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TWO VARIATIONS OF FRANKLIN'S THEOREM

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ABSTRACT. P. Franklin proved that each 3-connected plane cubic graph of minimum face size 5 contains a pentagon adjacent to two faces of size at most 6. In this paper we strengthen this theorem proving that there exists a pentagon adjacent to two faces of size at most 6 such that every vertex of these three faces is of degree at most 23; we prove also that there exists a triple of faces with size sum at most 17 such that they form an induced path in the dual graph.

1. Introduction

Throughout this paper, we consider connected plane graphs without loops or multiple edges. For a plane graph G, let $V(G), E(G)$ and $F(G)$ denote its set of vertices, edges and faces, respectively. The *size* of a face $\alpha \in F(G)$ (denoted by $\deg(\alpha)$) is the number of edges incident with α (the cut-edges are counted twice); a face of size k is called a k*-face*, a face of size $\geqslant k$ ($\leqslant k$) is called an $\geqslant k$*-face* (an $\leqslant k$-face). Similarly, a vertex of degree d is called a d*-vertex* (analogically for $\leqslant d$- and $\geqslant d$-vertex). A k-vertex path is a k*-path* P_k.

In [3], P. Franklin (with the connection to the Four Color Problem) proved that each 3-connected plane cubic graph of minimum face size 5 contains a 5-face adjacent to two $\leqslant 6$-faces. This theorem inspired many results on the structure of plane graphs, see [7], [1], [4], [5], [8]. This paper concentrates on the following variations which are not provided by the original theorem:

- to provide additional information about degrees of the vertices of three faces involved in the theorem
- to prove an analogy which would concern a chain of three faces of bounded size forming an induced path in the dual graph

The results are contained in the next two theorems:

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Theorem 1. *Each 3-connected plane graph G of minimum face size 5 contains a 5-face adjacent to two $\leqslant$ 6-faces such that every vertex incident with these faces is of degree $\leqslant 23$.*

Theorem 2. *Each 2-connected plane graph G of minimum degree at least 3 and minimum face size 5 contains a face α adjacent to two faces β, γ such that $\deg(\alpha)+\deg(\beta)+\deg(\gamma) \leqslant 17$ and α, β, γ form an induced path in the dual graph G^*. The bound 17 is the best possible.*

2. Proof of Theorem 1

The proof is by contradiction. Suppose that there exists a 3-connected plane graph G of minimum face size 5 such that each of its 5-faces is adjacent to at most one $\leqslant$ 6-face, or, if there exists a 5-face being adjacent to two $\leqslant$ 6-faces, then these three faces contain a vertex of degree $\geqslant 24$ (called *big* in the sequel). For the purposes of the proof, a triple of faces consisting of a 5-face adjacent to two $\leqslant$ 6-faces is called a *Franklin-chain*.

The following equation is an easy consequence of the Euler formula:

$$\sum_{v \in V(G)} (2\deg(v) - 6) + \sum_{f \in F(G)} (\deg(f) - 6) = -12\,.$$

According to this formula, assign to each vertex $v \in V(G)$ of the graph G the *initial charge* $\mu(v) = 2\deg(v) - 6$ and to each face $f \in F(G)$ the initial charge $\mu(f) = \deg(f) - 6$. Thus $\sum\limits_{x \in V(G) \cup F(G)} \mu(x) = -12$.

Now, the initial charges of vertices and faces are redistributed locally such that their total sum remains the same. This is done according to the following rules (they are executed strictly according to their order):

Rule 1. Each big d-vertex y sends $\frac{2d-6}{m(y)}$ to each incident 5- or 6-face; $m(y)$ is the number of 5- and 6-faces incident with y. If $m(y) = 0$, no charge is transferred.

Rule 2. Each d-vertex y, $3 \leqslant d \leqslant 23$ sends $\frac{2d-6}{m'(y)}$ to each incident 5-face whose charge is still negative after application of Rule 1; $m'(y)$ is the number of such faces incident with y. If $m'(y) = 0$, no charge is transferred.

Denote by $\widehat{\mu}(f)$ the charge of the face f after application of Rules 1 and 2. A face f is called *overcharged*, if $\widehat{\mu}(f) > 0$, and *undercharged*, if $\widehat{\mu}(f) < 0$.

Rule 3. Each overcharged face f sends $\frac{\widehat{\mu}(f)}{m(f)}$ to each undercharged adjacent 5-face; $m(f)$ is the number of undercharged 5-faces adjacent to f. If $m(f) = 0$, no charge is transferred.

