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A NOTE ON REGULAR MAPS WITH LARGE PLANAR WIDTH

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ABSTRACT. The planar width of a finite, non-spherical map is the smallest number of intersections of a non-contractible closed curve on the supporting surface of the map with the embedded graph.

We study the norm of the minimal polynomial of $2\cos(\pi/n)$ to improve the existing upper bounds on the size of the smallest regular map of type $\{m,n\}$ whose planar width is larger than r.

1. Introduction

A cellular embedding of a graph in a plane or on a compact orientable surface (briefly, a *map*) is *regular* if its orientation preserving automorphism group acts regularly on its directed edges (darts). Regular maps are interesting not only on their own but also because of numerous connections between various parts of mathematics such as complex functions, hyperbolic geometry, group theory and Galois theory; see [4] for a survey. In this place we only mention that finite regular maps of type $\{m,n\}$ (that is, of face length m and vertex valence n) are in a 1-1 correspondence with finite-index torsion-free normal subgroups of the triangle groups $T(2,m,n)=\langle a,b\,|\,a^n=b^m=(ba)^2=1\rangle$ (see [8] for details).

A natural measure of "density" of a map on a compact orientable surface $\mathcal{S}$ is the so-called *planar width* (also called representativity of a map), which is the least number of intersections of a non-contractible closed curve on $\mathcal{S}$ with the underlying graph of the map. The question of the existence of regular maps of a given hyperbolic type $\{m,n\}$ (that is, $1/m+1/n<1/2$) and of planar width larger than a given number r has an immediate affirmative answer since it is equivalent to residual finiteness of triangle groups. A covering space approach to this question was considered in [7], furnishing no explicit upper bound on

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the size (i.e., the number of darts) of the resulting maps. Using known faithful representations of triangle groups in certain 3-dimensional special linear groups over algebraic extensions of the integers, in [8] an upper bound of the form $C(m,n)^r$ was obtained for the smallest size $\delta(m,n,r)$ of a regular map of type $\{m,n\}$ with planar width at least r. This is, in some sense, the best possible type of bound, as it follows from [2] that $\delta(m,n,r) > c(m,n)^r$; here both $C(m,n)$ and $c(m,n)$ are independent of r. However, the current gap between the two "constants" is enormous; from [2] and [8] we know that

$$\sqrt{mn} < c(m,n) < C(m,n) < 2^{72(m+n)(mn)^3}.$$

Therefore it makes sense to look for better bounds on these parameters.

The result on $C(m,n)$ in [8] is obtained as a combination of two types of bounds. First, certain "norms" of polynomials over $\mathbb{Z}$ and $\mathbb{Z}(\gamma)$, where $\gamma = 2\cos(\pi/mn)$, based on modified Chebyshev polynomials, are derived. These are then combined with an estimate on a prime p for which the natural homomorphism $SL(3,\mathbb{Z}(\gamma)) \to SL(3,\mathbb{Z}_p(\gamma))$ has a sufficiently large "injectivity radius". Room for improvement in the above steps has been indicated in the last section of [8]. Considering iterated extensions of the form $\mathbb{Z}(\xi,\zeta)$ where $\xi = 2\cos(\pi/m)$ and $\zeta = 2\cos(\pi/n)$ in place of $\mathbb{Z}(\gamma)$ leads to an improvement [1] of the form

$$C(m,n) < 2^{[(m^2+n^2)-(m+n)+5]9(m+n)mn}.$$

The purpose of this note is to show that replacing (in the approach of [8]) the modified Chebyshev polynomials with the corresponding minimal polynomials for ξ and ζ yields further improvement on the upper bound on $C(m,n)$

$$C(m,n) < 2^{[(\varphi(m)^2+\varphi(n)^2)-(\varphi(m)+\varphi(n))+6]\frac{9}{16}(m+n)\varphi(m)\varphi(n)},$$

where $\varphi(n)$ is the *Euler function* giving the number of positive integers smaller than n and coprime with n.

