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COMPULSORY ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF SYSTEM OF THREE DISCRETE EQUATIONS

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ABSTRACT. This contribution is devoted to a discussion of the asymptotic behavior of solutions of systems of three difference equations. We connect two techniques – the so called retract type technique and Liapunov type approach. We show that under appropriate conditions containing conditions typical for the retract technique approach, and conditions typical for the Liapunov type approach, there exists at least one solution of the considered system the graph of which stays in a prescribed domain.

1. Introduction

In this contribution we investigate the asymptotic behavior for $k \to \infty$ of the solutions of the system of three difference equations

$$\begin{aligned}\Delta u_1(k) &= f_1\big(k, u_1(k), u_2(k), u_3(k)\big),\\ \Delta u_2(k) &= f_2\big(k, u_1(k), u_2(k), u_3(k)\big),\\ \Delta u_3(k) &= f_3\big(k, u_1(k), u_2(k), u_3(k)\big)\end{aligned} \tag{1}$$

with $f_i\colon N(a) \times \mathbb{R}^3 \to \mathbb{R}$, $\Delta u_i(k) = u_i(k+1) - u_i(k)$, $i = 1,2,3$, where $k \in N(a) = \{a, a+1, \dots\}$, $a \in \mathbb{N}$ is fixed and $\mathbb{N} = \{0, 1, \dots\}$. We recall some known facts. The *solution* of system (1) is defined as an infinite sequence of number vectors

$$\big\{u(a), u(a+1), u(a+2), \dots, u(a+n), \dots\big\}$$

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with $u = (u_1, u_2, u_3)$ such that for any $k \in N(a)$, equalities (1) hold. The *existence and uniqueness* of the solution of system (1) with a prescribed initial condition

$$\big(u_1(a), u_2(a), u_3(a)\big) = (u_1^a, u_2^a, u_3^a) \in \mathbb{R}^3 \tag{2}$$

on $N(a)$ is obvious. The sequence $\big\{(k, u_1(k), u_2(k), u_3(k))\big\}$, $k \in N(a)$, is called the *graph* of the solution $(u_1, u_2, u_3) = \big(u_1(k), u_2(k), u_3(k)\big)$ for $k \in N(a)$ of initial problem (1), (2). If for every fixed $k \in N(a)$ the right hand sides $f_i(k, u_1, u_2, u_3)$, $i = 1, 2, 3$, are continuous with respect to their arguments u_1, u_2, u_3, then the solution of initial problem (1), (2) *depends continuously* on the initial data.

1.1. Description of the considered problem and auxiliary notions

Let us define a set $\Omega \subset N(a) \times \mathbb{R}^3$ as

$$\Omega := \bigcup_{k \in N(a)} \Omega(k),$$

where

$$\Omega(k) := \big\{(k, u_1, u_2, u_3) : u_i \in \mathbb{R}, b_i(k) < u_i < c_i(k),\ i = 1, 2, 3\big\}$$

with $b_i, c_i \colon N(a) \to \mathbb{R}$, $i=1,2,3$, being auxiliary functions such that $b_i(k) < c_i(k)$ for each $k \in N(a)$. Such set Ω is called a *polyfacial set.*
Our aim is to find sufficient conditions with respect to the right-hand side of system (1) which *guarantee the existence of at least one solution* $u(k) = \big(u_1^*(k), u_2^*(k), u_3^*(k)\big)$, $k \in N(a)$ satisfying

$$\big(k, u_1^*(k), u_2^*(k), u_3^*(k)\big) \in \Omega(k), \tag{3}$$

for every $k \in N(a)$.
Since the asymptotic behavior of a solution satisfying (3) is in some sense conditioned by the shape of the set Ω, we call the asymptotic behavior of such solutions *compulsory* asymptotic behavior. The asymptotic behavior of discrete equations was studied (under different assumptions), e.g., in [1]–[5].

1.2. Preliminary

In paper [2] the above described problem is solved via Liapunov type technique. Here we will combine this technique with the retract type technique which was used in paper [1]. Before we start, recall some basic notions that will be used.