Let $\mu^*(x)$ be the charge of $x \in V(G) \cup F(G)$ after application of Rules 1, 2 and 3. We show that the $\mu^*(x) \geqslant 0$ for each $x \in V(G) \cup F(G)$, which is a contradiction. It is easy to see that $\mu^*(v) \geqslant 0$ for every $v \in V(G)$ and $\mu^*(f) \geqslant 0$ for every $f \in F(G)$, $\deg(f) \geqslant 6$ (notice that $\widehat{\mu}(f) > 0 \Rightarrow \mu^*(f) \geqslant 0$). Therefore, we examine only the case of 5-faces. Denote by $v_1, \dots, v_5$ the vertices incident with a 5-face f in the cyclical order, and denote by f_i, $i = 1, \dots, 5$ the face incident with the edge $v_i v_{i+1}$ (index modulo 5). Since G is 3-connected, it follows that $f_i \neq f_j$ for $i \neq j$.

The following lemmas will be useful in the analysis of 5-faces:

Lemma 2.1. *Each $\geqslant 7$-face g sends $\geqslant \frac{1}{4}$ by Rule 3 to each adjacent undercharged 5-face.*

Proof. The lemma is true for an $\geqslant 8$-face, since the rough estimation yields that g sends by Rule 3 $\geqslant \frac{8-6}{8} = \frac{1}{4}$ to each neighbouring 5-face. Thus, suppose that g is a 7-face; let g_i, $i = 1, \dots, 7$ be faces adjacent to g and let x_i be the vertex incident with g, g_i and g_{i+1} (indices are taken modulo 7). If $m(g) \leqslant 4$, then the lemma is also true, hence, suppose $m(g) \geqslant 5$. This implies that there exists at least one $j \in \{1, \dots, 7\}$ such that g_j, g_{j+1}, g_{j+2} are undercharged 5-faces.

If x_j (or x_{j+1}) is an $\geqslant 6$-vertex, then g_j and g_{j+1} (or g_{j+1} and g_{j+2}) receive $\geqslant 1$ from it by Rule 2, hence, they are not undercharged. If x_j (or x_{j+1}) is a 5-vertex, then it also sends $\geqslant \frac{2\cdot 5-6}{4} = 1$ to g_j and g_{j+1} (or g_{j+1} and g_{j+2}), because $m'(x_j) \leqslant 4$ (similarly for x_{j+1}). Thus, suppose that one of x_j, x_{j+1}— say, x_j—is a 4-vertex; let h be the fourth (i.e., different from g, g_j, g_{j+1}) face incident with x_j. Since g_j, g_{j+1} are undercharged 5-faces, h is either a 7-face, or a 6-face, or a 5-face incident with a big vertex (to avoid a light Franklin--chain); hence, h receives no charge from x_j, which subsequently sends 1 to g_j and g_{j+1}. Finally, if both g_j and g_{j+1} are 3-vertices, then g_j, g_{j+1}, g_{j+2} form a Franklin-chain, therefore, at least one of them is incident with a big vertex.

In each of these cases, we obtain that at least one of g_j, g_{j+1}, g_{j+2} is not undercharged after Rules 1 and 2, a contradiction. □

Lemma 2.2. *Each $\leqslant 6$-face g which is incident with a big vertex sends $\geqslant \frac{1}{4}$ by Rule 3 to each adjacent undercharged 5-face.*

Proof. If g is a 5-face incident with a big vertex, then $\widehat{\mu}(g) \geqslant -1 + \frac{2\cdot 24-6}{24} = \frac{3}{4}$ and $m(g) \leqslant 3$. Thus, g can send $\geqslant \frac{\frac{3}{4}}{3} = \frac{1}{4}$ to each adjacent undercharged 5-face. If g is a 6-face incident with a big vertex, then $\widehat{\mu}(g) \geqslant 0 + \frac{2\cdot 24-6}{24} = \frac{7}{4}$ and $m(g) \leqslant 4$. Hence, g can send $\geqslant \frac{\frac{7}{4}}{4} = \frac{7}{16} > \frac{1}{4}$ to each adjacent undercharged 5-face. □

If a 5-face f is incident with a $\geqslant 6$-vertex, or with at least two $\geqslant 4$-vertices, then $\widehat{\mu}(f) \geqslant -1 + \frac{2\cdot 6-6}{6} = 0$ or $\widehat{\mu}(f) \geqslant -1 + 2\cdot\frac{2\cdot 4-6}{4} = 0$. Thus, we consider three cases:

