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SOME FIXED POINTS THEOREMS FOR MULTIVALUED MAPS

IRENA DOMNIK

ABSTRACT. The main results presented in this paper concern multivalued maps. We consider a set of fixed points of multivalued maps. We will prove that this set is closed and the upper strong convergence preserves the fixed point property. There is given a condition under which the net of sets of fixed points of multivalued maps F_j converges to a set of fixed points of F.

Let $F\colon X \to Y$ be a multivalued map. The symbol GrF is used to denote the graph of F, i.e.,

$$GrF = \{(x,y) \in X \times Y : y \in F(x)\}.$$

We say that F has a closed graph if the set GrF is closed in the space $X \times Y$. Let $F\colon X \to X$. A point $x_o \in X$ is called a fixed point of F if $x_o \in F(x_o)$. The symbol $Fix(F)$ will be used to denote the set of all fixed points of the multivalued map F.

THEOREM 1. *Let $(X, \mathscr{T})$ be a topological space. If $F\colon X \to X$ is a multivalued map with the closed graph, then the set $Fix(F)$ of all fixed points of F is closed.*

Proof. Assume that a multivalued map $F\colon X \to X$ has a closed graph. Let $x_o \in \overline{Fix(F)}$. Hence there exists a net $\{x_j : j \in J\}$ of elements of $Fix(F)$ with the property $x_o \in \lim_{j\in J} x_j$. Then $x_j \in F(x_j)$ and $(x_j, x_j) \in GrF$ for every $j \in J$. From the assumption the graph of F is closed, so the implication

$$(x_j, x_j) \longrightarrow (x_o, x_o) \Longrightarrow (x_o, x_o) \in GrF$$

holds. It means that $x_o \in F(x_o)$ and $x_o \in Fix(F)$. In the consequence, the set of fixed points of the map F is closed. □

The concept of fixed points of functions and multifunctions was introduced by C. Godet-Thobie in [3] and I. Kupka in [5]. In the next proofs we will use the following theorem:

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Theorem 2. [3] *Let $(X,\mathscr{T})$ be a topological space and let $F\colon X\to X$ be a multivalued map with the closed graph. A map F has a fixed point if and only if for every open cover $\mathscr{A}$ of X there exists a set $W\in\mathscr{A}$ such that $W\cap F(W)\neq\emptyset$.*

Definition 1. A uniform space $(X,\mathscr{U})$ is said to have the Lebesgue property if for each open cover $\mathscr{A}$ of X there is $V\in\mathscr{U}$ such that $\{V[y]\colon y\in X\}$ is a refinement of $\mathscr{A}$.

Definition 2. [2] A topological space X is called paracompact if X is Hausdorff space and if for each open cover $\mathscr{A}$ of X there is an open locally finite cover $\mathscr{A}'$ such that $\mathscr{A}'$ is a refinement of $\mathscr{A}$.

Theorem 3. [4] *Let X be a uniformisable Hausdorff topological space. Then X possesses a uniform structure compatible with the topology of X for which X has the Lebesgue property, if and only if X is paracompact. Under this condition, this structure is the finest of all uniform structures compatible with the topology of X, and is unique.*

Theorem 4. *Let $(X,\mathscr{T})$ be a paracompact space, $\mathscr{U}$—the finest compatible uniformity in X and let $\mathscr{P}_{\mathscr{U}}$ be a family of pseudometrics which generates $\mathscr{U}$. Let $F\colon X\to X$ be a multivalued map with the closed graph. A map F has a fixed point if and only if*

$$\sup_{\rho\in\mathscr{P}_{\mathscr{U}}}\inf_{x\in X}\varrho\big(x,F(x)\big)=0\,.$$