1.2.1. Consequent point

A point $M_1^c(k, u_1, u_2, u_3)$ is called the *first consequent point* of a point $M = (k^*, u_1^*, u_2^*, u_3^*) \in N(a) \times \mathbb{R}^3$ (and we write $M_1^c = C[M]$) if

$$k = k^* + 1 \qquad \text{and} \qquad u_i = u_i^* + f_i(k^*, u_1^*, u_2^*, u_3^*) \qquad \text{for } i = 1, 2, 3.$$

1.2.2. Liapunov type polyfacial set

We say that a polyfacial set Ω is *Liapunov type* with respect to the discrete system (1) if

$$b_i(k+1) < u_i + f_i(k, u_1, u_2, u_3) < c_i(k+1)$$

for every $i = 1, 2, 3$ and every $(k, u) \in \Omega$.
The geometrical meaning of this property is this: If a point $M = (k, u_1, u_2, u_3)$ lies inside the set $\Omega(k)$, then its first consequent point $C[M]$ stays inside $\Omega(k+1)$. In this contribution we will deal with sets that need not be of Liapunov type, but they will have a similar property.

Definition 1.1. We say that a polyfacial set Ω is of *Liapunov type with respect to the jth variable with respect to the discrete system* (1) *if for every* $(k, u) \in \Omega$

$$b_j(k+1) < u_j + f_j(k, u_1, u_2, u_3) < c_j(k+1).$$

The geometrical meaning is that if $M = (k, u_1, u_2, u_3) \in \Omega(k)$, then the u_j-coordinate of its first consequent point stays between $b_j(k+1)$ and $c_j(k+1)$, meanwhile the other coordinates of $C[M]$ may be arbitrary.

1.2.3. Points of strict egress and their geometrical sense

An important role in the application of the retract type technique is played by the so called strict egress points.
Before we define these points, let us describe the boundary of the set Ω in detail. As one can easily see,

$$\partial\Omega = \left(\bigcup_{j=1,2,3} \Omega_B^j \right) \cup \left(\bigcup_{j=1,2,3} \Omega_C^j \right)$$

with

$$\Omega_B^j := \big\{(k, u) : k \in N(a), u_j = b_j(k), b_i(k) \le u_i \le c_i(k),\ i = 1, 2, 3,\ i \ne j\big\}$$

and

$$\Omega_C^j := \big\{(k, u) : k \in N(a), u_j = c_j(k), b_i(k) \le u_i \le c_i(k),\ i = 1, 2, 3,\ i \ne j\big\}.$$

In accordance with [1, Lemmas 1,2], a point $(k, u_1, u_2, u_3) \in \partial\Omega$ is a *point of the type of strict egress* for the polyfacial set Ω with respect to discrete system (1) if and only if for some $j \in \{1, 2, 3\}$

$$u_j = b_j(k) \quad \text{and} \quad f_j(k, u_1, u_2, u_3) < b_j(k+1) - b_j(k),$$

or

$$u_j = c_j(k) \quad \text{and} \quad f_j(k, u_1, u_2, u_3) > c_j(k+1) - c_j(k).$$

Geometrically these inequalities mean the following:
If a point $M = M(k, u_1, u_2, u_3) \in \partial\Omega$ is a point of the type of strict egress, then the first consequent point $C[M] \notin \overline{\Omega}$.

1.2.4. Retract and retraction

If $A \subset B$ are any two sets of a topological space and $\pi : B \to A$ is a continuous mapping from B onto A such that $\pi(p) = p$ for every $p \in A$, then π is said to be a *retraction* of B onto A. If there exists a retraction of B onto A, A is called a *retract* of B.

2. Main result

Let Ω be a polyfacial set. In this part we will solve our problem supposing that for one fixed index $j \in \{1, 2, 3\}$ the set Ω is of Liapunov type with respect to the jth variable and for the two remaining indices the corresponding parts of the set $\partial\Omega$ consist of points of the type of strict egress.

