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AN ALGORITHM FOR THE CONSTRUCTION OF THE FUNDAMENTAL SYSTEM OF SOLUTIONS FOR THE LINEAR DISCRETE SYSTEMS WITH CONSTANT COEFFICIENTS

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ABSTRACT. The construction of the fundamental system of solutions of the linear discrete systems with constant coefficients is presented. Firstly, the Weyr's theory of matrices is recalled. Secondly, the method based on Weyr's theory is described and then the results are compared with another method using Weyr's principle.

1. Introduction

This article deals with the construction of the fundamental system of solutions of the linear discrete systems with constant coefficients. The basic principle goes back to Eduard Weyr's original theory of matrices [1]. A method of solving homogeneous systems of linear difference equations with constant coefficients will be presented and it will be shown that the general solution of the mentioned system can be expressed with the aid of the system of vectors relative to the matrix of coefficients of the system.

The idea of the method presented here is due to Otakar Borůvka [2] who has derived the general solution of homogeneous systems of linear differential equations with constant coefficients from a theorem of Weyr's theory. According to his work, the way of the solving can be used also in the case of difference equations. The results are compared with another method using Weyr's principle.

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2. The principle of Weyr's theory

Let $\mathbf{A}$ be a square matrix of a size m with elements from the set of complex numbers, λ one of its characteristic roots and h the rank of the matrix $\mathbf{A}$. Then the number

$$\nu = m - h$$

is called the nullity of $\mathbf{A}$.

Assume that λ is of multiplicity k. Then the created sequence of the matrices

$$(\mathbf{A} - \lambda\,\mathbf{I})^0 = \mathbf{I},\ \ (\mathbf{A} - \lambda\,\mathbf{I})^1,\ \ (\mathbf{A} - \lambda\,\mathbf{I})^2,\ \ \ldots,\ \ (\mathbf{A} - \lambda\,\mathbf{I})^p$$

has a corresponding sequence of the ranks

$$h_0 > h_1 > h_2 > \cdots > h_p$$

and the corresponding sequence of nullities

$$0 = \nu_0 < \nu_1 < \nu_2 < \cdots < \nu_p = k,$$

where p is satisfying the condition

$$h_p = m - k.$$

Let

$$\sigma_1 = \nu_1 - \nu_0,\ \ \sigma_2 = \nu_2 - \nu_1,\ \ \ldots,\ \ \sigma_p = \nu_p - \nu_{p-1}$$

be characteristic numbers of $\mathbf{A}$ relative to the characteristic root λ (satisfying the inequalities $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_p$). Further it can be shown that there exists a system of nonzero vectors

$$\begin{array}{ll}
\mathbf{v}_{s1}, \ldots, \mathbf{v}_{s\sigma_p}, & s = 1, 2, \ldots, p, \\
\mathbf{v}_{s\sigma_p+1}, \ldots, \mathbf{v}_{s\sigma_{p-1}}, & s = 1, 2, \ldots, p-1, \\
\mathbf{v}_{s\sigma_{p-1}+1}, \ldots, \mathbf{v}_{s\sigma_{p-2}}, & s = 1, 2, \ldots, p-2, \\
\vdots & \vdots \\
\mathbf{v}_{s\sigma_2+1}, \ldots, \mathbf{v}_{s\sigma_1}, & s = 1
\end{array}$$

relative to the characteristic root λ consisting of $\sigma_1 + \sigma_2 + \ldots + \sigma_p = k$ linearly independent vectors. The system of vectors is determined by the characteristic numbers of $\mathbf{A}$ and a table of vectors with p rows and σ_1 columns can be constructed (see Table 1).

The number of vectors in each row is designated by the certain characteristic number (σ_s, $s = 1, 2, \ldots, p$) of $\mathbf{A}$. The computation of the system of vectors can be done according to the following formulas.

Firstly, the last (nonzero) vector in each column must be determined by the formula

$$(\mathbf{A} - \lambda\,\mathbf{I})^j\,\mathbf{v}_{js} = \mathbf{0}, \tag{1}$$

TABLE 1. The system of vectors relative to the characteristic root λ.

