Tatra Mt. Math. Publ. **36** (2007), 1–11

TATRA MOUNTAINS
Mathematical Publications

FUNDAMENTAL GROUPS OF PRIME 3-MANIFOLDS OF GENUS AT MOST TWO

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ABSTRACT. Present paper deals with fundamental groups of 3-manifolds represented by a certain family $\mathcal{S}$ of bipartite 4-edge-coloured graphs. List of fundamental groups of prime 3-manifolds with genus two represented by graphs in $\mathcal{S}$ with at most 42 vertices is produced.

The aim of the present paper is to deal with fundamental groups of 3-manifolds represented by a certain family $\mathcal{S}$ of bipartite 4-edge-coloured 4-valent graphs. The complexity of such a graph representation is half of the number of vertices of the graph. The list of fundamental and homology groups is derived for the set of 3-manifolds of complexity [2] at most 21. Fundamental groups are given in terms of presentations of rank 2. The presentations are derived from the graphs in $\mathcal{S}$ using the algorithm of Gagliardi [5]. The isomorphism problem is solved using some group theoretical techniques in combination with the software GAP (Groups, Algorithms and Programming [6]). For the group-theoretical terms, not defined in the paper, we refer the reader to [11].

Due to Pezzana [10] a closed connected 3-manifold can be represented by a particular 4-valent 4-edge-coloured graph called a crystallisation. The concept of crystallisations plays a crucial role in our study of 3-manifolds. We emphasize two approaches. First one is based on investigation of the structure of a crystallisation using some techniques based on dipole-moves [2], [4], [7], [9]. This approach can be considered as graph-theoretic. Using dipole moves one can recognize whether two 3-manifolds are homeomorphic without using any other knowledge about represented manifold.

The other approach is based on determining some topological invariants such as the fundamental or homology group from the structure of a crystallisation.

2000 Mathematics Subject Classification: Primary 57M05, 57M15, 57M27, 57N10, Secondary 57Q15.

Keywords: 3-manifold, fundamental group, first homology group, crystalisation.

Work was supported by Science and Technology Assistance Agency under the contract No. APVT-51-012502.

Next, group theory techniques to identify the isomorphism class of fundamental group of represented 3-manifold are used. Note that the knowledge about isomorphism of fundamental groups is not sufficient in order to claim that two 3-manifold are homeomorphic. On the other hand, the non-isomorphism of the fundamental groups guarantees that two respective 3-manifolds are not homeomorphic.

All considered 3-manifolds are assumed to be connected, compact and orientable with empty boundary. A 3-manifold is *prime* if it cannot be expressed as a connected sum of two non-trivial 3-manifolds. By Kneser Theorem (see [8, Theorem 3.15, 3.21]) an orientable compact connected 3-manifold has a unique decomposition into prime factors with respect to the connected sum. The corresponding operation on the respective fundamental groups is the free product of groups. In fact, combining some known non-trivial results (such as Thurston's Symmetry Theorem), one can prove that prime 3-manifolds of Heegaard genus two are characterized by the fact that the fundamental group cannot be expressed as a free product of two cyclic groups. List of isomorphism classes of fundamental groups of prime 3-manifolds of genus two represented by graphs in $\mathcal{S}$ with at most 42 vertices is produced in present note.

Definition 1. (4-coloured graph). [9] Let Γ be a finite bipartite, connected graph without loops. The **edge colouring** on a graph Γ is a map $\gamma\colon E(\Gamma) \to C_4 = \{0,1,2,3\}$ such that for every distinct $e, f \in E(\Gamma)\colon \gamma(e) \neq \gamma(f)$, if the edges e and f are adjacent. The graph (Γ,γ) is called a **4-coloured graph** if and only if all vertices of Γ are of degree four. The notation (Γ,γ) is used for such graphs.