Theorem 2.1 (Main result). *Let $b_i(k)$, $c_i(k)$, $b_i(k) < c_i(k)$, $i = 1, 2, 3$, be real functions defined on $N(a)$ and let $f_i : N(a) \times \mathbb{R}^3 \to \mathbb{R}$, $i = 1, 2, 3$, be functions that are continuous with respect to their last three arguments. Suppose that for a fixed $j \in \{1, 2, 3\}$ the set Ω is of the Liapunov type with respect to the jth variable, i.e., for every $(k, u_1, u_2, u_3) \in \Omega$*

$$b_j(k+1) < u_j + f_j(k, u_1, u_2, u_3) < c_j(k+1). \tag{4}$$

Further suppose that all the points of the sets Ω_B^i, Ω_C^i, where $i \in \{1, 2, 3\}$, $i \neq j$, are points of strict egress, i.e., if $(k, u_1, u_2, u_3) \in \Omega_B^i$ for some $i \neq j$, then

$$f_i(k, u_1, u_2, u_3) < b_i(k+1) - b_i(k), \tag{5}$$

and if $(k, u_1, u_2, u_3) \in \Omega_C^i$ for some $i \neq j$, then

$$f_i(k, u_1, u_2, u_3) > c_i(k+1) - c_i(k). \tag{6}$$

Then there exists a solution $u = \big(u_1^*(k), u_2^*(k), u_3^*(k)\big)$ *of system* (1) *satisfying the inequalities*

$$b_i(k) < u_i^*(k) < c_i(k), \qquad i = 1, 2, 3, \tag{7}$$

for every $k \in N(a)$.

2.1. Proof of the main result

The proof will be performed by a contradiction. We will suppose that there exists no solution satisfying inequalities (7) for every $k \in N(a)$. Under this supposition we prove that there exists a continuous mapping (a retraction) of a closed rectangle onto its boundary which is, by known facts, impossible.

2.1.1. Specification of the general scheme of the proof

Without the loss of generality we may suppose that the index j in Theorem 2.1 is equal to 3, i.e., all the points of the sets Ω_B^1, Ω_C^1, Ω_B^2 and Ω_C^2 are of the type of strict egress.

Each solution of system (1) is uniquely determined by the chosen initial condition

$$\big(u_1(a), u_2(a), u_3(a)\big) = (u_1^a, u_2^a, u_3^a).$$

For the following considerations, let u_3^a with $u_3^a \in \big(b_3(a), c_3(a)\big)$ be chosen arbitrarily but fixed. Now the solution of (1) is given just by the choice of u_1^a and u_2^a, we can write

$$u = u(k, u^a) = \big(u_1(k, u^a), u_2(k, u^a), u_3(k, u^a)\big), \qquad \text{where} \quad u^a = (u_1^a, u_2^a).$$

Define the closed rectangle

$$D := \big[b_1(a), c_1(a)\big] \times \big[b_2(a), c_2(a)\big].$$

Hereafter we show that, under the supposition that there exists no solution satisfying inequalities (7), there exists a retraction R (which will be a composition of two auxiliary mappings R_1 and R_2) of the set $B := D$ onto the set $A := \partial D$. This contradiction will prove our result.

2.1.2. Auxiliary mapping R_1

Let a point $\tilde{u} = (\tilde{u}_1, \tilde{u}_2) \in D$ be fixed. The initial condition $u_1(a) = \tilde{u}_1, u_2(a) = \tilde{u}_2$ defines the corresponding solution $u = u(k, \tilde{u}) = \big(u_1(k, \tilde{u}), u_2(k, \tilde{u}), u_3(k, \tilde{u})\big)$ which accordingly to our supposition does not satisfy inequalities (7) for every $k \in N(a)$. That means that there exists an index $s \in N(a)$ such that $\big(k, u(k, \tilde{u})\big) \in \Omega(k)$ for $a \le k < s$ and $\big(s, u(s, \tilde{u})\big) \notin \Omega(s)$, i.e., s is the first index

for which the graph of the solution is out of Ω.
If $\tilde{u} \in \partial D$ (i.e., $\tilde{u}_i = b_i(a)$ or $\tilde{u}_i = c_i(a)$ for some $i \in \{1,2\}$), then $s = a$, because $\big(a, u(a,\tilde{u})\big) \notin \Omega(a)$. Further, because of the strict egress property of the corresponding part of the boundary of Ω, either $u_i(a+1,\tilde{u}) < b_i(a+1)$—if $\tilde{u}_i = b_i(a)$—or $u_i(a+1,\tilde{u}) > c_i(a+1)$—if $\tilde{u}_i = c_i(a)$.
If $\tilde{u} \in \operatorname{int} D$, then, as the set Ω is of the Liapunov type with respect to the third variable and $\big(s-1, u(s-1,\tilde{u})\big) \in \Omega(s-1)$,