Case 1: f is incident with a 5-vertex v_1, the remaining vertices of f are of degree 3. We may suppose that all faces incident with v_1 are 5-faces (otherwise $m'(v_1) \leqslant 4$ and $\widehat{\mu}(f) \geqslant -1 + \frac{2\cdot 5-6}{4} = 0$). Then—to avoid light Franklin-chain—one of f_1, f_5—say, f_1—is incident with a big vertex v; we have $\widehat{\mu}(f_1) \geqslant -1 + \frac{2\cdot 24-6}{24} = -1 + \frac{7}{4} = \frac{3}{4} > 0$. Also, $m(f_1) \leqslant \deg(f_1) - 2$ (due to the presence of v) and $m'(v_1) \leqslant 4$. Then—counting just sole contribution of v_1 and f_1 to $f - \mu^*(f) \geqslant -1 + \frac{2.5-6}{4} + \frac{\frac{3}{4}}{3} = \frac{1}{4} > 0$.

Case 2: f is incident with a 4-vertex v_1, the remaining vertices of f are of degree 3. We may suppose that at most one face incident with v_1 is $\geqslant 7$-face (otherwise $m'(v_1) \leqslant 2$ and $\widehat{\mu}(f) \geqslant -1 + \frac{2\cdot 4-6}{2} = 0$).

Case 2.1: f_1 and f_5 are $\leqslant 6$-faces. Then at least one of them—say, f_1—is incident with a big vertex v; we have $\widehat{\mu}(f_1) \geqslant -1 + \frac{2\cdot 24-6}{24} = \frac{3}{4} > 0$. Also, $m(f_1) \leqslant \deg(f_1) - 2$ (due to the presence of v). Counting just sole contribution of v_1 and f_1 to f, we have $\mu^*(f) \geqslant -1 + \frac{2\cdot 4-6}{3} + \frac{\frac{3}{4}}{3} = \frac{-1}{12}$. To supply the missing $\frac{1}{12}$, consider the faces f_4, f_5. If f_5 is incident with a big vertex, then $\hat{\mu}(f_5) \geqslant -1 + \frac{2\cdot 24-6}{24} = \frac{3}{4} > 0$ and $m(f_5) \leqslant \deg(f_5) - 2$, which gives the contribution of $\geqslant \frac{3}{12}$ from f_5 to f. In the opposite case, either f_4 is a $\geqslant 7$-face (giving a contribution $\geqslant \frac{7-6}{7}$ to f), or it is a $\geqslant 6$-face necessarily incident with a big vertex; then $\hat{\mu}(f_4) \geqslant -1 + \frac{2\cdot 24-6}{24} = \frac{3}{4}$ and $m(f_4) \leqslant \deg(f_4) - 2$, which gives a contribution $\geqslant \frac{1}{4}$ to f. Hence, $\mu^*(f) > 0$.

Case 2.2: f_1 is a $\geqslant 7$-face. Then $\mu^*(f) \geqslant -1 + \frac{2\cdot 4-6}{3} + \frac{7-6}{7} = \frac{-4}{21}$. To supply this missing charge, consider the faces f_2, f_3, f_5. If f_2 and f_3 are $\geqslant 7$-faces, then they contribute $\geqslant 2\cdot\frac{7-6}{6} = \frac{1}{3} > \frac{4}{21}$. In the opposite case at least one of them is $\leqslant 6$-face, hence, at least one of f_5, f_2, f_3 is incident with a big vertex. This face then contributes $\geqslant \frac{-1+\frac{2\cdot 24-6}{24}}{3} = \frac{1}{4} > \frac{4}{21}$ to f. Thus, $\mu^*(f) > 0$.

Case 3: All vertices incident with f are 3-vertices.

Case 3.1: All f_i, $i = 1, \ldots, 5$ are $\geqslant 7$-faces. Then each of them sends $\geqslant \frac{1}{4}$ to f, hence, $\mu^*(f) \geqslant -1 + 5\cdot\frac{1}{4} > 0$.

Case 3.2: There exists an $\leqslant 6$-face adjacent to f. If at least one such a face is not incident with a big vertex, then each of the remaining faces is either an $\geqslant 7$-face, or an $\leqslant 6$-face incident with a big vertex. Thus, each of these remaining faces sends $\geqslant \frac{1}{4}$ to f and $\mu^*(f) \geqslant -1 + 4\cdot\frac{1}{4} = 0$. In the opposite case, each $\leqslant 6$-face adjacent to f is incident with a big vertex, thus, sends $\geqslant \frac{1}{4}$ to f; together with the contributions from (possible) $\geqslant 7$-faces adjacent to f, it gives $\mu^*(f) \geqslant -1 + 5\frac{1}{4} > 0$.

