph is edge 3-subcolorable and edge 2-subcolorable unicyclic graphs have a simple structure allowing an easy linear time recognition.

1. Introduction

Let $G = (V, E)$ be a graph. An *independent set* (a *clique*) is a set of pairwise nonadjacent (adjacent) vertices. For $W \subseteq V$ the subgraph of G induced by W is denoted by $G[W]$. For $F \subseteq E$ the symbol $V(F)$ denotes the set of endvertices of edges from F, and $G(F) = \big(V(F), F\big)$ is the subgraph of G induced by the edge set F.

A (proper) *vertex r-coloring* of G is a partition $V_1, \dots, V_r$ into disjoint independent sets called *color classes* of the coloring. The *chromatic number* $\chi(G)$ is the smallest number r for which G admits a vertex r-coloring. One of the most interesting generalizations of the classical vertex coloring is the notion of vertex subcoloring; see [2], [9], [12], [15]. A *vertex r-subcoloring* is a partition $V_1, \dots, V_r$ of V where each color class V_i consists of disjoint cliques (of various sizes). The smallest number r for which G has a vertex r-subcoloring is called the *subchromatic number* $\chi_s(G)$ of G.

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Note that a partition $V_1, \dots, V_r$ of V is a vertex r-coloring of $G = (V, E)$ if and only if for each i the graph $G[V_i]$ does not contain a P_2 as an (induced) subgraph. The partition is a vertex r-subcoloring if and only if for each i the graph $G[V_i]$ does not contain a P_3 as an induced subgraph. (P_k denotes the path on k vertices.)

A (proper) *edge r-coloring* is a partition $E_1, \dots, E_r$ of E into color classes E_i in which every two distinct edges do not have an endvertex in common, i.e., each E_i forms a matching. The *chromatic index* $\chi'(G)$ is the smallest number r for which G admits an edge r-coloring. Clearly, a partition $E_1, \dots, E_r$ of E is an edge r-coloring of $G = (V, E)$ if and only if for each i the subgraph $G(E_i)$ does not contain a P_3 as a (not necessarily induced) subgraph. This observation leads to the following natural generalization of the classical edge coloring.

Definition 1. An *edge r-subcoloring* of the edges of a graph $G = (V, E)$ is a partition $E_1, \dots, E_r$ of E into disjoint color classes E_i such that for each E_i the graph $G(E_i)$ contains no P_4 as a (not necessarily induced) subgraph. The *subchromatic index* $\chi'_s(G)$ is the smallest number r for which G admits an edge r-subcoloring.

Remark 1. Obviously, a partition $E_1, \dots, E_r$ of $E(G)$ is an edge r-subcoloring of G if and only if for each i the connected components of $G(E_i)$ are stars or triangles, where a star is a complete bipartite graph $K_{1,s}$ for some $s \geqslant 1$.

A related notion that has been studied in the literature is as follows. A partition $E_1, \dots, E_r$ of $E(G)$ is a *star partition* of G if for each i the connected components of $G(E_i)$ are stars. The *star index* $\chi^*(G)$ of G is the smallest number r for which G has a star partition into r subsets E_i; cf. [1], [3], [4], [5], [11], [17], [21].

Clearly, $\chi'_s(G) \leqslant \chi^*(G)$ for all graphs G, and it holds that $\chi'_s(G) = \chi^*(G)$ whenever G is triangle-free.

Recall that the *line graph* $L(G)$ of a graph G has the edges of G as vertices and two distinct edges e, e' are adjacent in $L(G)$ whenever they have an endvertex in common. It is well-known that proper edge colorings of G correspond to proper vertex colorings of $L(G)$ and vice versa. In particular, $\chi'(G) = \chi\big(L(G)\big)$. Likewise, the following fact is easy to see.

Fact 1. *Edge subcolorings of a graph G correspond to vertex subcolorings of the line graph $L(G)$ of G and vice versa. In particular, $\chi'_s(G) = \chi_s\big(L(G)\big)$.*

Our terminology of edge subcoloring is intended to recall this fact.

2. Basic properties and examples

The subchromatic index of a disconnected graph is the maximum subchromatic index among those of its connected components. Hence, without loss of generality we assume throughout this paper that all graphs are connected.

By the definition, the edge subchromatic index is monotone with respect to graph inclusion, i.e., $G' \subset G \Rightarrow \chi_s'(G') \leqslant \chi_s'(G)$. The next observation shows a close link between subchromatic indices of graphs where one is formed from the other by removal of a vertex.

Observation 1. *For any graph G and any vertex v of G it holds that*

$$\chi_s'(G) \leqslant \chi_s'(G \setminus v) + 1\,.$$

Proof. Any subcoloring of $G \setminus v$ can be extended to a subcoloring of G by using an extra new color on all edges incident with v. □

General lower and upper bounds for the subchromatic index are given below. Let $\Delta(G)$ be the maximum degree of a vertex in the graph G.

Proposition 1. *Any graph G with m edges on n vertices satisfies*

$$\frac{m}{n} \leqslant \chi_s'(G) \leqslant \Delta(G)\,.$$

Moreover, $\chi_s'(G) > \frac{m}{n}$ if G is triangle-free.