$\mathbf{v}_{11}$	$\mathbf{v}_{12}$	$\dots$	$\mathbf{v}_{1\sigma_p}$	$\mathbf{v}_{1,\sigma_p+1}$	$\dots$	$\mathbf{v}_{1,\sigma_{p-1}}$	$\dots$	$\mathbf{v}_{1\sigma_2}$	$\dots$	$\mathbf{v}_{1\sigma_1}$
$\mathbf{v}_{21}$	$\mathbf{v}_{22}$	$\dots$	$\mathbf{v}_{2\sigma_p}$	$\mathbf{v}_{2,\sigma_p+1}$	$\dots$	$\mathbf{v}_{2,\sigma_{p-1}}$	$\dots$	$\mathbf{v}_{2\sigma_2}$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			
$\mathbf{v}_{p-1,1}$	$\mathbf{v}_{p-1,2}$	$\dots$	$\mathbf{v}_{p-1,\sigma_p}$	$\mathbf{v}_{p-1,\sigma_p+1}$	$\dots$	$\mathbf{v}_{p-1,\sigma_{p-1}}$				
$\mathbf{v}_{p1}$	$\mathbf{v}_{p2}$	$\dots$	$\mathbf{v}_{p\sigma_p}$							

where $j = 1, 2, \dots, p$ and $s = \sigma_{j+1} + 1, \sigma_{j+1} + 2, \dots, \sigma_j$. If $j = p$ then we define $\sigma_{p+1} = 0$. The equation (1) is not defined in the case $\sigma_{j+1} + 1 > \sigma_j$.

Secondly, the previous vectors in each column can be computed. The formula

$$\mathbf{v}_{l-1,s} = (\mathbf{A} - \lambda\,\mathbf{I})\,\mathbf{v}_{ls}\,, \tag{2}$$

where $l = j, j-1, \dots, 2$, is valid for each couple of indices (j, s) and $j \in \{2, 3, \dots, p\}$, $s \in \{\sigma_{j+1} + 1, \sigma_{j+1} + 2, \dots, \sigma_j\}$. If $j = p$ then $\sigma_{p+1} = 0$.

Consider now the system of linear homogeneous differential equations in the matrix form

$$\mathbf{y}'(x) = \mathbf{A}\,\mathbf{y}(x), \tag{3}$$

where $\mathbf{A}$ is the $m \times m$ matrix of coefficients and $\mathbf{y}(x)$ is the vector of solutions which make up an m-dimensional vector space. The base of this space is the fundamental system of solutions which is a set of m solutions $\mathbf{y}_1(x)$, $\mathbf{y}_2(x), \dots, \mathbf{y}_m(x)$. The general solution is then a linear combination of the solutions of the fundamental system. If λ is an arbitrary characteristic root of $\mathbf{A}$ of multiplicity k and if Table 1 contains the system of vectors relative to it, then to the root λ there correspond k solutions of (3) of the form

$$\begin{aligned}
\mathbf{y}_{1s}(x) &= \mathbf{v}_{1s}\,\mathrm{e}^{\lambda x}, & (4)\\
\mathbf{y}_{2s}(x) &= (\mathbf{v}_{1s}x + \mathbf{v}_{2s})\,\mathrm{e}^{\lambda x}, & (5)\\
&\vdots \\
\mathbf{y}_{js}(x) &= \left(\mathbf{v}_{1s}\frac{x^{j-1}}{(j-1)!} + \mathbf{v}_{2s}\frac{x^{j-2}}{(j-2)!} + \dots + \mathbf{v}_{j-1,s}x + \mathbf{v}_{js}\right)\mathrm{e}^{\lambda x}, & (6)
\end{aligned}$$

for each couple of indices (j, s), where $j \in \{1, 2, \dots, p\}$ and $s \in \{\sigma_{j+1} + 1, \sigma_{j+1} + 2, \dots, \sigma_j\}$.

3. The discrete system

Now, the analogical case of linear discrete system will be discussed. Let

$$\mathbf{y}(n+1) = \mathbf{A}\,\mathbf{y}(n) \tag{7}$$

be the system of linear homogeneous difference equations with constant coefficients, where $\mathbf{A}$ is a constant $m \times m$ matrix of coefficients and $\mathbf{y}(n)$ is the vector of solutions.

In this case, the vectors from the Table 1 are computed according to the formulas different from (1), (2). Firstly, the first nonzero vector in each column must be determined as

$$(\mathbf{A} - \lambda\,\mathbf{I})\,\mathbf{v}_{js} = \mathbf{0}, \tag{8}$$

where $j = 1$ and $s = 1, 2, \ldots, \sigma_j$.

Secondly, the next vectors in each column can be computed. The formula

$$(\mathbf{A} - \lambda\,\mathbf{I})\,\mathbf{v}_{js} = \lambda\left(\frac{\mathbf{v}_{1s}}{(j-1)!} + \frac{\mathbf{v}_{2s}}{(j-2)!} + \cdots + \frac{\mathbf{v}_{j-2,s}}{2!} + \mathbf{v}_{j-1,s}\right), \tag{9}$$

where $j \in \{2, 3, \ldots, p\}$ and $s \in \{1, 2, \ldots, \sigma_j\}$, holds.