$$b_3(s) \;<\; u_3(s,\tilde{u}) \;<\; c_3(s)$$

and the validity of inequalities (7) has to be violated in the u_1 or u_2-coordinate. Now, two cases are possible: either $\big(s, u(s,\tilde{u})\big) \notin \overline{\Omega(s)}$ or $\big(s, u(s,\tilde{u})\big) \in \partial\Omega(s)$. In the first case $u_i(s,\tilde{u}) < b_i(s)$ or $u_i(s,\tilde{u}) > c_i(s)$ for some $i \in \{1,2\}$. In the second case there exists an $i \in \{1,2\}$ such that $u_i(s,\tilde{u}) = b_i(s)$ or $u_i(s,\tilde{u}) = c_i(s)$ and, due to (5) and (6), $u_i(s+1,\tilde{u}) < b_i(s+1)$ or $u_i(s+1,\tilde{u}) > c_i(s+1)$ for this i. Unfortunately, for the next consideration the index s would be of little use. What we will need is the last index for which the graph of the solution stays inside $\overline{\Omega}$. We will denote this index as k^* and will call it the *leaving index*. Obviously, $k^* = s-1$ if $\big(s, u(s,\tilde{u})\big) \notin \overline{\Omega(s)}$ and $k^* = s$ if $\big(s, u(s,\tilde{u})\big) \in \partial\Omega(s)$. We could write $k^* = k^*(\tilde{u})$ but we will mostly omit the argument $\tilde{u}$, unless it is necessary.
From the above considerations it follows that

$$b_i(k^*) \le u_i(k^*,\tilde{u}) \le c_i(k^*), \qquad i = 1,2,$$

and

$$u_i(k^*+1,\tilde{u}) < b_i(k^*+1) \quad \text{or} \quad u_i(k^*+1,\tilde{u}) > c_i(k^*+1) \qquad \text{for some } i \in \{1,2\}.$$

Now we construct the auxiliary mapping $R_1\colon D \to \mathbb{R}\times\mathbb{R}^2$:
First extend the functions b_i, c_i, $i = 1,2$ onto the whole interval $[a,\infty)$:

$$\begin{aligned} b_i(t) &:= b_i\big([t]\big) + \Big(b_i\big([t]+1\big) - b_i\big([t]\big)\Big)\big(t-[t]\big), \\ c_i(t) &:= c_i\big([t]\big) + \Big(c_i\big([t]+1\big) - c_i\big([t]\big)\Big)\big(t-[t]\big), \end{aligned}$$

$[t]$ being the integer part of t. Note that b_i, c_i are now continuous functions of a real variable t.
For any $k \in N(a)$ define the set $V_{k,k+1}$, "connecting" the sets $\Omega(k)$ and $\Omega(k+1)$, as

$$V_{k,k+1} := \big\{(t,u_1,u_2) : k \le t \le k+1, b_i(t) \le u_i \le c_i(t),\ i = 1,2\big\}.$$

Obviously, $V_{k,k+1}$ is for every $k \in N(a)$ a truncated pyramid (see Figure 1) and, as such, a convex set.

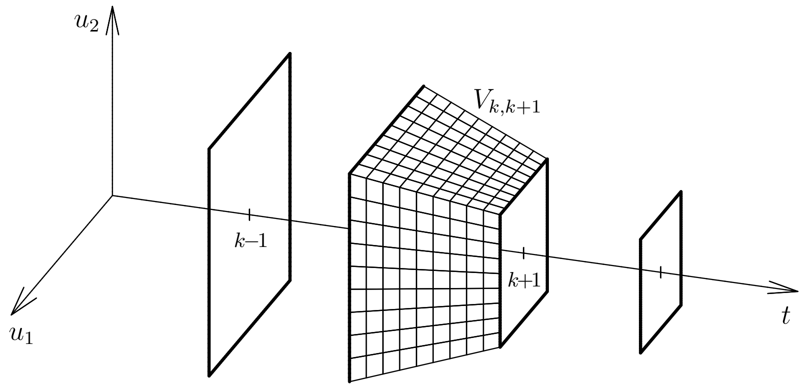

FIGURE 1. The "connecting set" $V_{k,k+1}$.