Since the colouring of (Γ,γ) is regular, a factor induced by two colours is a disjoint union of bicoloured cycles. Let $\mathcal{I}$ denote the set of 2-cell embeddings of (Γ,γ) into a closed orientable surface such that the local rotation of colours induced by the embedding in "black" vertices is the same, say ρ, while the local rotation of colours in "white" vertices is ρ^{-1}. Note that there are six possibilities for choosing ρ. It follows that faces of such embedding are bounded by bicoloured cycles. Out of these six possibilities for ρ we choose such ρ that the genus of the underlying surface is minimal in $\mathcal{I}$. The integer g is an invariant of a 3-manifold $\mathcal{M}$ represented by (Γ,γ) and it is called the *regular genus* of $\mathcal{M}$ (or shortly the genus of $\mathcal{M}$). It is known that regular genus of $\mathcal{M}$ is equal to the Heegaard genus of $\mathcal{M}$ [1].

Let $\Gamma_{\widehat{B}}$ be a subgraph of Γ such that $V(\Gamma) = V(\Gamma_{\widehat{B}})$ and every edge in $\Gamma_{\widehat{B}}$ is coloured by a colour from the set $C_4 \setminus B$. The *residual graph* $\Gamma_{\widehat{B}}$ can be simply obtained from Γ by deleting the edges coloured by colours from the set B. In particular, the graph $\Gamma_{\hat{c}}$ is the residual graph created by deleting the edges coloured by the colour c.

Definition 2. (Contracted graph). [1] The graph (Γ, γ) is **contracted** if for all colours $c \in C_4$ the residuals $\Gamma_{\hat{c}}$ are connected graphs.

Simple method of reconstruction of the underlying 3-dimensional simplicial complex given by a 4-coloured graph is introduced in [9]. Note that a general 4-coloured graph may determine a more general space of dimension 3 called a compact pseudomanifold of dimension 3.

Definition 3. (Crystallisation (Γ, γ)). [1] A 4-coloured graph (Γ, γ) will be called a **crystallisation** if and only if it is contracted and admits a 3-manifold.

A basis of the theory of crystallisations was found by Mario Pezzana [1]. In particular, we have the following theorem.

Theorem 1. (Pezzana's Existence Theorem). [1] *Every closed connected 3-manifold admits a crystallisation.*

It follows from [3] that each 3-manifold of genus at most two can be represented by a crystallisation which structure can be coded by a 6-tuple of integers satisfying the following conditions. Let $\widetilde{\mathcal{F}_2}$ be a set of 6-tuples:

$$f = (h_0, h_1, h_2; q_0, q_1, q_2), \; h_i, q_i \in \mathbb{N}.$$

satisfying

(i) $\forall i \in \mathbb{Z}_3 \colon h_i > 0$,

(ii) all h_i have the same parity,

(iii) $\forall i \in \mathbb{Z}_3 \colon 0 \leqslant q_i < h_{i-1} + h_i = 2l_i$,

(iv) all q_i have the same parity.

Remark. From here all operations with numbers q_i will be considered modulo $2l_i$, and according to (iii), q_i will be always the least non negative integer of the class.

Let us define the set $V(f)$ for a 6-tuple $f \in \widetilde{\mathcal{F}_2}$ to be

$$V(f) = \bigcup_{i \in \mathbb{Z}_3} \{i\} \times \mathbb{Z}_{2l_i}.$$

Define the following fixed-point-free involutory permutations acting on $V(f)$:

$$\alpha_0(i,j) = \big(i, j + (-1)^j\big),$$
$$\alpha_1(i,j) = \big(i, j - (-1)^j\big),$$
$$\alpha_2(i,j) = \begin{cases} (i+1, 2l_{i+1} - j - 1); & 0 \leqslant j < h_i, \\ (i-1, 2l_i - j - 1); & h_i \leqslant j < 2l_i, \end{cases}$$
$$\alpha_3(i,j) = \rho \circ \alpha_2 \circ \rho^{-1},$$

where $\rho\colon V(f) \to V(f)$ is a bijection defined by the rule

$$\rho(i,j) = (i, j + q_i)\,.$$

Now let $f \in \widetilde{\mathcal{F}_2}$ and satisfy the following conditions:

(v) $\forall i \in \mathbb{Z}_3\colon h_i + q_i$ is odd, i.e., h_i and q_i have different parity,

(vi) the group $\langle \alpha_2, \alpha_3 \rangle$ has exactly three orbits.