Theorem 1. *The system of all linearly independent solutions corresponding to the arbitrary characteristic root λ of $\mathbf{A}$ with the multiplicity k is created from k solutions of (7) in the form*

$$\mathbf{y}_{js}(n) = \left(\mathbf{v}_{1s}\frac{n^{j-1}}{(j-1)!} + \mathbf{v}_{2s}\frac{n^{j-2}}{(j-2)!} + \cdots + \mathbf{v}_{j-1,s}n + \mathbf{v}_{js}\right)\lambda^n, \tag{10}$$

for each couple of indices (j, s), where $j \in \{1, 2, \ldots, p\}$ and $s \in \{1, 2, \ldots, \sigma_j\}$.

Proof. Relations (8), (9) will be used in this proof. Only the vectors from the first column of Table 1 will be proved. Therefore, assume index $s = 1$. The vectors from other columns of Table 1 can be obtained analogically. If $j = \{1, 2, \ldots, p-1, p\}$ and λ is one of the characteristic roots of $\mathbf{A}$, the corresponding relations of the system of vectors from (9) are

$$\begin{aligned}
(\mathbf{A}-\lambda\mathbf{I})\,\mathbf{v}_{11} &= \mathbf{0}, && (11)\\
(\mathbf{A}-\lambda\mathbf{I})\,\mathbf{v}_{21} &= \lambda\,\mathbf{v}_{11}, && (12)\\
&\vdots\\
(\mathbf{A}-\lambda\mathbf{I})\,\mathbf{v}_{p-1,1} &= \lambda\Bigg(\frac{\mathbf{v}_{11}}{(p-2)!}+\frac{\mathbf{v}_{21}}{(p-3)!}+\cdots\\
&\qquad\cdots+\frac{\mathbf{v}_{p-3,1}}{2!}+\mathbf{v}_{p-2,1}\Bigg), && (13)\\
(\mathbf{A}-\lambda\mathbf{I})\,\mathbf{v}_{p1} &= \lambda\Bigg(\frac{\mathbf{v}_{11}}{(p-1)!}+\frac{\mathbf{v}_{21}}{(p-2)!}+\cdots\\
&\qquad\cdots+\frac{\mathbf{v}_{p-2,1}}{2!}+\mathbf{v}_{p-1,1}\Bigg). && (14)
\end{aligned}$$

The system of equations (11)–(14) can be expressed in the equivalent form

$$\begin{aligned}
\mathbf{A}\,\mathbf{v}_{11} &= \lambda\,\mathbf{v}_{11}, && (15)\\
\mathbf{A}\,\mathbf{v}_{21} &= \lambda\,(\mathbf{v}_{11}+\mathbf{v}_{21}), && (16)\\
&\vdots\\
\mathbf{A}\,\mathbf{v}_{p-1,1} &= \lambda\Bigg(\frac{\mathbf{v}_{11}}{(p-2)!}+\frac{\mathbf{v}_{21}}{(p-3)!}+\cdots\\
&\qquad\cdots+\frac{\mathbf{v}_{p-3,1}}{2!}+\mathbf{v}_{p-2,1}+\mathbf{v}_{p-1,1}\Bigg), && (17)\\
\mathbf{A}\,\mathbf{v}_{p1} &= \lambda\Bigg(\frac{\mathbf{v}_{11}}{(p-1)!}+\frac{\mathbf{v}_{21}}{(p-2)!}+\cdots\\
&\qquad\cdots+\frac{\mathbf{v}_{p-2,1}}{2!}+\mathbf{v}_{p-1,1}+\mathbf{v}_{p1}\Bigg). && (18)
\end{aligned}$$

Let $j=1$. According to the formula (10),

$$\mathbf{y}_{11}(n)=\mathbf{v}_{11}\lambda^n. \tag{19}$$

Substitution into (7) and modification according to (15) lead to

$$\mathbf{y}_{11}(n+1)=\mathbf{v}_{11}\lambda^{n+1}=\lambda\,\mathbf{v}_{11}\lambda^n=\mathbf{A}\,\mathbf{v}_{11}\lambda^n=\mathbf{A}\,\mathbf{y}_{11}(n).$$