3. Proof of Theorem 2

We prove the dual form of theorem, that is, the existence of an induced 3-path abc with $\deg(a)+\deg(b)+\deg(c) \leqslant 17$ in each connected plane graph of minimum degree 5. The proof is again by contradiction. Suppose that there exists a connected plane graph G of minimum degree 5 such that for each induced 3-path $P_3 \in G$ the sum of degrees of its vertices is $\geqslant 18$. For the purposes of the proof, a vertex x is called *diagonally incident* with a face α if there exists a 3-face $[xyz]$ such that the edge yz belongs to the boundary of α. A face of degree at least 4 is called *big*.

We use the discharging method. First, we assign to each vertex $v \in V(G)$ the initial charge $\mu(v) = \deg_G(v) - 6$ and to each face $\alpha \in F(G)$ the initial charge $\mu(\alpha) = 2\deg_G(\alpha) - 6$. By a consequence of the Euler theorem, $\sum_{x \in V(G) \cup F(G)} \mu(x) = -12$.

Next, the initial charges of the vertices and the faces of G are redistributed locally such that their total sum remains the same. This is performed by the following rules:

Rule 1: Each 7-vertex v sends $\frac{1}{m(v)}$ to each adjacent 5-vertex, where $m(v)$ is the number of 5-vertices adjacent to v. If $m(v)=0$, no charge is transferred.

Rule 2: Each k-vertex, $k \geqslant 8$, sends $\frac{k-6}{k}$ to each adjacent 5-vertex.

Rule 3: Let $[xyz]$ be a 3-face of G, let x be an $\geqslant 8$-vertex, y a 5-vertex and z a $\geqslant 6$-vertex. Then x sends additional $\frac{k-6}{2k}$ to y.

Rule 4: Each big face α sends $\frac{1}{2}$ to each incident 5-vertex. The remaining positive charge of α is then equally distributed among all 5-vertices diagonally incident with α.

Let $\overline{\mu}(v)$ be the charge of a vertex v after application of Rules 1–4.

Rule 5: The charge $\overline{\mu}(v)$ of each 5-vertex v with $\overline{\mu}(v) > 0$ is equally distributed among all its 5-neighbours whose charge after application of Rules 1–4 is negative.

We will show that after redistribution of the charges, the new charge assignment $\widetilde{\mu} : V(G) \cup F(G) \to \mathbb{Q}$ is a non-negative function, which is a contradiction. It is easy to see that for each 7-vertex v, by the Rule 1, we have $\widetilde{\mu}(v) \geqslant 0$; moreover, $m(v) \leqslant 3$ (otherwise there exist four 5-neighbours of v. Denoting them in the cyclical order around v by $v_1, \ldots, v_4$, there must be edges v_1v_3 and v_2v_4 in order to avoid light induced 3-paths, but this violates the planarity of G) which implies that each 7-vertex sends $\geqslant \frac{1}{3}$ to its 5-neighbour. For each k-vertex y, $k \geqslant 8$ there is $\widetilde{\mu}(y) \geqslant 0$ too (from the Rules 2 and 3; note that each

$\geqslant 8$-vertex can send $\geqslant \frac{1}{4}$ by Rule 2 and $\geqslant \frac{1}{8}$ by Rule 3). Also, for a big face α, $\widetilde{\mu}(\alpha) \geqslant 2 \cdot 4 - 6 - 4 \cdot \frac{1}{2} = 0$. Moreover, the only case when a big face has zero charge after the first phase of the Rule 4, is the case of 4-face incident only with 5-vertices. Thus, every 5-vertex diagonally incident with a big face consisting not only of 5-vertices may receive $\geqslant \frac{2 \cdot 4 - 6 - 3 \cdot \frac{1}{2}}{4} = \frac{1}{8}$, and it may receive $\geqslant \frac{1}{4}$ if it is incident with at least two $\geqslant 6$-vertices). Hence, it is enough to perform the analysis of 5-vertices.

Case 1: Let v be incident with at least two big faces. Then $\widetilde{\mu}(v) \geqslant -1 + 2 \cdot \frac{1}{2} = 0$ by Rule 4.