Definition 4. (Admissible 6-tuple). The elements of the set $\mathcal{F}_2 \subset \widetilde{\mathcal{F}_2}$ satisfying conditions (i)–(vi) will be called **admissible 6-tuples**.

Given admissible 6-tuple f there is an associated 4-coloured graph $\Gamma(f)$, called the derived graph, which vertices are elements of $V(f)$ and xy is an edge coloured by a colour $i \in \{0,1,2,3\}$ if and only if $y = \alpha_i(x)$.

Theorem 2. [3] *For every orientable compact connected 3-manifold $\mathcal{M}$ of genus 2 there exists $f \in \mathcal{F}_2$ such that $\Gamma(f)$ represents $\mathcal{M}$.*

Definition 5. (Complexity of 6-tuple). Let f be an admissible 6-tuple. The number $z(f) = h_0 + h_1 + h_2$ will be called a **complexity of** f.

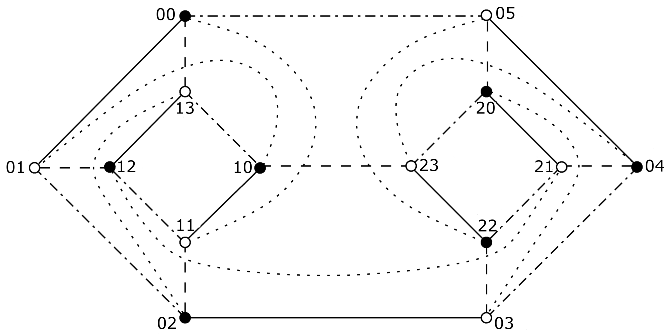

Figure 1. The graph $\Gamma(f)$ represented by 6-tuple $f = (3,1,3;0,2,2)$.

In what follows we describe an algorithm constructing a presentation of the fundamental group of a 3-manifold represented by $\Gamma(f)$, $f \in \mathcal{F}_2$ (see [5]).

Let f be an admissible 6-tuple and $\Gamma(f)$ be the derived graph. The fundamental group $\pi_1(f)$ of the 3-manifold given by $\Gamma(f)$ has a presentation of the form $\pi_1(f) = \langle a, b, c \mid R_0, R_1, R_2 \rangle$. Set a, b, c to be the three cycles of the 2-factor induced by edges coloured by $\{0,1\}$. Hence we have a mapping $\beta\colon V(\Gamma) \to \{a,b,c\}$. By the definition, the $\{2,3\}$-factor has 3-cycles C_0, C_1, C_2. The relator R_i is given by $C_i = (v_{0,i}v_{1,i}v_{2,i}\dots v_{k_i,i})$, $i = 0,1,2$. We assume that the edge $v_{0,i}v_{1,i}$ is coloured by the colour 2. We set

$$R_i = \prod_{j=0}^{(k_i-1)/2} \beta(v_{2j,i})\beta(v_{2j+1,i})^{-1}, \quad i = 0,1,2\,. \tag{1}$$

Add the relator $c = 1$ into the presentation of the group after obtaining the relators. Thus the group can be considered as a group of rank 2.