Then the vector (19) is one of the solutions of (7).
Let $j=2$. According to the formula (10),

$$\mathbf{y}_{21}(n)=(\mathbf{v}_{11}n+\mathbf{v}_{21})\,\lambda^n. \tag{20}$$

Analogically, the vector (20) is substituted into (7) and the modification is made according to (15), (16):

$$\begin{aligned}
\mathbf{y}_{21}(n+1) &= \big(\mathbf{v}_{11}(n+1) + \mathbf{v}_{21}\big)\lambda^{n+1} \\
&= (\lambda\,\mathbf{v}_{11}n + \lambda\,\mathbf{v}_{11} + \lambda\,\mathbf{v}_{21})\,\lambda^{n} \\
&= (\,\mathbf{A}\,\mathbf{v}_{11}n + \mathbf{A}\,\mathbf{v}_{21})\,\lambda^{n} \\
&= \mathbf{A}\,(\,\mathbf{v}_{11}n + \mathbf{v}_{21})\,\lambda^{n} = \mathbf{A}\,\mathbf{y}_{21}(n).
\end{aligned}$$

The vector (20) is one of the solutions of (7), too.
Now, the case with $j = p$ is considered. Then the vector

$$\mathbf{y}_{p1}(n) = \left(\mathbf{v}_{11}\frac{n^{p-1}}{(p-1)!} + \mathbf{v}_{21}\frac{n^{p-2}}{(p-2)!} + \cdots + \mathbf{v}_{p-1,1}n + \mathbf{v}_{p1}\right)\lambda^{n}$$

should be one of the solutions of (7):

$$\begin{aligned}
\mathbf{y}_{p1}(n+1) &= \Bigg(\mathbf{v}_{11}\frac{(n+1)^{p-1}}{(p-1)!} + \mathbf{v}_{21}\frac{(n+1)^{p-2}}{(p-2)!} + \cdots \\
&\quad \cdots + \mathbf{v}_{p-1,1}(n+1) + \mathbf{v}_{p1}\Bigg)\lambda^{n+1} \\
&= \Bigg(\lambda\,\mathbf{v}_{11}\frac{(n+1)^{p-1}}{(p-1)!} + \lambda\,\mathbf{v}_{21}\frac{(n+1)^{p-2}}{(p-2)!} + \cdots \\
&\quad \cdots + \lambda\,\mathbf{v}_{p-1,1}(n+1) + \lambda\,\mathbf{v}_{p1}\Bigg)\lambda^{n} \\
&= \Bigg[\frac{\lambda\,\mathbf{v}_{11}}{(p-1)!}\Bigg[n^{p-1} + \binom{p-1}{1}n^{p-1-1} + \binom{p-1}{2}n^{p-1-2} + \cdots \\
&\quad \cdots + \binom{p-1}{p-2}n + \binom{p-1}{p-1}\Bigg] \\
&\quad\quad + \frac{\lambda\,\mathbf{v}_{21}}{(p-2)!}\Bigg[n^{p-2} + \binom{p-2}{1}n^{p-2-1} + \binom{p-2}{2}n^{p-2-2} + \cdots \\
&\quad \cdots + \binom{p-2}{p-3}n + \binom{p-2}{p-2}\Bigg] + \cdots \\
&\quad \cdots + \lambda\,\mathbf{v}_{p-1,1}(n+1) + \lambda\,\mathbf{v}_{p1}\Bigg]\lambda^{n} \\
&= \Bigg[\frac{\lambda\,\mathbf{v}_{11}}{(p-1)!}\Bigg(n^{p-1} + (p-1)n^{p-2} + \frac{(p-1)(p-2)}{2}n^{p-3} + \cdots
\end{aligned}$$

$$
\begin{aligned}
&\cdots + (p-1)n + 1\Big) \\
&+\frac{\lambda\,\mathbf{v}_{21}}{(p-2)!}\Big(n^{p-2} + (p-2)n^{p-3} + \frac{(p-2)(p-3)}{2}n^{p-4} + \cdots \\
&\cdots + (p-2)n + 1\Big) + \cdots \\
&\cdots + \lambda\,\mathbf{v}_{p-1,1}(n+1) + \lambda\,\mathbf{v}_{p1}\Big]\lambda^n \\
= \quad &\Big[n^{p-1}\frac{\lambda\,\mathbf{v}_{11}}{(p-1)!} + n^{p-2}\Big(\frac{\lambda\,\mathbf{v}_{11}(p-1)}{(p-1)!} + \frac{\lambda\,\mathbf{v}_{21}}{(p-2)!}\Big) + \cdots \\
&\cdots + n\Big(\frac{\lambda\,\mathbf{v}_{11}(p-1)}{(p-1)!} + \frac{\lambda\,\mathbf{v}_{21}(p-2)}{(p-2)!} + \cdots + \lambda\,\mathbf{v}_{p-1,1}\Big) \\
&+\Big(\frac{\lambda\,\mathbf{v}_{11}}{(p-1)!} + \frac{\lambda\,\mathbf{v}_{21}}{(p-2)!} + \cdots + \lambda\,\mathbf{v}_{p-1,1} + \lambda\,\mathbf{v}_{p1}\Big)\Big]\lambda^n \\
= \quad &\Big[\lambda\,\mathbf{v}_{11}\frac{n^{p-1}}{(p-1)!} + (\lambda\,\mathbf{v}_{11} + \lambda\,\mathbf{v}_{21})\frac{n^{p-2}}{(p-2)!} + \cdots \\
&\cdots + \Big(\frac{\lambda\,\mathbf{v}_{11}}{(p-2)!} + \frac{\lambda\,\mathbf{v}_{21}}{(p-3)!} + \cdots + \lambda\,\mathbf{v}_{p-1,1}\Big)n \\
&+\Big(\frac{\lambda\,\mathbf{v}_{11}}{(p-1)!} + \frac{\lambda\,\mathbf{v}_{21}}{(p-2)!} + \cdots + \lambda\,\mathbf{v}_{p-1,1} + \lambda\,\mathbf{v}_{p1}\Big)\Big]\lambda^n.
\end{aligned}
$$