P r o o f. First, assume that F has a fixed point. Let ϱ be an arbitrary pseudometric from the family $\mathscr{P}_{\mathscr{U}}$. Then there exists a point $x_o\in X$ satisfying the condition $x_o\in F(x_o)$. Consequently $\varrho\big(x_o,F(x_o)\big)=0$, so $\inf_{x\in X}\varrho\big(x,F(x)\big)=0$ for every $\varrho\in\mathscr{P}_{\mathscr{U}}$. Thus $\sup_{\varrho\in\mathscr{P}_{\mathscr{U}}}\inf_{x\in X}\varrho\big(x,F(x)\big)=0$.
Conversely, let $\sup_{\varrho\in\mathscr{P}_{\mathscr{U}}}\inf_{x\in X}\varrho\big(x,F(x)\big)=0$. Then $\inf_{x\in X}\varrho\big(x,F(x)\big)=0$ for every $\varrho\in\mathscr{P}_{\mathscr{U}}$. Let $\mathscr{A}$ be an open cover of the space X. The space X has the Lebesgue property, so we can choose such $r>0$, $\varrho\in\mathscr{P}_{\mathscr{U}}$ that the open cover $\mathscr{A}'=\big\{\mathscr{K}_\varrho^o(x,r)\colon x\in X\big\}$ is a refinement of $\mathscr{A}$. From the assumption there is a point $x_o\in X$ with the property $\varrho\big(x_o,F(x_o)\big)<r$. Thus, for some $y\in F(x_o)$ we obtain $\varrho(x_o,y)<r$, so $y\in\mathscr{K}_\varrho^o(x_o,r)$. Since $\mathscr{A}'$ is the refinement of $\mathscr{A}$ there exists $G\in\mathscr{A}$ with the condition $\mathscr{K}_\varrho^o(x_o,r)\subset G$. The fact $x_o\in G$ implies the inclusion $F(x_o)\subset F(G)$. Then $y\in F(G)\cap G\neq\emptyset$. Using Theorem 2 the map F has a fixed point. $\square$

Corollary. *Let (X, ϱ) be a metric space with the Lebesgue property. Suppose that $F\colon X \to X$ is a multivalued map with the closed graph. The map F has a fixed point if and only if $\inf_{x \in X} \varrho\big(x, F(x)\big) = 0$.*

If we assume that (X, ϱ) is a compact metric space, then we have:

Theorem 5. [3] *Let (X, ϱ) be a compact metric space and let $F\colon X \to X$ be a multivalued map with the closed graph. The map F has a fixed point if and only if $\inf_{x \in X} \varrho\big(x, F(x)\big) = 0$.*

It is known that a uniform convergence doesn't preserve fixed points. Now we will consider a strong convergence which is formulated in terms of open covers. We will prove that this kind of convergence preserves the fixed point property.

A set $St(A, \mathscr{A}) = \bigcup\{B \in \mathscr{A} \colon A \cap B \neq \emptyset\}$ is called a star of a set A with respect to a cover $\mathscr{A}$ of a space Y.

Definition 3. [1] Let X be an arbitrary set, Y—a topological space. A net $\{F_j \colon j \in J\}$ of multivalued maps $F_j\colon X \to Y$ is said to be upper (respectively, lower) strongly convergent to a multivalued map $F\colon X \to Y$ if for each open cover $\mathscr{A}$ of the space Y there is $j_o \in J$ such that the inclusion $F_j(x) \subset St\big(F(x), \mathscr{A}\big)$ (respectively, $F(x) \subset St\big(F_j(x), \mathscr{A}\big)$) holds for each $j \in J$, $j \geqslant j_o$ and $x \in X$.

Theorem 6. *Let $(X, \mathscr{T})$ be a T_1-space. Assume that a net $\{F_j \colon j \in J\}$ of multivalued maps $F_j\colon X \to X$ is upper strongly convergent to a multivalued map $F\colon X \to X$ with the closed graph. If F_j has fixed points for $j \in J$, then F has a fixed point.*

P r o o f . Suppose that $\mathscr{A}$ is an open cover of the space X. From the upper strong convergence of the net $\{F_j \colon j \in J\}$ there exists $j_o \in J$ such that $F_j(x) \subset St\big(F(x), \mathscr{A}\big)$ for each $j \geqslant j_o$ and $x \in X$. For the multivalued map F_j, where $j \geqslant j_o$ there is $x_j \in X$ with the property $x_j \in F_j(x_j)$. In particular, $F_j(x_j) \subset St\big(F(x_j), \mathscr{A}\big)$. We can choose such $W \in \mathscr{A}$ that $x_j \in W$ and $F(x_j) \cap W \neq \emptyset$. Let $y \in F(x_j) \cap W$. Hence we obtain $y \in F(x_j) \subset F(W)$. Consequently, $F(W) \cap W \neq \emptyset$ for some set $W \in \mathscr{A}$. Moreover, F has closed graph, so by virtue of Theorem 2, F has a fixed point. $\square$