2. The norm of the minimal polynomial of $2\cos(\pi/n)$

Let $\mathbb{Z}[x]$ denote the ring of polynomials in x with integer coefficients. We define the *norm* $\|f(x)\|$ of a polynomial $f(x) \in \mathbb{Z}[x]$ as the largest absolute value of all its coefficients.

Let $f_n(x)$ be the minimal polynomial of $2\cos(\pi/n)$ (that means, monic and irreducible annihilating integer polynomial); for a recurrence relation for $f_n(x)$ we refer to [3]. If $n \geqslant 3$, the degree of $f_n(x)$ is $d_n = \varphi(n)/2$ (see [5]). The bound on the norm of $f_n(x)$ is in the following lemma.

Lemma 1. *Let $n \geqslant 3$. The norm of the minimal polynomial $f_n(x)$ of the algebraic integer $2\cos(\pi/n)$ satisfies*

$$\|f_n(x)\| \leqslant 3^{\frac{\varphi(n)}{2}}. \tag{1}$$

P r o o f. For any integer $k \geq 3$ we denote by k^+ the set of all primitive roots of unity of the form $\cos(2\pi d/k) + i\sin(2\pi d/k)$ where $1 \leq d < k/2$ and $(d,k) = 1$, and let

$$\Phi_k(x) = \prod_{\omega \in k^+} \left(x - (\omega + \omega^{-1})\right). \tag{2}$$

The polynomial $\Phi_k(x)$ is monic and irreducible of degree $\varphi(k)/2$, with integer coefficients, whose roots have the form $2\cos(2\pi d/k)$, cf. [5].

Since $|\omega+\omega^{-1}| \leqslant 2$ and the cardinality of the set k^+ is $d_k = \varphi(k)/2$, then the absolute value of the coefficient on the jth power of x in the polynomial $\Phi_k(x)$ is, by Viète's formula, smaller than the corresponding coefficient of $(x+2)^{d_k}$, which is $2^{d_k - j}\binom{d_k}{j} < (2+1)^{d_k} = 3^{d_k}$.

If we substitute $k = 2n$ to the polynomial (2), then we obtain $\Phi_{2n}(x) = f_n(x)$, the minimal polynomial of $2\cos(\pi/n)$. Since $f_n(x)$ satisfies the above conditions, we have the bound on the norm of the polynomial $f_n(x)$ in the form

$$\|f_n(x)\| < 3^{\frac{\varphi(n)}{2}}.$$

□

3. Linear representations of triangle groups

In this section we present a new upper bound on the smallest order of a finite group that is a homomorphic image of a triangle group $T(l,m,n) = \langle a,b,c \,|\, a^n = b^m = c^l = bac = 1\rangle$ with injectivity radius at least r_0.

Let $1/l + 1/m + 1/n \leqslant 1$ and let r_0 be a positive integer. Denote by S_{r_0} the set of all non-identity elements $u \in T(l,m,n)$ with the property that u can be expressed as a word of length at most r_0 in the symbols a, b and c. We will say that representation $\vartheta\colon T(l,m,n) \to H$ in a finite group H has *injectivity radius at least* r_0 if $\vartheta(u) \neq 1_H$ for each $u \in S_{r_0}$.

Let $g(x,y,z) \in \mathbb{Z}[x,y,z]$ be a polynomial in three variables over $\mathbb{Z}$. We define the *norm* $\|g(x,y,z)\|$ as the largest absolute value of all coefficients of $g(x,y,z)$. Further, we define $w(g)$, the *width* of $g(x,y,z)$, as the number of non-zero coefficients of $g(x,y,z)$.

Let $f_1(x)$, $f_2(y)$ and $f_3(z)$ be arbitrary monic polynomials of ξ, ζ and η, respectively. We define the quotient ring $\mathbb{Z}(\xi,\zeta,\eta)$ as the ring of polynomials $\mathbb{Z}[x,y,z]/\left(f_1(x), f_2(y), f_3(z)\right)$.