Case 2: Let v be incident with exactly one big face. If v is adjacent to at least two $\geqslant 7$-vertices, then $\widetilde{\mu}(x) \geqslant -1 + \frac{1}{2} + 2 \cdot \frac{8-6}{8} = 0$. Otherwise, at least four neighbours of v – say, $v_j, v_{j+1}, v_{j+2}, v_{j+3}$—are of degree at most 6. Then there must exist edges $v_j v_{j+2}$ and $v_{j+1} v_{j+3}$ (in order to avoid light induced paths $v_j v v_{j+2}$ and $v_{j+1} v v_{j+3}$), but this contradicts the planarity of G.

Case 3: Let v be incident only with 3-faces. Denote $v_1, \dots, v_5$ the neighbours of v. Also, denote by γ, ζ, η and ξ the faces incident with the edges $v_1 v_2, v_2 v_3, v_3 v_4$ and $v_4 v_5$, respectively, which are not incident with v.

Case 3.1: Let v be adjacent to at least three $\geqslant 7$-vertices. An easy calculation shows that these vertices send together to v a charge $\geqslant 1$ by Rules 1, 2 and 3, except of the following cases:

- **a**) v_j, v_{j+1} are 7-vertices and v_{j+3} is an 8-vertex, for some $j \in \{1, \dots, 5\}$
- **b**) v_j is a 7-vertex and v_{j+1}, v_{j+3} are 8-vertices.

In the former case, observe that at least one of v_{j+2}, v_{j+4} is $\geqslant 6$-vertex (otherwise, in order to avoid light 3-paths $v_j v v_{j+2}$ and $v_{j+1} v v_{j+4}$, there must exist edges $v_j v_{j+2}$ and $v_{j+1} v_{j+4}$, which contradicts the planarity of G); thus, by Rules 1, 2 and 3, $\overline{\mu}(v) \geqslant -1 + 2 \cdot \frac{1}{3} + \frac{8-6}{8} + \frac{8-6}{16} > 0$. In the latter case, we can assume that v_{j+2}, v_{j+4} are 5-vertices (otherwise $\overline{\mu}(x) \geqslant -1 + \frac{1}{3} + 2 \cdot \frac{8-6}{8} + 2 \cdot \frac{8-6}{16} > 0$ by Rules 1, 2 and 3). Then there must exist the edges $v_j v_{j+2}$ and $v_{j+2} v_{j+4}$; from this follows that v_{j+2} is incident with at least two big faces and $\overline{\mu}(v_{j+2}) \geqslant -1 + 2 \cdot \frac{1}{2} + 2 \cdot \frac{8-6}{8} = \frac{1}{2}$ by Rules 2 and 4. Then, by Rules 1, 2 and 5, $\widetilde{\mu}(v) \geqslant -1 + \frac{1}{3} + 2 \cdot \frac{8-6}{8} + \frac{\frac{1}{2}}{2} > 0$.

Case 3.2: Let v be adjacent to two $\geqslant 7$-vertices such that they share no common 3-face with v; let these vertices be v_1, v_4. Then there must exist the edges $v_2 v_5, v_3 v_5$.

Case 3.2.1: Suppose first that v_5 is a 5-vertex. Then the edge $v_1 v_5$ belongs to a big face α and the edge $v_4 v_5$ belongs to a big face ξ. This gives $\overline{\mu}(v_5) \geqslant -1 + 2 \cdot \frac{1}{2} + 2 \cdot \frac{8-6}{8} = \frac{1}{2}$. Note also that α, ξ send to v by the second part of Rule 4 at least

$\frac{1}{8}$. If at least one of v_2, v_3 is a 6-vertex, then $\widetilde{\mu}(v) \geqslant -1+2\cdot\frac{1}{3}+\frac{-1+2\cdot\frac{1}{2}+2\cdot\frac{1}{3}}{3}+2\cdot\frac{1}{8} > 0$ (if both v_1, v_4 are 7-vertices), or $\widetilde{\mu}(v) \geqslant -1+\frac{1}{3}+\frac{8-6}{8}+\frac{-1+2\cdot\frac{1}{2}+\frac{1}{3}+\frac{8-6}{8}}{3}+2\cdot\frac{1}{8} = \frac{37}{36} > 0$ (if one of them is a 7-vertex and another one is an $\geqslant$ 8-vertex) or $\widetilde{\mu}(v) \geqslant -1+2\cdot\frac{8-6}{8}+\frac{8-6}{16}+\frac{-1+2\cdot\frac{1}{2}+\frac{1}{2}}{3}+2\cdot\frac{1}{8} > 0$ (if both v_1, v_4 are $\geqslant$ 8-vertices). In the opposite case, both v_2, v_3 are 5-vertices; then v_1, v_4 must be $\geqslant$ 8-vertices.