Theorem 7. *Let $(X, \mathscr{T})$ be a T_1-space. Suppose that a net $\{F_j \colon j \in J\}$ of multivalued maps $F_j\colon X \to X$ is upper strongly convergent to a multivalued map $F\colon X \to X$. If F has the closed graph, then for every open set G such that $Fix(F) \subset G$ there is $j_o \in J$ with the property $Fix(F_j) \subset G$ for each $j \geqslant j_o$.*

P r o o f . Assume that there exists an open set $G \subset X$ such that $Fix(F) \subset G$ and for each $j_o \in J$ there is $j \geqslant j_o$ with

$$Fix(F_j) \cap (X \setminus G) \neq \emptyset. \tag{1}$$

Let us denote $U = (X \times X) \setminus \big(GrF \cap \big((X \setminus G) \times X\big)\big)$. The set U is open in $X \times X$ and contains the diagonal. Hence for $x \in X$ there is an open set $O_x \subset X$ such that $(x,x) \in O_x \times O_x \subset U$. The family $\mathscr{A} = \{O_x \colon x \in X\}$ forms an open cover of X. The upper strong convergence of the net $\{F_j \colon j \in J\}$ to F implies the existence of $j_o \in J$ such that

$$F_j(x) \subset St\big(F(x), \mathscr{A}\big) \text{ for every } j \geqslant j_o \text{ and } x \in X. \tag{2}$$

Using (1) we can choose $j \in J$, $j \geqslant j_o$ with the property $Fix(F_j) \cap (X \setminus G) \neq \emptyset$. Let $x_j \in Fix(F_j) \cap (X \setminus G)$. This means that x_j is a fixed point of F_j and from (2)

$$F_j(x_j) \subset St\big(F(x_j), \mathscr{A}\big).$$

Thus we can choose a set $O \in \mathscr{A}$ such that $x_j \in O$ and $O \cap F(x_j) \neq \emptyset$. For $y \in O \cap F(x_j)$ we have shown conditions $(x_j, y) \in O \times O \subset U$, $(x_j, y) \in GrF$ and $(x_j, y) \in (X \setminus G) \times X$ which imply

$$U \cap GrF \cap \big((X \setminus G) \times X\big) \neq \emptyset.$$

This contradiction completes the proof. □

Remark. The conclusion of the Theorem 7 can be formulated as follows: the net $\big\{Fix(F_j) \colon j \in J\big\}$ converges to $Fix(F)$ pointwise with respect to the upper Vietoris topology.

Definition 4. [2] A topological space $(X, \mathscr{T})$ is called a perfect space if every closed set is G_δ.

Let $J = \{\alpha \colon \alpha < \Omega\}$, where Ω is the first uncountable ordinal number.

Theorem 8. *Let $(X, \mathscr{T})$ be a perfect T_1-space and let a transfinite sequence $\{F_\alpha \colon \alpha < \Omega\}$ of multivalued maps $F_\alpha \colon X \to X$ be upper strongly convergent to a multivalued map $F \colon X \to X$ with the closed graph. Then there exists $\alpha_o < \Omega$ such that*

$$\bigcup_{\alpha \geqslant \alpha_o, \alpha < \Omega} Fix(F_\alpha) \subset Fix(F).$$

Proof. Because F has the closed graph, so by Theorem 1 the set $Fix(F)$ of all fixed points of the map F is closed. The space X is perfect, thus $Fix(F)$ is G_δ, i.e., $Fix(F) = \bigcap_{n \in \mathbb{N}} G_n$, where G_n is an open set for every $n \in \mathbb{N}$. The transfinite sequence $\{F_\alpha \colon \alpha < \Omega\}$ upper strongly converges to F, so according to Theorem 7 for each $n \in \mathbb{N}$ there is $\alpha_n < \Omega$ such that $Fix(F_\alpha) \subset G_n$ for every $\alpha \geqslant \alpha_n$, $\alpha < \Omega$. Let $\alpha_0 = \sup\{\alpha_n \colon n \in \mathbb{N}\}$. Then $\alpha_o < \Omega$ and $Fix(F_\alpha) \subset G_n$ for each $\alpha \geqslant \alpha_o$ and $n \in \mathbb{N}$. Consequently, $Fix(F_\alpha) \subset \bigcap_{n \in \mathbb{N}} G_n = Fix(F)$ for all $\alpha \geqslant \alpha_o$ and the proof is finished. □

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Department of Mathematics
Academy Pomeraniensis
Arciszewskiego 22 b
PL–76-200 Słupsk
POLAND
E-mail: domnik@pap.edu.pl