Define $R_1(\tilde{u})$ as the point of intersection of the line segment defined by its end points $\big(k^*, u_1(k^*, \tilde{u}), u_2(k^*, \tilde{u})\big)$, $\big(k^* + 1, u_1(k^* + 1, \tilde{u}), u_2(k^* + 1, \tilde{u})\big)$ with the boundary of V_{k^*,k^*+1} (see Figure 2). The mapping R_1 is well defined on D: A line segment with one end point inside of a convex set and the other end point outside of this set has to have just one intersection with the boundary of that set. If the initial point of the line segment lies on the boundary of the convex set, in general the intersection could be a line segment, but this cannot happen in our case and the only intersection has to be the point $\big(k^*, u_1(k^*, \tilde{u}), u_2(k^*, \tilde{u})\big)$ itself.

Prove that the mapping R_1 is continuous. Let $\{v_n\}_{n=1}^{\infty}$ be any sequence with $v_n \in D$ such that $v_n \to \tilde{u}$. We will show that $R_1(v_n) \to R_1(\tilde{u})$. Because of the continuity of the functions f_i, $i = 1, 2, 3$,

$$u_1(k, v_n) \to u_1(k, \tilde{u}_1), \quad u_2(k, v_n) \to u_2(k, \tilde{u}_1) \qquad \text{for every fixed } k \in N(a). \tag{8}$$

We have to consider two cases:

I) $\big(k^*, u(k^*, \tilde{u})\big) \in \Omega(k^*)$,

II) $\big(k^*, u(k^*, \tilde{u})\big) \in \partial\Omega(k^*)$.

I) In this case also

$$\big(k^*, u(k^*, v_n)\big) \in \Omega(k^*) \quad \text{and} \quad \big(k^* + 1, u(k^* + 1, v_n)\big) \notin \Omega(k^* + 1)$$

for n sufficiently large. That means that the leaving index $k^*(v_n)$ is the same as k^* given by $\tilde{u}$ and thus we deal with the same set V_{k^*,k^*+1}. As $\big(k^*, u(k^*, v_n)\big) \to$

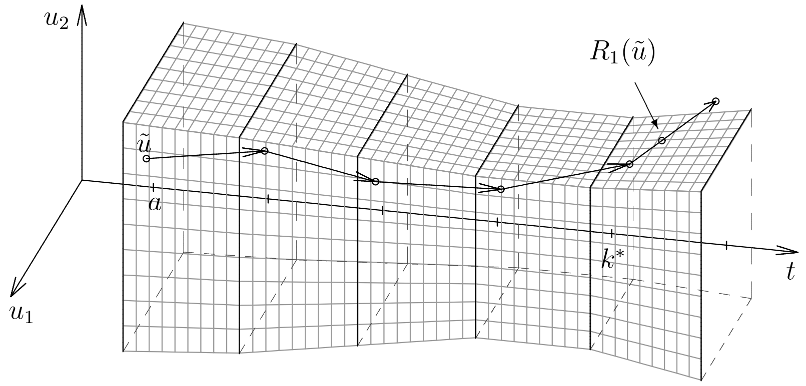

FIGURE 2. Construction of the mapping R_1.

$\big(k^*, u(k^*, \tilde{u})\big)$ and $\big(k^*+1, u(k^*+1, v_n)\big) \to \big(k^*+1, u(k^*+1, \tilde{u})\big)$, also the point of intersection of the line segment with the end points $\big(k^*, u_1(k^*, v_n), u_2(k^*, v_n)\big)$, $\big(k^*+1, u_1(k^*+1, v_n), u_2(k^*+1, v_n)\big)$ with $\partial V_{k^*,k^*+1}$ has to converge to the point of intersection of the line segment with the end points $\big(k^*, u_1(k^*, \tilde{u}), u_2(k^*, \tilde{u})\big)$, $\big(k^*+1, u_1(k^*+1, \tilde{u}), u_2(k^*+1, \tilde{u})\big)$ with $\partial V_{k^*,k^*+1}$.