Proof. Since every color class consists of stars and triangles, it may contain at most n edges. In the subcoloring each edge has to be colored and the lower bound follows.

Note that a color class in a graph on n vertices can have n edges if and only if n is a multiple of 3 and the class itself is a covering of the vertices by $\frac{n}{3}$ disjoint triangles. In triangle-free graphs no such class exists and the lower bound can be made sharp.

How to obtain the upper bound we have from Fact 1: $\chi_s'(G) = \chi_s\big(L(G)\big) \leqslant \left\lceil \frac{\Delta(L(G))}{2} \right\rceil + 1 \leqslant \left\lceil \frac{2\Delta(G)-2}{2} \right\rceil + 1 = \Delta(G)$ where the inequality for the subchromatic number was shown in [2].

To find a valid subcoloring using at most $\Delta(G)$ edge colors efficiently, we may proceed greedily on the vertex set: with each new vertex u assign colors to its adjacent edges as follows: for an edge (u,v) pick a color that is not used on an already colored edge incident with v. Such a subcoloring is triangle-free and all stars have the property that the center of the star is the latest vertex of the star in the order. □

Observe that the upper bound is attained e.g., for the 5-cycle $\chi'_s(C_5) = 2 = \Delta(C_5)$ or the Petersen graph $\chi'_s(P) = 3 = \Delta(P)$. The last property follows for any cubic graph which contains C_5 as an induced subgraph: it is impossible to extend a valid 2-subcoloring to all edges incident to the cycle C_5. See also Figure 1.

Corollary 1. *Any r-regular graph G satisfies $\frac{r}{2} \leqslant \chi'_s(G) \leqslant r$. Moreover, $\frac{r}{2} < \chi'_s(G)$ if G is triangle-free.*

2.1 Trees and cycles

For trees and cycles the subchromatic index can be determined explicitly as follows:

Proposition 2.

(i) *For any tree T, $\chi'_s(T) = \chi^*(T) \leqslant 2$; $\chi'_s(T) = 2$ if and only if T is not a star;*

(ii) $\chi'_s(C_3) = 1$ *and* $\chi'_s(C_n) = \chi^*(C_n) = 2$ *for all* $n \geqslant 4$.

Proof. Color greedily. □

2.2 Cacti

A *cactus* is a connected graph in which every block (maximal 2-connected subgraph) is an edge or a cycle. Equivalently, a graph G is a cactus if and only if every two cycles in G are edge-disjoint.

Proposition 3. *For all cacti G we have $\chi'_s(G) \leqslant \chi^*(G) \leqslant 3$. Moreover, an edge 3-subcoloring can be found in linear time.*

Proof. Let T be a breadth-first search (bfs) tree of G rooted at vertex v. We claim that all edges of G outside T form a matching.

During the search we arrange vertices into levels based on the distance from the initial vertex. Since G is a cactus, the tree T misses from each odd cycle the edge connecting the two vertices at the highest level. Similarly, for an even cycle one of the two edges incident with the vertex at the highest level remains outside T.

Now observe the given two edges e, e' of $E(G) \setminus E(T)$, either e is separated from e' by the lowest vertex of the cycle containing e, or vice-versa.

We color T with two colors and use the matching $E(G) \setminus E(T)$ as the third color. This shows $\chi^*(G) \leqslant 3$. Since a bfs-tree can be performed in linear time, Proposition 3 follows. □

We leave it as an open problem whether cacti with subchromatic index at most 2 allow a simple structural description. Since all cacti have treewidth upperbounded by 2, their subchromatic index can be computed in polynomial time as we will show later in Section 3.3.

Observe that in any edge 2-subcolored cactus G each vertex of degree at least 3 is either a center of a monochromatic star or it belongs to a monochromatic triangle. Let us direct the edges of monochromatic stars $K_{1,k}$, $k \geqslant 2$ towards their centers. (The other edges remain undirected.) Clearly, no vertex of degree at least 3 in G is of outdegree 2 or more. Each directed cycle is of even length, since the colors of stars must alternate.

Figure 1 shows some cacti G that do allow such orientation of their cycles and hence have $\chi'_s(G) = 3$.

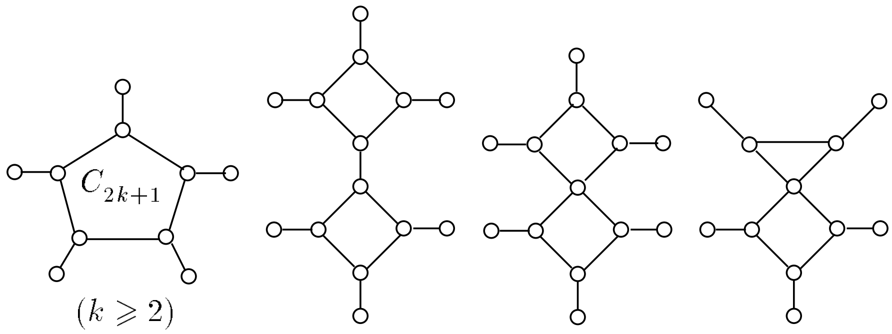

FIGURE 1. Some cacti with subchromatic index 3.