Let z be the vertex different from v_1, v, v_3, v_5 which is incident with v_2 and, similarly, let y be the vertex different from v_2, v, v_4, v_5 which is incident with v_3. We will distinguish several subcases according to the position of y and z.

a) z is inside the 3-cycle $[v_2vv_5]$, y is inside the 3-cycle $[v_3vv_5]$. Then y, z are both $\geqslant$ 8-vertices. Since y is incident with ξ and z is incident with α, by Rules 1 and 4 we get $\overline{\mu}(v) \geqslant -1+2\cdot\frac{8-6}{8}+2\cdot\frac{2-2\cdot\frac{1}{2}}{4} = 0$.

b) z is outside the 3-cycle $[v_2vv_5]$, y is outside the 3-cycle $[v_3vv_5]$. Then the face α contains both edges v_1v_2, v_1v_5 and the face ξ contains both edges v_4v_3 and v_4v_5. By Rules 1 and 4, $\overline{\mu}(v) \geqslant -1+2\cdot\frac{8-6}{8}+4\cdot\frac{2-2\cdot\frac{1}{2}}{4} > 0$.

c) z is outside the 3-cycle $[v_2vv_5]$, y is inside the 3-cycle $[v_3vv_5]$ (or vice versa). Then at least three of the edges $v_1v_2, v_1v_5, v_4v_3, v_4v_5$ are incident with a big face (in fact, α or ξ) and, as in previous subcase, $\overline{\mu}(v) \geqslant -1+2\cdot\frac{8-6}{8}+2\cdot\frac{2-2\cdot\frac{1}{2}}{4}+\frac{2-3\cdot\frac{1}{2}}{4} > 0$.

Case 3.2.2: Suppose now that v_5 is a 6-vertex. Then at least one of the edges v_5v_1, v_5v_4 belongs to a big face. If both v_1, v_4 are $\geqslant$ 8-vertices, then $\overline{\mu}(v) \geqslant -1+2\cdot\frac{8-6}{8}+2\cdot\frac{8-6}{16}+2\cdot\frac{2-2\cdot\frac{1}{2}}{4} > 0$; hence, at least one of them has to be a 7-vertex, say v_1. Then v_2, v_3 are necessarily 6-vertices and we get $\overline{\mu}(v) \geqslant -1+2\cdot\frac{1}{3}+\frac{2-\frac{1}{2}}{4} > 0$.

Case 3.3: Let v be adjacent to two $\geqslant$ 7-vertices such that they share a common 3-face with v; let these vertices be v_1, v_5. Then there must exist the edge v_2v_4.

Case 3.3.1: Let v_1, v_5 be 7-vertices. Then $v_1vv_4, v_1vv_3, v_5vv_3$ are induced 3-paths (otherwise the planarity of G is violated), which implies that v_3 is a 6-vertex and at least one of v_2, v_4 is a 6-vertex, say v_4. Now, if there exists a 5-neighbour z of v_1 such that $z \neq v, v_2$, then zv_1v is an induced 3-path of weight 17, a contradiction. Hence, $m(v_1) \leqslant 2$. The similar argument holds also for v_5; we get $\widetilde{\mu}(v) \geqslant -1+2\cdot\frac{1}{2} = 0$.

Case 3.3.2: Let v_1 be a 7-vertex and v_5 be an $\geqslant$ 8-vertex. Suppose first that v_4 is a 5-vertex; then there must be an edge v_1v_4 and the edges v_3v_4, v_4v_5 are incident with big faces η, ξ. Thus, we have $\widetilde{\mu}(v) \geqslant -1+\frac{1}{3}+\frac{8-6}{8}+\frac{8-6}{16}+2\cdot\frac{1}{8}+\frac{-1+2\cdot\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}{2} = \frac{30}{24} > 0$. So, let v_4 be a 6-vertex. Then again, by the argumentation of the previous case, v_1 may have only two 5-neighbours and we have $\overline{\mu}(v) \geqslant -1+\frac{1}{2}+\frac{8-6}{8}+2\cdot\frac{98-6}{16} = 0$.