EXAMPLE 1. Let us determine the fundamental group given by the 6-tuple (3,1,3;0,2,2) (see Figure 1). Define the mapping $\beta\colon V(\Gamma) \to \{a, b, c\}$ as follows: $\beta(v) = a, b, c$ depending on whether v belongs to C_0, C_1 or C_2, respectively. The $\{2, 3\}$-factor is a union of the cycles:

```
C1 = ([0,0][1,3][0,2][1,1]),
C2 = ([0,1][1,2][2,1][0,4][2,3][1,0]),
C3 = ([0,3][2,2][0,5][2,0])
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forming orbits of $\alpha_2\alpha_3$. Using (1) these cycles transform to the relators $R_1 := ab^{-1}ab^{-1} = 1$, $R_2 := ab^{-1}ca^{-1}cb^{-1} = 1$ and $R_3 := ac^{-1}ac^{-1} = 1$. Setting $c = 1$ we get

$$\pi_1\big((3,1,3;0,2,2)\big) = \langle a, b \,|\, (ab^{-1})^2 = 1, ab^{-1}a^{-1}b^{-1} = 1, a^2 = 1\rangle\,.$$

Since a and ab^{-1} generate the group, we can rewrite the presentation of this group in generators $x = a$, $y = ab^{-1}$ in following way:

$$\pi_1((3,1,3;0,2,2)) = \langle x, y \,|\, x^2 = y^2 = 1\rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2\,.$$

Since $\pi_1\big((3,1,3;0,2,2)\big)$ is a free product of two cyclic groups, the corresponding 3-manifold is decomposable [12].

EXAMPLE 2. Our next example is slightly more complicated than previous one. The 6-tuple (3,3,3;2,2,2) is a representative of the class F.3 in our catalogue. Computing the action of α's on the set of vertices we get the table showed as Table 1.

The cycles of $\alpha_2\alpha_3$ follow:

```
C1=([ 0, 0][ 1, 5][ 2, 2][ 0, 3][ 2, 4][ 1, 1]),
C2=([ 0, 1][ 1, 4][ 2, 3][ 1, 2][ 0, 5][ 2, 0]),
C3=([ 0, 2][ 1, 3][ 0, 4][ 2, 1][ 1, 0][ 2, 5]).
```

By (1) the derived relators are $R_1 := ab^{-1}ca^{-1}cb^{-1} = 1$, $R_2 := ab^{-1}cb^{-1}ac^{-1} = 1$ and $R_3 := ab^{-1}ac^{-1}bc^{-1} = 1$. Setting $c = 1$ we get the presentation

$$\pi_1\big((3,3,3;2,2,2)\big) = \langle a, b \,|\, ab^{-1}a^{-1}b^{-1} = 1, ab^{-2}a = 1, ab^{-1}ab = 1\rangle\,.$$

One can easily see that this is the presentation of the group of quaternions Q_8.

Next step of processing the presentation of the fundamental group is to apply Tieze transformations to get the relators as short as possible.

Factorizing obtained fundamental group by the derived subgroup one gets the first homology group $H_1(f)$. It can be done by adding the commutator of the

TABLE 1. Actions of α's in the case of $f = (3, 3, 3; 2, 2, 2)$.

[i, j]	α_0 [i, j]	α_1 [i, j]	α_2 [i, j]	α_3 [i,j]
[0, 0]	[0, 1]	[0, 5]	[1, 3]	[1, 1]
[0, 1]	[0, 0]	[0, 2]	[1, 2]	[1, 0]
[0, 2]	[0, 3]	[0, 1]	[1, 1]	[1, 3]
[0, 3]	[0, 2]	[0, 4]	[2, 2]	[2, 0]
[0, 4]	[0, 5]	[0, 3]	[2, 1]	[2, 3]
[0, 5]	[0, 4]	[0, 0]	[2, 0]	[2, 2]
[1, 0]	[1, 1]	[1, 3]	[2, 3]	[0, 1]
[1, 1]	[1, 0]	[1, 2]	[0, 2]	[0, 0]
[1, 2]	[1, 3]	[1, 1]	[0, 1]	[2, 1]
[1, 3]	[1, 2]	[1, 0]	[0, 0]	[0, 2]
[2, 0]	[2, 1]	[2, 3]	[0, 5]	[0, 3]
[2, 1]	[2, 0]	[2, 2]	[0, 4]	[1, 2]
[2, 2]	[2, 3]	[2, 1]	[0, 3]	[0, 5]
[2, 3]	[2, 2]	[2, 0]	[1, 0]	[0, 4]

two generators into a presentation of the fundamental group. The set of 6-tuples splits into homology classes. Of course if $H_1(f) \ncong H_1(e)$, then $\pi_1(f) \ncong \pi_1(e)$.

The finiteness of homology groups is tested by solving a system of linear equations over the ring $\mathbb{Z}$ given by the relators of the homology group. The computation is done using GAP [6] in the finite case. In the infinite case the homology group has a form of a direct given product $\mathbb{Z} \times \mathbb{Z}_n$. We have an exact formula determining n.

Having homology classes we analyse fundamental groups in each homology class first determining some group invariants such as centers, low index subgroups, normal low index subgroups and the respective quotients. If the invariants of given two groups in the same homology class coincide, we test the isomorphism of the groups. As a result, we have the following theorem.

THEOREM 3. *Prime 3-manifolds of genus at most two, represented by admissible 6-tuples of complexity $\leqslant 21$, have fundamental groups of one of the following types:*

- *trivial group,*
- *cyclic groups $\mathbb{Z}_n, 2 \leqslant n \leqslant 29$,*
- *infinite cyclic group $\mathbb{Z}$,*
- *acyclic groups.*

The list includes **78** *isomorphism classes of acyclic fundamental groups of prime 3-manifolds of genus two. Among them, there are* **39** *elliptic manifolds with finite groups,* **4** *Euclidean manifolds and* **35** *other manifolds with infinite fundamental groups.*