2.3 Unicyclic graphs

A (connected) graph is *unicyclic* if it contains exactly one cycle. As we show now, all edge 2-subcolorable unicyclic graphs have a simple structure and hence can be easily recognized in linear time.

THEOREM 1. *For any unicyclic graph G we have $\chi'_s(G) \leqslant 3$. Moreover, $\chi'_s(G) = 3$ if and only if the only cycle C of G has length $2k+1$, $k \geqslant 2$ and all vertices of C are of degree at least 3 in G.*

Proof. Since unicyclic graphs are cacti, the first part follows from Proposition 3. We have shown above that if an odd cycle C_{2k+1}, $k \geqslant 2$ contains no vertex of degree 2, then the graph G cannot be edge 2-subcolorable.

It suffices to construct an edge 2-subcoloring in all other cases. First consider the case when C is a triangle. We make it monochromatic (say white) and then distribute all remaining edges of G into two color classes as follows: two edges belong to the same class if their distance from C has the same parity modulo 2. (The edge-distance is viewed as the distance between the corresponding vertices in the line graph.)

Now assume that C_{2k+1} contains a vertex of degree 2. Let us denote the vertices of C by $v_1, v_2, \dots, v_{2k+1}$ where $\deg(v_1) = 2$. We delete the edge (v_1, v_2) from G and obtain a tree T. We use an edge 2-subcoloring of T obtained by rooting T in the vertex v_2 and by using white colors on odd levels and black on even. In particular, the edge (v_{2k+1}, v_1) is black and it is plausible to color (v_1, v_2) white to obtain an edge 2-subcoloring of G.

For an even cycle C we delete the edge (v_1, v_2) as color edges of the resulting tree as in the previous case. Then v_1 is the center of a black star and it is feasible to color the remaining edge (v_1, v_2) black. □

Clearly, C can be found in G in linear time. Theorem 1 also implies that unicyclic graphs with subchromatic index at most 2 can be recognized in linear time.

2.4 Complete bipartite graphs

Since complete bipartite graphs contain no triangle, subchromatic index coincides with the star chromatic index. We adopt the following results straightforwardly:

Proposition 4. *For any $n \geqslant 1$ it holds that*

$$\chi'_s(K_{n,n}) = \chi^*(K_{n,n}) = \begin{cases} n\,, & n \leqslant 4\,, \\ 4\,, & n = 5\,, \\ \lceil \frac{n}{2} \rceil + 2\,, & n \geqslant 6\,. \end{cases}$$

2.5 Cubes

Recall that the d-dimensional Cube Q_d has all $0, 1$ d-tuples as vertices and two such d-tuples are adjacent in Q_d if and only if they differ in exactly one position. Note that Q_d is d-regular, bipartite and has 2^d vertices.

Proposition 5 ([21]).

(i) *For $k \geqslant 2$, $\chi'_s(Q_{2^k-2}) = \chi^*(Q_{2^k-2}) = 2^{k-1}$;*

(ii) *For $n \geqslant 3$, $\lceil \frac{n}{2} \rceil + 1 \leqslant \chi'_s(Q_n) = \chi^*(Q_n) \leqslant \lceil \frac{n}{2} \rceil + \log_2 n$.*

2.6 Complete graphs

Proposition 6. *For any $n \geqslant 1$ the subchromatic index of the complete graph K_n is bounded by:*

$$\left\lfloor \frac{n}{2} \right\rfloor \leqslant \chi'_s(K_n) \leqslant \chi^*(K_n) = \left\lceil \frac{n}{2} \right\rceil + 1\,.$$

Moreover, the lower bound is attained if and only if $n = 6k+3$ *for some integer* k.

P r o o f. The upper bound by $\chi^*(K_n) = \lceil \frac{n}{2} \rceil + 1$ was given in [1]. For an odd n a subcoloring of K_n attaining the lower bound on color classes given by Proposition 1 $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$ is equivalent to a Kirkman triple system on n vertices. Its existence has been shown exactly for all $n = 6k+3$ in [19]. □

This allows us to classify the subchromatic index for all complete graphs of order at most ten. First we show that $\chi'_s(K_6) = 4$. Assume for a contradiction that a valid edge 3-subcoloring of K_6 exists. Then each color class may have at most 6 edges. Hence, at least two of these two classes must have at least 5 edges because $|E_{K_6}| = 15$. By a case study it is easy to determine that a color class with at least five edges may only induce a subgraph in K_6 of one of the possible three types: $K_3 \cup K_3$ or $K_3 \cup P_3$ or $K_{1,5}$. (Here $\cup$ stands for disjoint union.)

If $K_3 \cup K_3$ is a color class, the remaining edges form $K_{3,3}$ which has no edge 2-subcoloring. Obviously, $K_{1,5}$ can appear only once as a color class. Hence, if an edge 3-subcoloring exists, it must contain two color classes of type $K_3 \cup P_3$. Up to an isomorphism there is only one way to combine such two classes, but the remaining edges form $C_4 \cup K_2$, hence no edge 3-subcoloring of K_6 exists.