II) In this case there can be $\big(k^*, u(k^*, v_n)\big) \in \Omega(k^*)$ for some members of the sequence $\{v_n\}$ and $\big(k^*, u(k^*, v_n)\big) \notin \Omega(k^*)$ for the others. In general there could be three subsequences $\{v_{k_n}\}$, $\{v_{l_n}\}$ and $\{v_{m_n}\}$ such that

$$\big(k^*, u(k^*, v_{k_n})\big) \in \Omega(k^*),\ \big(k^*, u(k^*, v_{l_n})\big) \in \partial\Omega(k^*) \text{ and } \big(k^*, u(k^*, v_{m_n})\big) \notin \overline{\Omega(k^*)}.$$

If $k^* > a$, all three subsequences come into account, if $k^* = a$, the subsequence $\{v_{m_n}\}$ can be left out from our considerations.
For the subsequence $\{v_{k_n}\}$ the situation is the same as in the case I) and the convergence is guaranteed.
For the subsequence $\{v_{l_n}\}$ there is no problem, too. If $\big(k^*, u_1(k^*, v_{l_n}), u_2(k^*, v_{l_n})\big)$ belongs to $\partial V_{k^*,k^*+1}$, then (because of (8))

$$R_1(v_{l_n}) = \big(k^*, u_1(k^*, v_{l_n}), u_2(k^*, v_{l_n})\big) \to \big(k^*, u_1(k^*, \tilde{u}), u_2(k^*, \tilde{u})\big) = R_1(\tilde{u}).$$

As for the subsequence $\{v_{m_n}\}$, the leaving index $k^*(v_{m_n})$ is different from k^* given by $\tilde{u}$ because $\big(k^*, u(k^*, v_{m_n})\big)$ is already out of $\overline{\Omega\big(k^*(\tilde{u})\big)}$. For n sufficiently large,

$$k^*(v_{m_n}) = k^* - 1$$

because $\big(k^*-1, u(k^*-1, \tilde{u})\big) \in \Omega(k^*-1)$ and thus also $\big(k^*-1, u(k^*-1, v_{m_n})\big) \in \Omega(k^*-1)$. Now, $R_1(\tilde{u}) = \big(k^*, u_1(k^*, \tilde{u}), u_2(k^*, \tilde{u})\big)$ can be also seen as the

point of intersection of the line segment given by the end points $\big(k^*-1, u_1(k^*-1,\tilde{u}), u_2(k^*-1,\tilde{u})\big)$, $\big(k^*, u_1(k^*,\tilde{u}), u_2(k^*,\tilde{u})\big)$ with the boundary of the set V_{k^*-1,k^*}. The desired convergence $R_1(v_{m_n}) \to R_1(\tilde{u})$ can be proved with help of similar considerations as in case I).
We have shown that $R_1(v_{k_n}) \to R_1(\tilde{u})$, $R_1(v_{l_n}) \to R_1(\tilde{u})$ and $R_1(v_{m_n}) \to R_1(\tilde{u})$ and thus $R_1(v_n) \to R_1(\tilde{u})$.

2.1.3. Auxiliary mapping R_2

When we unite all the sets $V_{k,k+1}$ defined in the previous paragraph, we get an unbounded set

$$V := \bigcup_{k\in N(a)} V_{k,k+1}$$

the boundary (in $\mathbb{R}^3$) of which consists of two parts:

$$\partial V = V_a \cup V_{b,c}\,,$$

where

$$\begin{aligned} V_a &= \big\{(a,u_1,u_2) : b_i(a) < u_i < c_i(a),\ i=1,2\big\},\\ V_{b,c} &= \big\{(t,u_1,u_2) : (t,u_1,u_2)\in V,\ \text{there exists an } i\in\{1,2\}.\\ &\qquad\qquad \text{such that } u_i = b_i(t) \text{ or } u_i = c_i(t)\big\}. \end{aligned}$$