The modification with the aid of equations (15)–(18) shows that

$$
\begin{aligned}
\mathbf{y}_{p1}(n+1) \quad &= \quad \Big(\mathbf{A}\,\mathbf{v}_{11}\frac{n^{p-1}}{(p-1)!} + \mathbf{A}\,\mathbf{v}_{21}\frac{n^{p-2}}{(p-2)!} + \cdots \\
&\qquad \cdots + \mathbf{A}\,\mathbf{v}_{p-1,1}n + \mathbf{A}\,\mathbf{v}_{p1}\Big)\lambda^n \\
&= \quad \mathbf{A}\Big(\mathbf{v}_{11}\frac{n^{p-1}}{(p-1)!} + \mathbf{v}_{21}\frac{n^{p-2}}{(p-2)!} + \cdots \\
&\qquad \cdots + \mathbf{v}_{p-1,1}n + \mathbf{v}_{p1}\Big)\lambda^n = \mathbf{A}\,\mathbf{y}_{p1}(n).
\end{aligned}
$$

The Theorem 1 is proved. There is a corresponding set of solutions to every characteristic root of $\mathbf{A}$, their number is always equal to the multiplicity of the respective root.

From the general theory of linear systems (7) obviously the next theorem follows. The proof is omitted. □

Theorem 2. *Let all distinct characteristic roots of* $\mathbf{A}$ *be* $\lambda_1, \lambda_2, \ldots, \lambda_r$ *of multiplicities* $k_1, k_2, \ldots, k_r$*. All the solutions of the form* (10) *corresponding to these characteristic roots make up the fundamental system of solutions of* (7)*.*

4. Conclusion

The method which was used to solve the linear homogeneous difference equations with constant coefficients gives the solution of (7) with the aid of the formulas (8), (9). These equations are different from the equations (1), (2) in Weyr's theory.

The solution of the discrete system with constant coefficients with the aid of Weyr's theory was already investigated in [3] where the fundamental system of the system (7) was found in the form

$$\begin{aligned} \mathbf{y}_{js} &= \Bigg(\mathbf{v}_{js} + \mathbf{v}_{j+1,s}\frac{n}{1!} + \mathbf{v}_{j+2,s}\frac{n(n-1)}{2!} + \cdots \\ &\qquad \cdots + \mathbf{v}_{p,s}\frac{n(n-1)\ldots(n-(p-j-1))}{(p-j)!}\Bigg)\lambda^n \end{aligned} \tag{21}$$

for $j \in \{1, 2, \ldots, p-1\}$, $s \in \{1, 2, \ldots, \sigma_{p-j+1}\}$ and

$$\mathbf{y}_{js} = \mathbf{v}_{js}\lambda^n \tag{22}$$

for $j = p$, $s \in \{1, 2, \ldots, \sigma_1\}$. The formulas used to compute the system of vectors in (21), (22) according to [3] are:

$$\frac{1}{\lambda}(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{v}_{js} = \mathbf{0}$$

for $j = p$ and

$$\frac{1}{\lambda}(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{v}_{js} = \mathbf{v}_{j+1,s}$$

for $j = 1, 2, \ldots, p-1$. In both cases $s = 1, 2, \ldots, \sigma_{p-j+1}$. Analyzing formulas (21), (22), it is obvious they are more complicated than formula (10). An advantage of the presented method are simple formulas (10) which directly copy the formulas (4)–(6) known from Weyr's theory.

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