We summarize the values of subchromatic index of small complete graphs the following table:

Graph G	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9	K_{10}
$\chi'_s G$	0	1	1	2	3	4	4	4	4	5

The values for K_7 and K_8 are majorized by $\chi'_s(K_9)$ given by Proposition 6. The remaining values follow from the monotone property of χ'_s and other bounds which were discussed above.

3. Computational complexity

Formally, we define EDGE k-SUBCOLORABILITY as a decision problem which for a given graph G (the instance) answers the question: "Is $\chi'_s(G) \leqslant k$?"

In the following sections we will show that the problem is NP-complete in general. (Obviously, EDGE k-SUBCOLORABILITY $\in$ NP.) On the other hand, a linear time algorithm exists for a restricted class of graphs of bounded treewidth. This also implies that cacti of subchromatic index at most 2 can be recognized in linear time.

3.1 NP -hardness of EDGE 2-SUBCOLORABILITY

In this section we prove the following negative result.

THEOREM 2. EDGE 2-SUBCOLORABILITY *is* NP-*complete even when restricted to triangle-free graphs of maximum degree three.*

As all graphs with maximum degree at most 2 are edge 2-subcolorable, Theorem 2 is the best possible with respect to degree constraint.

Proof. We prove Theorem 2 by showing a reduction from the NOT-ALL-EQUAL 3-SATISFIABILITY (NAE-3SAT) problem which has been shown to be NP-complete by Schaefer [20] (see also [14, Problem LO3]).

This problem decides whether a Boolean formula Φ in conjunctive normal form satisfying that each clause is a disjunction of literals, allows a satisfying assignment for Φ such that each clause in Φ contains at least one negatively valued literal. This problem can be reduced to the case when all literals are positive (i.e., with no negations) and when each clause contains exactly three not necessarily distinct literals. (Also known as SET SPLITTING or HYPERGRAPH 2-COLORABILITY [14, Problem SP4].) We denote the class of all formulas that allow such an assignment by NAE-3SAT.

Let a formula Φ be an instance for the NAE-3SAT problem. Assume that Φ consists of m clauses $C_1, C_2, \dots, C_m$ over variables $x_1, x_2, \dots, x_n$ such that every clause C_j contains exactly three positive literals $C_j = (x_{j_1} \vee x_{j_2} \vee x_{j_3})$.

We will construct a triangle-free graph $H = H(\Phi)$ of maximum degree three such that H has an edge 2-subcoloring if and only if $\Phi \in$ NAE-3SAT.

Clause gadget. Consider the graph G_C depicted in Figure 2 (left) with three labeled vertices a, b, c.

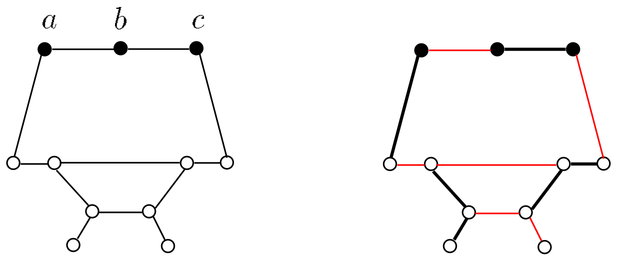

FIGURE 2. Clause gadget G_C (left) with an edge 2-subcoloring (right).

FACT 2.

(i) *The graph G_C is edge 2-subcolorable. In any edge 2-subcoloring the edges (a, b), (b, c) receive different colors.*

(ii) *Every coloring of the two edges (a, b), (b, c) with two distinct colors can be extended into an edge 2-subcoloring of the entire graph G_C such that the two edges incident with the vertex a and the two edges incident with c receive different colors (cf. Figure 2 (right)).*

Fact 2 can be seen quickly by inspection.

Variable gadget. Let $k \geqslant 2$ be an integer. Let G_V^k be the graph depicted in Figure 3 with k labeled vertices $a_1, \dots, a_k$.

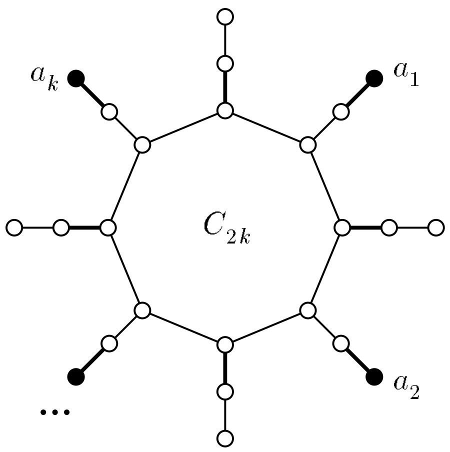

FIGURE 3. Variable gadget G_V^k.

FACT 3. *The graph G_V^k is edge 2-subcolorable. In any edge 2-subcoloring the thick edges receive the same color.*

For the construction of H we take the incidence graph G_Φ. Vertices of G_Φ represent variables and clauses of Φ, and edges represent the incidence relation. Since the same variables may appear in the same clause as a literal, multiple edges may appear in G_Φ.

For the construction of H we replace each clause-representing vertex v_j of degree three with a unique copy of G_C such that the three edges incident with v are one-to-one incident with the vertices a, b and c (i.e., each edge chooses exactly one vertex).