Case 3.3.3: Let v_1, v_5 be $\geqslant$ 8-vertices. If both v_2, v_4 are 6-vertices, then $\overline{\mu}(v) \geqslant -1+2\cdot\frac{1}{4}+4\cdot\frac{1}{8} = 0$. Similarly, if both γ, ξ are big, then $\overline{\mu}(v) \geqslant -1+2\cdot\frac{1}{4}+2\cdot\frac{1}{8}+2\cdot\frac{1}{8} = 0$. Thus, up to the symmetry, we can consider the following possibilities:

a) v_2 is a 5-vertex and γ is a 3-face, $\gamma = [v_1 v_2 z]$, $z \neq v_5, v_3$. Then vv_2z is an induced 3-path, hence z is an $\geqslant$ 8-vertex. Note that ζ is now a big face. Let y be a vertex adjacent to v_3 such that yv_3, v_3v_2 are two consecutive edges on ζ. Then y is an $\geqslant$ 7-vertex. If the edge v_3v_4 is incident with ζ, then $\overline{\mu}(v) \geqslant -1 + 2\cdot\frac{1}{4} + 2\cdot\frac{1}{8} + 2\cdot\frac{2-3\cdot\frac{1}{2}}{4} = 0$; hence, we can suppose that there is an other neighbour of v_4 inside the 3-cycle $v_2v_3v_4$ (it may be y). Now, if v_4 is a 5-vertex, then consider the face ξ. Since ξ cannot be a 3-face (otherwise $z = v_5$), we get $\overline{\mu}(x) \geqslant -1 + 2\cdot\frac{1}{4} + 2\cdot\frac{1}{8} + 2\cdot\frac{1}{8} = 0$. So suppose that v_4 is a 6-vertex. Then $\overline{\mu}(x) \geqslant -1 + 2\cdot\frac{1}{4} + 2\cdot\frac{1}{8} + \frac{1}{4} = 0$.

b) v_2 is a 5-vertex and γ is a 3-face, $\gamma = [v_1v_2v_4]$. If ξ is a big face, then $\overline{\mu}(v) \geqslant -1+2\cdot\frac{1}{4}+2\cdot\frac{1}{8}+\frac{1}{4} = 0$; hence, suppose that it is a 3-face. Then its third vertex y is different from v_1, which gives that v_4 is a 6-vertex and the face η incident with v_3v_4, $\eta \neq [v_3vv_4]$ is big. Moreover, the neighbour w of v_2 which is different from v_1, v, v_3, v_4 is an $\geqslant$ 8-vertex (see the induced 3-path vv_2w). This gives $\overline{\mu}(v) \geqslant -1 = 2\cdot\frac{1}{4}+2\cdot\frac{1}{8}+\frac{1}{4} = 0$.

c) v_2 is a 5-vertex and ξ is a 3-face, $\xi = [v_4v_5z]$, $z \neq v_1, v_2$. Consider the vertex v_4. If it is a 5-vertex, then the face η is big; moreover, it is incident with at least one $\geqslant$ 7-vertex (since every neighbour of v_3 different from v, v_2, v_4 is an $\geqslant$ 7-vertex). If it is incident with at least two $\geqslant$ 6-vertices or if it is an $\geqslant$ 5-face, then it sends $\geqslant \frac{1}{4}$ to v and $\overline{\mu}(v) \geqslant -1+2\cdot\frac{1}{4}+2\cdot\frac{1}{8}+\frac{1}{4} = 0$. Thus, suppose that it is a 4-face incident with exactly three 5-vertices v_2, v_3, v_4. Then the face γ is not a 3-face, hence, it is big and sends $\geqslant \frac{1}{8}$ to v; we have $\overline{\mu}(v) \geqslant -1+2\cdot\frac{1}{4}+2\cdot\frac{1}{8}+2\cdot\frac{1}{8} = 0$.

Suppose now that v_4 is a 6-vertex. We can assume that the face η is a 3-face (otherwise it is big and incident with at least two $\geqslant$ 6-vertices: v_4 and an $\geqslant$ 7-neighbour of v_3). Now γ is necessarily a big face incident with at least two $\geqslant$ 6-vertices (v_1 and v_4), thus, $\overline{\mu}(v) \geqslant -1 + 2\cdot\frac{1}{4} + 2\cdot\frac{1}{8} + \frac{1}{4} = 0$.

d) v_2 is a 5-vertex and ξ is a 3-face, $\xi = [v_4v_5v_2]$. Then the face γ is big and we get $\overline{\mu}(v) \geqslant -1 + 2\cdot\frac{1}{4} + 2\cdot\frac{1}{8} + \frac{1}{4} = 0$.

4. Concluding remarks

1. In [8], the following problem was formulated:

Problem 4.1. *Find the maximum number* n_0 *and the minimum numbers* $k(n_0)$, $t(n_0)$ *with the property that each plane graph of minimum face size 5 contains a chain (that is, the path in the dual graph) of* n_0 *faces of size* $\leqslant k(n_0)$ *such that each of their vertices is of degree at most* $t(n_0)$. *Determine the corresponding values* $k(n)$, $t(n)$ *for each integer* $n \leqslant n_0$.