Note that there are also non-prime 3-manifolds of genus two, which fundamental groups are connected sums of two lens spaces, or of a lens space and $S^1 \times S^2$.

Since the classification of 3-manifolds of genus less than two is known, cyclic groups are "less interesting" for further study. The list of all acyclic fundamental groups of prime 3-manifolds of genus two follows. The list is divided into three parts including at first prime elliptic 3-manifolds, at second prime Euclidean 3-manifolds and at the end, prime 3-manifolds with other geometries according to [12].

Fundamental groups of prime elliptic 3-manifolds of genus two (Classes F).

	Presentation	Abelianisation
F.1	$\pi_1(f) = \langle a, b \mid a^5 = b^3 = (ab)^2 \rangle$	$H_1 = 1$
F.2	$\pi_1(f) = \langle a, b \mid a^4 = b^3 = (ab)^2 \rangle$	$H_1 = \mathbb{Z}_2$
F.3	$\pi_1(f) = \langle a, b \mid a^2 = b^2 = (ab)^2 \rangle$	$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$
F.4	$\pi_1(f) = \langle a, b \mid a^4 = b^2 = (ab)^2 \rangle$	$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$
F.5	$\pi_1(f) = \langle a, b \mid a^6 = b^2 = (ab)^2 \rangle$	$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$
F.6	$\pi_1(f) = \langle a, b \mid a^8 = b^2 = (ab)^2 \rangle$	$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$
F.7	$\pi_1(f) = \langle a, b \mid a^3 = b^3 = (ab)^2 \rangle$	$H_1 = \mathbb{Z}_3$
F.8	$\pi_1(f) = \langle a, b \mid a^3 = b^2 = (ab)^2 \rangle$	$H_1 = \mathbb{Z}_4$
F.9	$\pi_1(f) = \langle a, b \mid a^5 = b^2 = (ab)^2 \rangle$	$H_1 = \mathbb{Z}_4$
F.10	$\pi_1(f) = \langle a, b \mid a^7 = b^2 = (ab)^2 \rangle$	$H_1 = \mathbb{Z}_4$
F.11	$\pi_1(f) = \langle a, b \mid a^2 = b^2 = (ab)^{-2} \rangle$	$H_1 = \mathbb{Z}_6 \times \mathbb{Z}_2$
F.12	$\pi_1(f) = \langle a, b \mid a^4 = b^2 = (ab^{-1})^2 \rangle$	$H_1 = \mathbb{Z}_6 \times \mathbb{Z}_2$
F.13	$\pi_1(f) = \langle a, b \mid a^5 = b^3 = (a^2b)^2 \rangle$	$H_1 = \mathbb{Z}_7$
F.14	$\pi_1(f) = \langle a, b \mid a^3 = b^2 = (a^2b)^2 \rangle$	$H_1 = \mathbb{Z}_8$
F.15	$\pi_1(f) = \langle a, b \mid a^5 = b^2 = (a^2b)^2 \rangle$	$H_1 = \mathbb{Z}_8$
F.16	$\pi_1(f) = \langle a, b \mid a^7 = (a^4b)^2, b^2 = (a^2b)^2 \rangle$	$H_1 = \mathbb{Z}_8$
F.17	$\pi_1(f) = \langle a, b \mid a^3 = b^3 = (a^{-1}b)^2 \rangle$	$H_1 = \mathbb{Z}_9$
F.18	$\pi_1(f) = \langle a, b \mid a^4 = b^2 = (ab)^3 \rangle$	$H_1 = \mathbb{Z}_{10}$
F.19	$\pi_1(f) = \langle a, b \mid a^2 = b^2 = (a^3b^3)^2 \rangle$	$H_1 = \mathbb{Z}_{10} \times \mathbb{Z}_2$
F.20	$\pi_1(f) = \langle a, b \mid a^4 = b^2 = (ab)^{-2} \rangle$	$H_1 = \mathbb{Z}_{10} \times \mathbb{Z}_2$
F.21	$\pi_1(f) = \langle a, b \mid a^6 = b^2 = (ab^{-1})^2 \rangle$	$H_1 = \mathbb{Z}_{10} \times \mathbb{Z}_2$
F.22	$\pi_1(f) = \langle a, b \mid a^5 = b^{-2} = (ab^{-1})^3 \rangle$	$H_1 = \mathbb{Z}_{11}$