Let further each vertex u_i representing a variable x_i with $k_i \geqslant 2$ occurrences be replaced by a unique copy G_i of $G_V^{k_i}$. Note that variables with only one occurrence can be either represented by G_V^2 or alternatively recursively eliminated from the formula by a preprocessing. Similarly as in the previous case, the k_i edges incident with u_i become one-to-one incident with the vertices $a_1, \dots, a_{k_i}$.

The construction of H is completely described. Since all gadgets have maximum degree three and no triangles, and since labeled vertices in the gadgets have degree one or two, H has maximum degree three and no triangles as well.

Suppose now that the edges of H can be subcolored by two colors, say red and blue. For each i we define $\phi(x_i) = \texttt{true}$ if the thick edges of G_i are red; and $\phi(x_i) = \texttt{false}$ otherwise. By Fact 3 the assignment ϕ is well defined. For each variable gadget the edges leaving the gadget G_i from vertices $a_1, \dots, a_k$ must obtain the same color, complementary to the color of the thick edges of G_i. On the other side, the edges pendant from a clause gadget cannot have all the same color, since then a monochromatic P_4 would appear by Fact 2. Hence, in each clause at least one variable is positively valued and at least one is valued negatively by ϕ and $\Phi \in \mathrm{NAE}\text{-}3\mathrm{SAT}$.

In the opposite direction assume that $\Phi \in \mathrm{NAE}\text{-}3\mathrm{SAT}$ for an assignment ϕ. We derive an edge 2-subcoloring of H as follows. If $\phi(x_i) = \texttt{true}$ ($\texttt{false}$) we color the thick edges of the variable gadget G_i red (blue, respectively). Then we extend this coloring into an edge 2-subcoloring of G_i. This is always possible by Fact 3. As it was mentioned in the previous paragraph, the edges stemming from the clause gadgets (i.e., the original edges of G_Φ) allow a unique 2-subcoloring extension. Finally, we complete the 2-subcoloring of H on the clause gadgets according to Fact 2 (ii).

This argument completes the proof of Theorem 2. □

Corollaries. Since $\chi'_s(G) = \chi^*(G)$ holds for triangle-free graphs G, Theorem 2 implies

COROLLARY 2. *Deciding if the star index of a given graph is two is* NP *-complete even for triangle-free graphs with maximum degree three.*

We remark that it was first proved in [17] that deciding if the star index of a triangle free graph is two is NP-complete. However, the graph constructed in [17] does not have bounded degree while our NP-complete result for the star index is the best possible with respect to degree constraint.

In [12] it was shown that recognizing vertex 2-subcolorable graphs is NP-complete even for triangle-free planar graphs with maximum degree 4. Fact 1 (recall Section 1) and Theorem 2 imply

COROLLARY 3. *Recognizing vertex* 2*-subcolorable graphs is* NP*-complete even for line graphs* (*of triangle-free graphs*) *with maximum degree* 4.

An $(r_1, \dots, r_k)$-*subcoloring* of a graph $G = (V, E)$ is a partition $V_1, \dots, V_k$ of V such that each V_i consists of disjoint cliques, each of which has cardinality at most r_i. In [2] it was shown that all cubic graphs are $(2,2)$-subcolorable, and in [18] it was shown that deciding if a cubic graph is $(1,3)$-subcolorable is NP-complete. Theorem 2 and Fact 1 imply that a similar result holds if we restrict ourselves to line graphs.

COROLLARY 4. *Recognizing vertex* $(3,3)$*-subcolorable graphs is* NP*-complete even for line graphs* (*of triangle-free graphs*) *with maximum degree* 4.

3.2 NP-hardness of Edge 3-subcoloring

This section deals with the proof that Edge 3-subcoloring is NP-complete. Given a graph G we construct a graph H as follows. Take three copies G_1, G_2, G_3 of G, take further a triangle (v_1, v_2, v_3), and for each $i = 1, 2, 3$ connect v_i with all vertices in G_i. Finally, take two vertices x, y and connect x to v_1, v_2, v_3 and y. See Fig. 4.

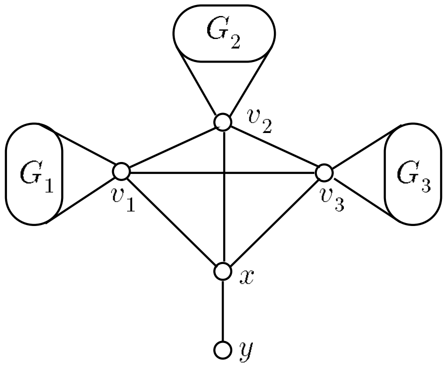

Figure 4. The graph H obtained from the copies G_1, G_2, G_3 of the given graph G.

In the graph H the set of edges between v_i and all vertices of G_i is called the (v_i, G_i)-star. We now point out that $\chi'_s(G) \leqslant 2$ if and only if $\chi'_s(H) \leqslant 3$.

Assume that $\chi'_s(G) \leqslant 2$ and consider an edge 2-subcoloring of G with colors c_1, c_2. In H we color the three copies G_i with this coloring and the (v_i, G_i)-stars with the third color c_3. We color the triangle (v_1, v_2, v_3) with color c_1 and the 4-star at x with color c_2. This yields an edge 3-subcoloring for H.