If $A \in SL(q, \mathbb{Z}[\xi,\zeta,\eta])$ is a matrix with entries $A_{ij} = A_{ij}(\xi,\zeta,\eta) \in \mathbb{Z}[\xi,\zeta,\eta]$, then the width $w(A)$ and the norm $||A||$ of A are defined as the maximum of $w(A_{ij}(\xi,\zeta,\eta))$ and $\|A_{ij}(\xi,\zeta,\eta)\|$, respectively, taken over all indices $1 \leq i, j \leq q$.

The next theorem from [1] gives a construction of homomorphisms of hyperbolic triangle groups into finite groups of order less than G^{r_0} and with injectivity radius at least r_0.

Theorem 1. *Let $\mathbb{Z}(\xi,\zeta,\eta) = \mathbb{Z}[x,y,z]/(f_1(x), f_2(y), f_3(z))$, where $f_1(x) \in \mathbb{Z}[x]$, $f_2(y) \in \mathbb{Z}[y]$ and $f_3(z) \in \mathbb{Z}[z]$ are monic polynomials of degree $d_1 > 0$, $d_2 > 0$ and $d_3 > 0$; let $s_1 = 1 + \|f_1(x)\|$, $s_2 = 1 + \|f_2(y)\|$ and $s_3 = 1 + \|f_3(y)\|$. Assume that we have an injective homomorphism $\vartheta\colon T(l,m,n) \to SL(q, \mathbb{Z}(\xi,\zeta,\eta))$ such that all the matrices $\vartheta(a)$, $\vartheta(b)$ and $\vartheta(c)$ have norm $\leqslant t$ and width $\leqslant w$. Then, for any positive integer r_0 there exists a homomorphism of $T(l,m,n)$ with injectivity radius at least r_0 into a finite group H such that*

$$|H| < \left(qwts_1^{d_1-1}s_2^{d_2-1}s_3^{d_3-1}\right)^{q^2 d_1 d_2 d_3 r_0}. \tag{3}$$

The most suitable linear representation of $T(l,m,n)$ to work with seems to be Mennicke's injective representation [6] ϑ of the triangle group $T(l,m,n)$ in the group $SL(3, \mathbb{Z}(\xi,\zeta,\eta))$ given by

$$A = \vartheta(a) = \begin{pmatrix} -1 & -\xi & 0 \\ \xi & \xi^2 - 1 & 0 \\ \zeta & \zeta\xi + \eta & 1 \end{pmatrix}, \quad B = \vartheta(b) = \begin{pmatrix} \zeta^2 - 1 & 0 & \zeta \\ \eta\zeta + \xi & 1 & \eta \\ -\zeta & 0 & -1 \end{pmatrix},$$

$$C = \vartheta(c) = \begin{pmatrix} 1 & \xi & \eta\xi + \zeta \\ 0 & -1 & -\eta \\ 0 & \eta & \eta^2 - 1 \end{pmatrix}, \tag{4}$$

where $\xi = 2\cos(\pi/n)$, $\zeta = 2\cos(\pi/m)$ and $\eta = 2\cos(\pi/l)$.

In order to apply Theorem 1 we use monic and also irreducible polynomials of ξ, ζ and η, mentioned in the beginning of Section 2. Degrees of this polynomials are $d_1 = \varphi(n)/2$, $d_2 = \varphi(m)/2$ and $d_3 = \varphi(l)/2$. Lemma 1 enables us to take $s_1 = 1 + 3^{\varphi(n)/2}$, $s_2 = 1 + 3^{\varphi(m)/2}$ and $s_3 = 1 + 3^{\varphi(l)/2}$ as bounds on the norms of $f_1(x)$, $f_2(y)$ and $f_3(z)$. The norm and the width of the matrices A, B and C are $t = 1$ and $w = 2$. Using all this information together with $q = 3$, we obtain the following result from Theorem 1 as an improvement of the result in [1].