F.23	$\pi_1(f) = \langle a, b \mid a^5 = b^2 = (a^2b^{-1})^2 \rangle$	$H_1 = \mathbb{Z}_{12}$
F.24	$\pi_1(f) = \langle a, b \mid a^5 = b^3 = (a^2b^{-1})^2 \rangle$	$H_1 = \mathbb{Z}_{13}$
F.25	$\pi_1(f) = \langle a, b \mid a^4 = b^3 = (ab^{-1})^2 \rangle$	$H_1 = \mathbb{Z}_{14}$
F.26	$\pi_1(f) = \langle a, b \mid a^6 = b^2 = (ab)^{-2} \rangle$	$H_1 = \mathbb{Z}_{14} \times \mathbb{Z}_2$
F.27	$\pi_1(f) = \langle a, b \mid a^3 = b^3 = (a^{-2}b)^2 \rangle$	$H_1 = \mathbb{Z}_{15}$
F.28	$\pi_1(f) = \langle a, b \mid a^3 = b^2 = (ab)^{-2} \rangle$	$H_1 = \mathbb{Z}_{16}$
F.29	$\pi_1(f) = \langle a, b \mid a^5 = b^2 = (ab^{-1})^2 \rangle$	$H_1 = \mathbb{Z}_{16}$
F.30	$\pi_1(f) = \langle a, b \mid a^5 = b^3 = (a^2b^2)^3 \rangle$	$H_1 = \mathbb{Z}_{17}$
F.31	$\pi_1(f) = \langle a, b \mid a^5 = b^3 = (ab^{-1})^2 \rangle$	$H_1 = \mathbb{Z}_{19}$
F.32	$\pi_1(f) = \langle a, b \mid a^3 = b^2 = (a^2b)^{-2} \rangle$	$H_1 = \mathbb{Z}_{20}$
F.33	$\pi_1(f) = \langle a, b \mid a^3 = b^3 = (ab)^{-2} \rangle$	$H_1 = \mathbb{Z}_{21}$
F.34	$\pi_1(f) = \langle a, b \mid a^4 = b^3 = (ab^{-2})^2 \rangle$	$H_1 = \mathbb{Z}_{22}$
F.35	$\pi_1(f) = \langle a, b \mid a^5 = b^2 = (ab)^{-2} \rangle$	$H_1 = \mathbb{Z}_{24}$
F.36	$\pi_1(f) = \langle a, b \mid a^7 = b^2 = (ab^{-1})^2 \rangle$	$H_1 = \mathbb{Z}_{24}$
F.37	$\pi_1(f) = \langle a, b \mid a^4 = b^3 = (ab)^{-2} \rangle$	$H_1 = \mathbb{Z}_{26}$
F.38	$\pi_1(f) = \langle a, b \mid a^3 = b^3 = (a^2b)^{-2} \rangle$	$H_1 = \mathbb{Z}_{27}$
F.39	$\pi_1(f) = \langle a, b \mid a^5 = b^3 = (ab)^{-2} \rangle$	$H_1 = \mathbb{Z}_{31}$

Fundamental groups of prime Euclidean 3-manifolds of genus two (Classes E).

	Presentation	Abelianisation
E.1	$\pi_1(f) = \langle a, b \mid a^2 = (ab^2)^2, b^2 = (a^2b)^2 \rangle$	$H_1 = \mathbb{Z}_4 \times \mathbb{Z}_4$
E.2	$\pi_1(f) = \langle a, b \mid a^6 = b^3 = (ab)^2 \rangle$	$H_1 = \mathbb{Z}$
E.3	$\pi_1(f) = \langle a, b \mid a^4 = b^4 = (ab)^2 \rangle$	$H_1 = \mathbb{Z} \times \mathbb{Z}_2$
E.4	$\pi_1(f) = \langle a, b \mid a^3 = (ab)^3 = (ab^2)^3 \rangle$	$H_1 = \mathbb{Z} \times \mathbb{Z}_3$

Fundamental groups of prime 3-manifolds of genus two endowed with other geometries (Classes I).

	Presentation	Abelianisation
I.1	$\pi_1(f) = \langle a, b \mid a^7 = b^3 = (ab)^2 \rangle$	$H_1 = 1$
I.2	$\pi_1(f) = \langle a, b \mid a^5 = b^4 = (ab)^2 \rangle$	$H_1 = \mathbb{Z}_2$
I.3	$\pi_1(f) = \langle a, b \mid a^8 = b^3 = (ab)^2 \rangle$	$H_1 = \mathbb{Z}_2$

I.4	$\pi_1(f) = \langle a, b \mid a^6 = b^4 = (ab)^2 \rangle$	$H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$
I.5	$\pi_1(f) = \langle a, b \mid a^4 = b^3 = (ab^{-2})^{-3} \rangle$	$H_1 = \mathbb{Z}_3$
I.6	$\pi_1(f) = \langle a, b \mid a^3 = b^3 = (ab)^3 \rangle$	$H_1 = \mathbb{Z}_3 \times \mathbb{Z}_3$
I.7	$\pi_1(f) = \langle a, b \mid a^4 = b^2 a^{-1} b^2, a^3 = b^3 a^{-3} b \rangle$	$H_1 = \mathbb{Z}_4$
I.8	$\pi_1(f) = \langle a, b \mid a^2 = (a^{-1} b^2)^2, b^2 = (a^2 b)^2 \rangle$	$H_1 = \mathbb{Z}_4 \times \mathbb{Z}_4$