Assume that $\chi'_s(H) \leqslant 3$ and consider an edge 3-subcoloring of H with colors c_1, c_2, c_3.

Claim 1. For $i = 1, 2, 3$ one of the (v_i, G_i)-stars is monochromatic.

Proof. Assume that the claim is false. Then no two edges of the triangle (v_1, v_2, v_3) have the same color: If (v_1, v_2) and (v_2, v_3) are colored with c_1, say, then the (v_1, G_1)-star and the (v_3, G_3)-star must be colored with c_2 and c_3. This implies that the edge (v_1, v_3) is also colored with c_1 and the (v_2, G_2)-star is also colored with c_2 and c_3. Now, the edges (x, v_i), $i = 1, 2, 3$, must have color c_2 or c_3 and there exists a P_4 with color c_2 or c_3, a contradiction.

Let's assume, without loss of generality that (v_1, v_2) is colored with c_1, (v_2, v_3) with c_2 and (v_1, v_3) with c_3. If c_1 does not appear in the (v_3, G_3)-star, then, by assumption, the (v_3, G_3)-star is colored with c_2 and c_3. Hence c_2 cannot appear in the (v_2, G_2)-star; otherwise there is a P_4 colored with c_2. Therefore, the (v_2, G_2)-star is colored with c_1 and c_3, implying c_1 cannot appear in the (v_1, G_1)-star; otherwise there is a P_4 colored with c_1. Thus, the (v_1, G_1)-star is colored with c_2 and c_3. But then there exists a P_4 colored with c_3. This contradiction shows that c_1 must appear in the (v_3, G_3)-star, and by symmetry, c_2 must appear in the (v_1, G_1)-star and c_3 must appear in the (v_2, G_2)-star.

In particular, for each v_i, each color appears in the star at v_i minus the edge (v_i, x).

Hence, the edge (x, y) is colored differently with each of the edges (v_i, x) for $i = 1, 2, 3$; otherwise there is a monochromatic P_4. As only two colors are available for the edges $(x, v_1), (x, v_2)$ and (x, v_3), at least two of these edges must have the same color. By symmetry, let (x, v_1) and (x, v_2) have color c, say. Then $c = c_1$, since otherwise there is a monochromatic P_4 in the K_4 induced by the v_i and x. Therefore both the (v_1, G_1)-star and the (v_2, G_2)-star are colored with c_2 and c_3. Since c_2 or c_3 appears in the (v_3, G_3)-star, there exists a monochromatic P_4. This last contradiction proves the claim. □

By the claim we may assume that the (v_1, G_1)-star is colored with c_1. If G_1 has less than three vertices, then clearly $\chi'_s(G) \leqslant 2$. If G_1 has at least three vertices, then c_1 cannot appear in G_1; otherwise there is a P_4 colored with c_1. Therefore, the restriction of the edge 3-subcoloring of H on G_1 is an edge 2-subcoloring for G.

From the reduction above and Theorem 2 we obtain

Theorem 3. Edge 3-subcolorability *is* NP-*complete.*

3.3 Graphs of bounded treewidth

We note first that for a fixed k the Edge k-subcolorability problem can be expressed in Monadic Second Order Logic (MSOL) and a linear-time algorithm for graphs of bounded treewidth exists due to [10].

The disadvantage of this general scheme is a huge hidden constant in the time complexity. We follow here the usual scheme for constructing linear-time algorithms for graphs of bounded treewidth, cf. [6], [7] and outline the main aspects of the dynamic programming algorithm.

A *nice tree decomposition* of width at most t of a graph G is a rooted tree T where nodes X_i of T are subsets of vertices of G chosen according to the following rules:

- For each edge $(u, v) \in E(G)$ there exists a node X_i such that $\{u, v\} \subseteq X_i$.
- For each vertex $u \in V(G)$ the nodes X_i containing u induce a connected subgraph (i.e., a subtree) in T.
- The size of each node $|X_i| \leqslant t + 1$.
- Each node X_i has at most two children and

– it is called a *leaf node* if it has no children and $|X_i| = 1$,
– or X_i has one child X_j, then X_i is either an *introduce node* if $X_i = X_j \cup \{u\}$ for some vertex $u \notin X_j$, or a *forget node* when $X_i = X_j \setminus \{u\}$ for some $u \in X_j$,
– or X_i has two children $X_j, X_{j'}$. Then it is called a *join node* and it holds that $X_i = X_j = X_{j'}$.