As we have seen, for $\tilde{u}\in D$, $R_1(\tilde{u}) \in V_{b,c}$.
There exists a continuous mapping $\tilde{R}$ of the set V onto the set D (remind that $D = \big[b_1(a), c_1(a)\big] \times \big[b_2(a), c_2(a)\big] = \{(U_1,U_2) : b_i(a) \le U_i \le c_i(a),\ i=1,2\}$), $\tilde{R} : (t,u_1,u_2) \mapsto (U_1,U_2)$, where

$$\begin{aligned} U_1 &= b_1(a) + \frac{c_1(a)-b_1(a)}{c_1(t)-b_1(t)}\big(u_1 - b_1(t)\big),\\ U_2 &= b_2(a) + \frac{c_2(a)-b_2(a)}{c_2(t)-b_2(t)}\big(u_2 - b_2(t)\big). \end{aligned}$$

It is easy to see that the image of $\partial V_{b,c}$ is the boundary of D. Thus we define the mapping R_2 as the restriction of the mapping $\tilde{R}$ to the set $\partial V_{b,c}$,

$$R_2 = \tilde{R}|_{V_{b,c}}\,.$$

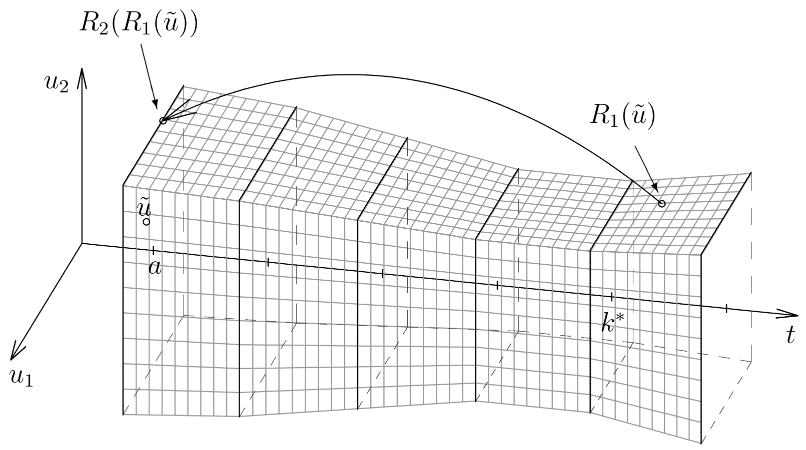

FIGURE 3. Construction of the mapping R.

2.1.4. Resulting mapping R and its properties

Define $R := R_2 \circ R_1$ (see Figure 3). Due to construction we have

$$R(D) = \partial D.$$

The mapping R is continuous because of the continuity of the partial mappings R_1 and R_2, so it is the sought retraction of D onto ∂D. But such a retraction cannot exist and thus we get a contradiction and the proof is complete.

3. Example

EXAMPLE 3.1. Consider the system

$$\begin{aligned}
\Delta u_1(k) &= f_1\big(k, u_1(k), u_2(k), u_3(k)\big) := u_1(k) + \frac{1}{15}\, u_2(k) - \frac{1}{5}\, u_3(k) - k,\\
\Delta u_2(k) &= f_2\big(k, u_1(k), u_2(k), u_3(k)\big) := \frac{1}{4}\, u_1(k) + u_2(k) + \frac{1}{16}\, u_3(k) - 2k,\\
\Delta u_3(k) &= f_3\big(k, u_1(k), u_2(k), u_3(k)\big) := \frac{2}{3}\, u_1(k) - \frac{1}{18}\, u_2(k) - \frac{1}{3}\, u_3(k)
\end{aligned} \tag{9}$$

with $k \in N(8)$. We prove that system (9) has a solution $u^* = \big(u_1^*(k), u_2^*(k), u_3^*(k)\big)$ such that for every $k \in N(8)$

$$k < u_1^*(k) < 2k, \quad k < u_2^*(k) < 3k, \quad k < u_3^*(k) < 4k. \tag{10}$$

Put for every $k \in N(8)$

$$b_i(k) := k,\ i = 1, 2, 3, \quad c_1(k) := 2k, \quad c_2(k) := 3k, \quad c_3(k) := 4k.$$