I.9	$\pi_1(f) = \langle a, b \mid a^5 = b^5 = (ab)^2 \rangle$	$H_1 = \mathbb{Z}_5$
I.10	$\pi_1(f) = \langle a, b \mid a^7 = b^2 = (a^{-2} b)^3 \rangle$	$H_1 = \mathbb{Z}_5$
I.11	$\pi_1(f) = \langle a, b \mid a^3 b^{-1} a = b a^{-1} b^3 = (a^2 b^2)^2 \rangle$	$H_1 = \mathbb{Z}_5 \times \mathbb{Z}_5$
I.12	$\pi_1(f) = \langle a, b \mid a^5 = b^4 = (a^2 b)^2 \rangle$	$H_1 = \mathbb{Z}_6$
I.13	$\pi_1(f) = \langle a, b \mid a^5 = b^3 = (ab^{-2})^{-3} \rangle$	$H_1 = \mathbb{Z}_6$
I.14	$\pi_1(f) = \langle a, b \mid a^3 = b^3 = (a^2 b)^3 \rangle$	$H_1 = \mathbb{Z}_6 \times \mathbb{Z}_3$
I.15	$\pi_1(f) = \langle a, b \mid a^4 = b^4 = (ab)^3 \rangle$	$H_1 = \mathbb{Z}_8$
I.16	$\pi_1(f) = \langle a, b \mid a^4 = b^4 = (a^{-1} b)^2 \rangle$	$H_1 = \mathbb{Z}_8 \times \mathbb{Z}_2$
I.17	$\pi_1(f) = \langle a, b \mid a^4 = b^3 = (ab)^3 \rangle$	$H_1 = \mathbb{Z}_9$
I.18	$\pi_1(f) = \langle a, b \mid a^5 = b^3 = (ab)^3 \rangle$	$H_1 = \mathbb{Z}_9$
I.19	$\pi_1(f) = \langle a, b \mid a^3 = b^3 = (ab)^{-3} \rangle$	$H_1 = \mathbb{Z}_9 \times \mathbb{Z}_3$
I.20	$\pi_1(f) = \langle a, b \mid a^6 = b^3 = (a^{-1} b)^2 \rangle$	$H_1 = \mathbb{Z}_{12}$
I.21	$\pi_1(f) = \langle a, b \mid a^4 = b^3 = (ab^{-1})^3 \rangle$	$H_1 = \mathbb{Z}_{15}$
I.22	$\pi_1(f) = \langle a, b \mid a^4 = b^4 = (a^{-1} b)^3 \rangle$	$H_1 = \mathbb{Z}_{16}$
I.23	$\pi_1(f) = \langle a, b \mid a^4 = b^4 = (ab)^{-2} \rangle$	$H_1 = \mathbb{Z}_{16} \times \mathbb{Z}_2$
I.24	$\pi_1(f) = \langle a, b \mid a^5 = b^4 = (a^{-1} b)^2 \rangle$	$H_1 = \mathbb{Z}_{18}$
I.25	$\pi_1(f) = \langle a, b \mid a^2 = b^2 a b^2, a^3 = b a^{-3} b^3 \rangle$	$H_1 = \mathbb{Z}_{20}$
I.26	$\pi_1(f) = \langle a, b \mid a^5 = b^3 = (ab^{-1})^3 \rangle$	$H_1 = \mathbb{Z}_{21}$
I.27	$\pi_1(f) = \langle a, b \mid a^5 = b^4 = (ab^{-1})^2 \rangle$	$H_1 = \mathbb{Z}_{22}$
I.28	$\pi_1(f) = \langle a, b \mid a^6 = b^3 = (ab^{-1})^2 \rangle$	$H_1 = \mathbb{Z}_{24}$
I.29	$\pi_1(f) = \langle a, b \mid a = b^2 a^2 b^2, b = a^3 b a^3 \rangle$	$H_1 = \mathbb{Z}_{24}$
I.30	$\pi_1(f) = \langle a, b \mid a^{-3} = b a^3 b^{-1}, a^4 = b^2 a^{-1} b^2 \rangle$	$H_1 = \mathbb{Z}_{24}$
I.31	$\pi_1(f) = \langle a, b \mid a^4 = b^3 = (ab)^{-3} \rangle$	$H_1 = \mathbb{Z}_{33}$
I.32	$\pi_1(f) = \langle a, b \mid a^2 b a^{-1} = b^{-1} a b^2, b^{-1} a^2 b^{-1} = a b^{-2} a \rangle$	$H_1 = \mathbb{Z}$
I.33	$\pi_1(f) = \langle a, b \mid a^3 = b^{-1} a b^2 a b^{-1}, a b^{-1} a b = b^{-1} a^{-1} b a^{-1} \rangle$	$H_1 = \mathbb{Z}$
I.34	$\pi_1(f) = \langle a, b \mid a^3 = b^2 a^{-1} b^2, b^3 = a^2 b^{-1} a^2 \rangle$	$H_1 = \mathbb{Z} \times \mathbb{Z}_4$

I.35 $\pi_1(f) = \langle a, b \mid ab^{-2}a^2 = b^2a^{-2}b, ba^{-2}b = a^{-1}b^2a^{-1} \rangle \qquad H_1 = \mathbb{Z} \times \mathbb{Z}_5.$

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Received September 2, 2004

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