THEOREM 4. *For any fixed k and t the* EDGE k-SUBCOLORING *problem can be solved in linear time for graphs of treewidth at most t.*

Proof. For the dynamic programming we compute with each node X_i of T a table $\mathtt{Tab}_i$ of the following contents: each entry $(\phi, r) \in \mathtt{Tab}_i$ is a pair such that: ϕ is an edge k-subcoloring of the graph G_i, the subgraph of G induced by the vertex set X_i. The symbol r stands for a ranking $r : X_i \times \{1, \dots, k\} \to \{0, 1, 2, 3, 4\}$ of the following meaning: for any edge k-subcoloring ψ extending ϕ to the subgraph of G induced by the union of descendants of X_i we define

- $r(u, c) = 0$ if no edge of color c in ψ contains the vertex u.
- $r(u, c) = 1$ if u is incident with exactly one edge (u, v) of color c, and v is incident to no other edge of color c in ψ.
- $r(u, c) = 2$ if u is incident with exactly one edge (u, v) of color c, but v is incident also with other edges of this color in ψ.
- $r(u, c) = 3$ if u belongs to a monochromatic triangle of color c in ψ.
- $r(u, c) = 4$ if u is the center of a monochromatic star $K_{1,k}$, $k \geqslant 2$ of color c in ψ.

The algorithm uses usual bottom-up dynamic programming for computing the tables $\mathtt{Tab}_i$. For the table evaluation it is important which of the above cases applies, so that we obtain all correct rankings r for a subcoloring ϕ. Firstly, we describe how these rules are applied (i.e., how the tables $\mathtt{Tab}_i$ are evaluated). Then we show by a case study that the values were assigned correctly.

The algorithm proceeds as follows:

1. If $X_i = \{u\}$ is a leaf node, we let $\mathtt{Tab}_i = (\emptyset, r(u, c) = 0)$ for $1 \leqslant c \leqslant k$.
2. If X_i is a forget node with child X_j, then we store all pairs (ϕ, r), in $\mathtt{Tab}_i$ where for some $(\phi', r') \in \mathtt{Tab}_j$ the subcoloring ϕ is a restriction of ϕ' to the subgraph induced by X_i, and r is a restriction of the ranking r' to the set X_i.
3. If X_i is an introduce mode with child X_j and $\{u\} = X_i \setminus X_j$, we consider all entries $(\phi', r') \in X_j$, and all possible extensions ϕ of ϕ'. Here a pair (ϕ, r) is feasible if for every color c

   3a) either u is incident with no edge of color c, and then we store $r(u, c) = 0$, 3b) or u is incident with only one edge (u,v) of color c, then $r(u,c) = r(v,c) = 1$ if $r'(v,c) = 0$, or $r(u,c) = 2$ and $r(v,c) = 4$ if $r'(v,c) \in \{1,4\}$,

3c) or u is incident with two edges $(u,v),(u,w)$ of color c, where $\phi(v,w) = c$ and $r'(v,c) = r'(w,c) = 1$, then we let $r(u,c) = r(v,c) = r(w,c) = 3$,

3d) or u is incident with $q \geqslant 2$ edges (u,v_s) of color c where $r'(v_s,c) = 0$ for $1 \leqslant s \leqslant q$. Then we let $r(u,c) = 4$ and $r(v_s,c) = 2$ for $1 \leqslant s \leqslant q$.

If not specified above, we let $r(v,c) = r'(v,c)$ for all other $v \in X_j$, and store all feasible pairs (ϕ,r) in $\mathtt{Tab}_i$.

4. If X_i is a join node with children $X_j, X_{j'}$, in $\mathtt{Tab}_i$, we will keep all pairs (ϕ,r) for which there exist $(\phi',r') \in \mathtt{Tab}_j$ and $(\phi'',r'') \in \mathtt{Tab}_{j'}$ such that $\phi = \phi' = \phi''$ and moreover for each color c and for every vertex $u \in X_i$ at least one of the following cases applies:

4a) either $r(u,c) = \max\{r'(u,c), r''(u,c)\}$ and $\min\{r'(u,c), r''(u,c)\} = 0$,

4b) or $r(u,c) = \max\{r'(u,c), r''(u,c)\}$ if there exists unique vertex $v \in X_i$, such that (u,v) is of color c, and $r'(u,c), r''(u,c) \in \{1,2\}$,

4c) or $r(u,c) = 3$ if

4ca) either there are $v,w \in X_i$, such that u,v,w form a monochromatic triangle in ϕ (then $r(u,c) = r'(u,c) = \cdots = r''(w,c)$),

4cb) or there is a vertex $v \in X_i$, such that $\phi(u,v) = c$ and $r'(u,c) = r'(v,c) \neq r''(u,c) = r''(v,c)$ where $\{r'(u,c), r''(u,c)\} = \{1,3\}$

4d) or $r(u,c) = 4$ if

4da) either $\max\{r'(u,c), r''(u,c)\} = 4$ and $\min\{r'(u,c), r''(u,c)\} \in \{1,4\}$

4db) or $r'(u,c) = r''(u,c) = 1$ and no edge of color c is incident with u in ϕ.

In order to argue the correctness of these rules, consider the subgraph of G induced by the union of descendants of some node X_i. Let ψ be any edge k-subcoloring ψ of G. We show that for ψ there exists a record $(\phi,r) \in \mathtt{Tab}_i$ if the table entries were evaluated recursively according to the above rules. For leaf and forget nodes the statement is correct straightforwardly.

Let X_i be an introduce node for the vertex u. If u is not incident with an edge of color c in ψ, then we get $r(u,c) = 0$.