Theorem 2. *Let $1/l + 1/m + 1/n < 1$. Then for any positive integer r_0 there exists a homomorphic image of the triangle group $T(l,m,n)$ in a finite group of order G^{r_0} with injectivity radius at least r_0, where*

$$G < 2^{[(\varphi(l)^2+\varphi(m)^2+\varphi(n)^2)-(\varphi(l)+\varphi(m)+\varphi(n))+6]\frac{9}{16}\varphi(l)\varphi(m)\varphi(n)}.$$

4. Application to regular maps

An *automorphism* of a map M is an orientation preserving homeomorphism of the supporting surface of M onto itself which preserves the cells of M and their incidence. Any automorphism of M will be regarded as a permutation of the set of all darts of M. The set of all automorphism of M forms the *automorphism group* $\mathrm{Aut}(M)$; it acts freely on the dart set of the map. As stated in Introduction, M is *regular* if $\mathrm{Aut}(M)$ acts regularly on the dart set of M. Since the automorphism group of a regular map is transitive on faces, on darts and on vertices, every vertex must have the same degree m and the number n of edges around a face must be the same for all faces; in this case we say that the map is of type $\{m,n\}$. The automorphism group of each regular map M of type $\{m,n\}$ is always a quotient of the triangle group $T(2,m,n)$ and hence has a presentation of the form

$$\mathrm{Aut}(M) = \langle a, b \,|a^n = b^m = (ba)^2 = \cdots = 1\rangle .$$

Here, the automorphism a rotates M clockwise around a fixed vertex v by the angle $2\pi/n$ and b rotates M clockwise around the centre of a fixed face F incident with v by the angle $2\pi/m$.

Let M be a map on a compact surface $\mathcal{S}$ of positive genus with planar width larger than r. It is shown in [S] that the injectivity radius r_0 of the natural homomorphism $T(2,m,n) \to \mathrm{Aut}(M)$ is at most $r(m+n)/2$.

If we take $l=2$, in the triangle group $T(l,m,n)$, and modify the presentation (4) accordingly (with $\eta = 0$), we obtain a faithful representation ϑ of the triangle group $T(2,m,n) = \langle a,b|a^n = b^m = (ba)^2 = 1\rangle$ in the group $SL\big(3,\mathbb{Z}(\xi,\zeta)\big)$ given by

$$A = \vartheta(a) = \begin{pmatrix} -1 & -\xi & 0 \\ \xi & \xi^2-1 & 0 \\ \zeta & \zeta\xi & 1 \end{pmatrix}, \quad B = \vartheta(b) = \begin{pmatrix} \zeta^2-1 & 0 & \zeta \\ \xi & 1 & 0 \\ -\zeta & 0 & -1 \end{pmatrix}.$$

By Theorem 1, for each r_0 there is a faithful representation of $T(2,m,n)$ in a finite group H with injectivity radius at least r_0 where the order of H is bounded above by (3). The minimal polynomial of $\eta = 0$ is $f_3(z) = z$ of norm 1 (therefore $s_3 = 2$) and degree $d_3 = 1$. For the remaining parameters in (3) we use the values listed before the statement of Theorem 2, which yields the following bound on the smallest number $\delta(m,n,r)$ of darts of a regular map of type $\{m,n\}$ and planar width at least r.

Theorem 3. *For every r and for each m, n such that $1/m + 1/n < 1/2$ there exists a regular map of type $\{m,n\}$ with planar width larger than r such that*

$$\delta(m,n,r) < 2^{[(\varphi(m)^2+\varphi(n)^2)-(\varphi(m)+\varphi(n))+6]\frac{9}{16}(m+n)\varphi(m)\varphi(n)r}.$$

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