In the same paper it was proved that $k(2) = 6$, $6 \leqslant t(2) \leqslant 9$ and that $t(n) < +\infty$ implies $n \leqslant 6$. Theorem 1 provides $k(3) = 6$, $t(2) \leqslant 23$. We believe that both bounds 9 and 23 can be substantially reduced and that for $n \in \{4, 5, 6\}$ the value $t(n)$ is finite.

2. In [6], the following definition was introduced:

Let $\mathcal{H}$ be a family of graphs and let H be a connected graph such that infinitely many members of $\mathcal{H}$ contain a subgraph isomorphic to H. Let $\varphi(H, \mathcal{H})$ be the smallest integer with the property that each graph $G \in \mathcal{H}$ which contains a subgraph isomorphic to H, contains also a subgraph $K \cong H$ such that, for every vertex $v \in K$,

$$\deg_G(v) \leqslant \varphi(H, \mathcal{H}).$$

Similarly, let $w(H, \mathcal{H})$ be the smallest integer with the property that each graph $G \in \mathcal{H}$ which contains a subgraph isomorphic to H, contains also a subgraph $K \cong H$ such that, for every vertex $v \in K$,

$$\sum_{v \in V(K)} \deg_G(v) \leqslant w(H, \mathcal{H}).$$

If such a finite $\varphi(H, \mathcal{H})$ does not exist, we write $\varphi(H, \mathcal{H}) = +\infty$ (similarly for $w(H, \mathcal{H})$). We say that the graph H is *light* in the family $\mathcal{H}$ if $\varphi(H, \mathcal{H}) < +\infty$ (or, equivalently, $w(H, \mathcal{H}) < +\infty$).

By Theorem 1, the following graphs (and their subgraphs) are light in the family $\mathcal{P}_3(3, 5)$ of 3-connected plane graphs of minimum face size 5 (since they are subgraphs of every of the nine possible Franklin-chains):

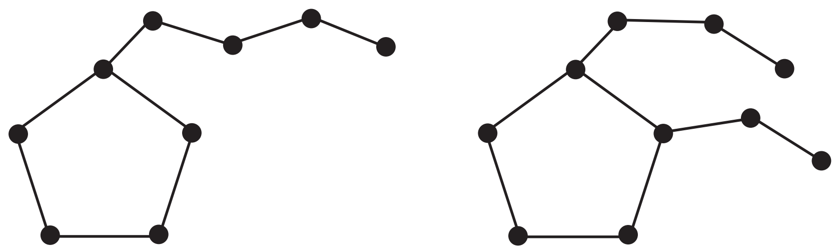

Figure 1. Some light graphs in $\mathcal{P}_3(3, 5)$.

3. Motivated by Theorem 2, the definition of light graphs in a family can be modified to cover the induced subgraphs:

Let $\mathcal{H}$ be a family of graphs and let H be a connected graph such that infinitely many members of $\mathcal{H}$ contain an induced subgraph isomorphic to H. Let $\varphi_I(H,\mathcal{H})$ be the smallest integer with the property that each graph $G \in \mathcal{H}$ which contains an induced subgraph isomorphic to H, contains also an induced subgraph $K \cong H$ such that, for every vertex $v \in K$,

$$\deg_G(v) \leqslant \varphi_I(H,\mathcal{H})$$

(The parameter $w_I(H,\mathcal{H})$ is defined analogically) If such a finite $\varphi_I(H,\mathcal{H})$ does not exist, we write $\varphi_I(H,\mathcal{H}) = +\infty$ (similarly for $w_I(H,\mathcal{H})$). We say that the graph H is *induced-light* in the family $\mathcal{H}$ if $\varphi_I(H,\mathcal{H}) < +\infty$ (or, equivalently, $w_I(H,\mathcal{H}) < +\infty$).

By Theorem 2, $w_I\big(P_3, \mathcal{P}_3(5,3)\big) = 17$, where $\mathcal{P}_3(5,3)$ stands for the family of 3-connected plane graphs of minimum degree 5. Until now, this is the only exact finite value of w_I for a graph on $\geqslant 3$ vertices. With R. Soták, we recently proved that for the family $\mathcal{P}_3(3,3)$ of all 3-connected plane graphs, the values $\varphi_I\big(P_k, \mathcal{P}_3(3,3)\big)$ are finite for every integer k.

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