If in ψ the vertex u is incident exactly with one edge (u, v) of color c, then the value of $r(u, c)$ depends on whether v is incident with some other edges or not. If v does not appear on X_i, then we keep the value $r'(c, u)$ according to the rule 3a). Else, the edge (u, v) is colored also by ϕ and the value of $r(u, c)$ depends on whether v has been incident with an edge of color c before, by the rule 3b). Note that in this case v cannot be a member of a monochromatic triangle nor a leaf of a monochromatic star of color c.

If in ψ the vertex u becomes a part of a monochromatic triangle u, v, w, then (v, w) is the only edge of color c in ψ incident with v (or w resp.). This is the only case how a monochromatic triangle may appear and is captured by the case 3c).

Finally, the case 3d) shows how a monochromatic star may appear with a new vertex. Clearly all q leaves of this star cannot be incident with any other edge of the same color.

Assume now that X_i is a join node. Again we distinguish five cases according to the presence of edges colored in ψ by a color c around a vertex u. If there is no such an edge, then we get $r(u, c) = 0$ and rankings r', r'' satisfy the same also on the children nodes, hence rule 4a).

Consider the case when u is incident with exactly one edge (u, v) of color c, which is also the only edge of color c incident with v. Then either $v \in X_i$ and the case 4b) applies or $v \notin X_i$ and we follow 4a). (Here v appears either in the subtree rooted in X_j or in the subtree below $X_{j'}$, but not in both. Clearly, ϕ cannot contain any other edge incident with u of color c.)

We handle similarly the case when the edge (u, v) belongs to a monochromatic star of color c in ψ. Then neither the center v nor any other edge of the star appears on X_i—case 4a). Or, alternatively, v belongs to X_i and the star also appears in some of the two children of X_i—case 4b).

Now we discuss the case when u is a member of a triangle in ψ. The entire triangle may appear in only one subtree below X_j or $X_{j'}$—case 4a). It may also completely lie in X_i and we get case 4ca). It is also possible that only one edge (u, v) of the triangle appears in X_i. This last case is captured in 4cb). Here the edge (u, v) must be recognized. The only edge of color c incident with u and v is in one of the two subtree, while in the other subtree we must find the remaining vertex of the triangle.

Finally, u can be a center of a monochromatic star of color c. Then the star may be completely placed in one of the subtrees and case 4a) applies. Alternatively, the star may appear as a union of a star with another star or with a new edge—case 4da). The star can also be formed out of two edges, each coming from different subtrees below X_j and $X_{j'}$ and we get the case 4db).

The dynamic programming algorithm evaluates the tables $\texttt{Tab}_i$ in a bottom-up manner. An edge k-subcoloring of the entire graph G exists if and only if the table $\texttt{Tab}_r$ for the root node X_r is nonempty. For each node the table may contain at most $k^{O(t^2)} \cdot 5^{(t+1)k}$ entries, each of length $O(t^2 \log k + kt)$. Both these values are bounded by a constant, since the treewidth t and the number of colors k are also bounded by a constant.

The evaluation of each table can be performed in time depending on k and t only. Hence, the entire complexity of the dynamic programming algorithm depends only on the number of nodes of the nice tree decomposition T. As it is mentioned in [7] a nice decomposition of width at most t containing at most $O(|V(G)|)$ nodes exists for any graph of treewidth at most t, and can be found in linear time [8]. $\square$

If we restrict the rankings r only to values $\{0, 1, 2, 4\}$, then the dynamic programming will check for the existence of an edge k-subcoloring without monochromatic triangles, i.e., for a star partition with at most k subsets. We can conclude that

COROLLARY 5. *For any fixed parameters k and t the test whether $\chi^*(G) \leqslant k$ can be performed in time linear in $|V(G)|$ for any graph of treewidth at most t.*

4. Conclusion

The concept of edge subcoloring of graphs is introduced for the first time in this paper, motivated by the study of vertex subcolorings.

Among many interesting open questions we pose the following.

1. What is the exact value for $\chi'_s(K_n)$?
   (Known for $n \leqslant 10$; see Section 2.)
2. What is the computational complexity of EDGE 3-SUBCOLORABILITY for planar graphs?
   (Note that for all planar graphs G we have $\chi'_s(G) \leqslant \chi^*(G) \leqslant 5$; cf. [17]. Recently Gonçalves and Ochem [16] showed that the EDGE 2-SUBCOLORABILITY is NP-complete for planar graphs by a reduction from 2-SUBCOLORABILITY problem of planar graphs. Moreover, the complexity of finding the ordinary chromatic index is not yet determined for planar graphs. It is widely expected to be a nontrivial problem since already the fact that any bridgeless cubic planar graph has chromatic index 3 is equivalent to the four color theorem, see, e.g., [13].)
3. What is the computational complexity of EDGE k-SUBCOLORABILITY for fixed $k \geqslant 4$?
   (We have proved that EDGE 2-SUBCOLORABILITY is NP-complete for triangle-free graphs with maximum degree 3; see Theorem 2 and that EDGE 3-SUBCOLORABILITY is NP-complete; see Theorem 3)

4. What is the computational complexity of EDGE k-SUBCOLORABILITY for graphs of bounded cliquewidth? Note that the straightforward expression of the property in MSOL uses quantification over edge subsets.

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