We prove that all the conditions of Theorem 2.1 with $j = 3$ are fulfilled.
The functions f_1, f_2 and f_3 are obviously continuous with respect to u_1, u_2 and u_3.
We show that all the points of the sets Ω_B^1 and Ω_C^1 are points of strict egress, i.e., if $u_1(k) = b_1(k) = k$ and $k < u_2(k) < 3k$, $k < u_3(k) < 4k$, then inequality (5) with $i = 1$ holds. The left-hand side of inequality (5) in our case is

$$f_1(k, u_1, u_2, u_3) = k + \frac{1}{15}\,u_2 - \frac{1}{5}\,u_3 - k = \frac{1}{15}\,u_2 - \frac{1}{5}\,u_3 < \frac{1}{15}\,3k - \frac{1}{5}\,k = 0,$$

meanwhile the right-hand side is

$$b_1(k+1) - b_1(k) = k + 1 - k = 1$$

and hence (5) is fulfilled.
Similarly, for $u_1(k) = c_1(k) = 2k$, the left hand-side of inequality (6) (again for $i = 1$) is

$$f_1(k, u_1, u_2, u_3) = 2k + \frac{1}{15}\,u_2 - \frac{1}{5}\,u_3 - k > k + \frac{1}{15}\,k - \frac{1}{5}\,4k = \frac{4}{15}\,k,$$

the right-hand side is

$$c_1(k+1) - c_1(k) = 2k + 2 - 2k = 2$$

and as $4k/15 > 2$ for $k > 7$, inequality (6) holds for $k \in N(8)$.
Now show the strict egress property for the points of the sets Ω_B^2 and Ω_C^2. The left-hand side of inequality (5) (for $i = 2$) in our case is

$$\frac{1}{4}\,u_1 + k + \frac{1}{16}\,u_3 - 2k = -k + \frac{1}{4}\,u_1 + \frac{1}{16}\,u_3 < -k + \frac{1}{4}\,2k + \frac{1}{16}\,4k = -\frac{1}{4}\,k < 0,$$

meanwhile the right-hand side is

$$b_2(k+1) - b_2(k) = k + 1 - k = 1$$

and hence (5) is fulfilled.
Similarly, for $u_2(k) = c_2(k) = 3k$, the left hand-side of inequality (6) is

$$\frac{1}{4}\,u_1 + 3k + \frac{1}{16}\,u_3 - 2k > k + \frac{1}{4}\,k + \frac{1}{16}\,k = \frac{21}{16}\,k,$$

the right-hand side is

$$c_1(k+1) - c_1(k) = 3k + 3 - 3k = 3$$

and as $21k/16 > 3$ for $k > 2$, inequality (6) holds for $k \in N(3)$ (and thus for $k \in N(8)$, too).

Now verify that the set Ω is of the Liapunov type with respect to the third variable, i.e., that for every (k, u_1, u_2, u_3) such that $k < u_1 < 2k$, $k < u_2 < 3k$, $k < u_3 < 4k$ inequalities (4) with $j = 3$ are fulfilled.
Let us find the upper and lower estimate for the expression $u_3 + f_3(k, u_1, u_2, u_3)$:

$$u_3 + f_3(k, u_1, u_2, u_3) = \frac{2}{3}\,u_1 - \frac{1}{18}\,u_2 + \frac{2}{3}\,u_3 < \frac{2}{3}\,2k - \frac{1}{18}\,k + \frac{2}{3}\,4k = \frac{71}{18}\,k$$

and

$$u_3 + f_3(k, u_1, u_2, u_3) = \frac{2}{3}\,u_1 - \frac{1}{18}\,u_2 + \frac{2}{3}\,u_3 > \frac{2}{3}\,k - \frac{1}{18}\,3k + \frac{2}{3}\,k = \frac{7}{6}\,k\,.$$

Inequalities (4) with $j = 3$ hold because

$$b_3(k+1) = k + 1 < \frac{7}{6}k \quad \text{for } k > 6 \quad \text{and} \quad \frac{71}{18}\,k < 4k + 4 = c_3(k+1) \quad \text{for } k \in \mathbb{N}.$$

Thus all the conditions of Theorem 2.1 are fulfilled and there exists a solution of system (9) for which inequalities (10) hold.

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