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OSCILLATION OF NONLINEAR DELAY INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. Oscillation criteria for the nonlinear delay intgro-differential equations of the form

$$\dot{x}(t) + \int_0^{\tau(t)} p(s)|x(t-s)|^\alpha \operatorname{sgn}\bigl[x(t-s)\bigr]\,\mathrm{d}s = 0, \quad t \ge t_0,$$

where $\alpha > 1$, are developed. Existence of a nonoscillatory solution is also treated.

1. Introduction

The equation

$$\dot{x}(t) + \int_{t-r}^{t} p(t-s)g\bigl(x(s)\bigr)\,\mathrm{d}s = 0$$

was encountered by Ergen [2] in the theory of a circulating fuel nuclear reactor and has been studied extensively by Levin and Nohel [5] and Hale [3]. In this model x is the neutron density. It is also a good model in one dimensional viscoelasticity in which x is the strain and p is the relaxation function. The cognate equations have recently found a variety of applications in several fields of natural sciences.

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The motivation of the problem arose from the above discussion. We consider the nonlinear functional differential equation of the form

$$\dot{x}(t) + \int\limits_{t-\tau(t)}^{t} p(t-s)|x(s)|^{\alpha} \operatorname{sgn}\big[x(s)\big]\,\mathrm{d}s = 0, \qquad t \ge t_0,$$

which can be transformed to the equation

$$\dot{x}(t) + \int\limits_{0}^{\tau(t)} p(s)|x(t-s)|^{\alpha} \operatorname{sgn}\big[x(t-s)\big]\,\mathrm{d}s = 0, \qquad t \ge t_0, \tag{1}$$

where

$$t_0 \in (0,\infty),\ \alpha > 1,\ p \in C\big([0,\infty),[0,\infty)\big),\ \tau \in C\big([t_0,\infty),(0,\infty)\big)$$

is a nondecreasing function, $\tau(t) < t,\ t \ge t_0$.

By a solution of Eq. (1) we mean a continuous function $x(t)$ which satisfies Eq. (1) on the interval $[t_0,\infty)$.

2. Main results

The next lemma will be useful in the following consideration.

Lemma 1. *Suppose that* $\alpha > 1$,

$$\sigma \in C\big([t_0,\infty),(0,\infty)\big),\ \sigma(t) < \tau(t),\ t \ge t_0,\ \lim_{t\to\infty}\big[t-\tau(t)\big] = \infty, \tag{2}$$

$q \in C\big([t_0,\infty),[0,\infty)\big),\ q(t) \not\equiv 0$ *on any interval and*

$$q(t) \le \int\limits_{\sigma(t)}^{\tau(t)} p(s)\,\mathrm{d}s, \qquad t \ge t_0.$$

If Eq. (1) *has a nonoscillatory solution, then also equation*

$$\dot{x}(t) + q(t)|x\big(t-\sigma(t)\big)|^{\alpha} \operatorname{sgn}\big[x(t-\sigma(t))\big] = 0, \qquad t \ge t_0, \tag{3}$$

has a nonoscillatory solution.

Proof. Assume that $v(t)$ is a nonoscillatory solution of Eq. (1) such that $v\big(t-\tau(t)\big) > 0$ on $[T,\infty),\ T > t_0$. Then

$$\dot{v}(t) = -\int\limits_{0}^{\tau(t)} p(s)\big[v(t-s)\big]^{\alpha}\,\mathrm{d}s, \qquad t \ge T, \tag{4}$$

i.e., $v(t)$ is decreasing for $t \geq T$. From (4) it follows that

$$v(t) \geq \int_t^{\infty} \int_0^{\tau(s)} p(\xi) \big[v(s-\xi)\big]^{\alpha} \, \mathrm{d}\xi \, \mathrm{d}s, \qquad t \geq T. \tag{5}$$

By $C_{loc}\big([T,\infty), R\big)$ we denote the space of continuous functions $x : [T,\infty) \to R$ endowed with the topology of local uniform convergence. We define the set $S \subset C_{loc}\big([T,\infty), R\big)$ of functions x which satisfy the inequalities

$$0 \leq x(t) \leq v(t) \qquad \text{for } t \geq T.$$

The operator $F : S \to C_{loc}\big([T,\infty), R\big)$ is defined by

$$F(x)(t) = \begin{cases} \int_t^{\infty} q(s) \big[x(s-\sigma(s))\big]^{\alpha} \, \mathrm{d}s & \text{for} \quad t \geq t_1, \\ v(t) - v(t_1) + F(x)(t_1) & \text{for} \quad t \in [T, t_1), \end{cases}$$

where $t_1 > T$ is such that $t - \sigma(t) \geq T$ for $t \geq t_1$.

If $x \in S$, then using decreasing character of $v(t)$ and (5) we have

$$\begin{aligned} 0 \quad \leq \quad & F(x)(t) = \int_t^{\infty} q(s) \big[x(s-\sigma(s))\big]^{\alpha} \, \mathrm{d}s \\ \leq \quad & \int_t^{\infty} \int_{\sigma(s)}^{\tau(s)} p(\xi) \, \mathrm{d}\xi \, \big[x(s-\sigma(s))\big]^{\alpha} \, \mathrm{d}s \\ \leq \quad & \int_t^{\infty} \int_{\sigma(s)}^{\tau(s)} p(\xi) \, \mathrm{d}\xi \, \big[v(s-\sigma(s))\big]^{\alpha} \, \mathrm{d}s \\ \leq \quad & \int_t^{\infty} \int_{\sigma(s)}^{\tau(s)} p(\xi) \big[v(s-\xi)\big]^{\alpha} \, \mathrm{d}\xi \, \mathrm{d}s \\ \leq \quad & \int_t^{\infty} \int_0^{\tau(s)} p(\xi) \big[v(s-\xi)\big]^{\alpha} \, \mathrm{d}\xi \, \mathrm{d}s \leq v(t), \qquad t \geq t_1. \end{aligned}$$

Thus $F(S) \subset S$. We note that S is a nonempty closed convex subset of $C_{loc}\big([T,\infty), R\big)$ and the operator F is continuous. The functions belonging to the set $F(S)$ are equicontinuous on every compact subinterval of $[T,\infty)$. Then according to the Schauder-Tychonoff fixed point Theorem (cf., e.g., [1, p. 231]),

F has an element $x \in S$ such that $x = F(x)$. It is easy to see that x satisfies equation (3) on $[t_1, \infty)$.

Now we show that x is positive on $[t_1, \infty)$. Obviously $v(t) > v(t_1)$ on $[T, t_1)$, x is nonnegative on $[t_1, \infty)$, $x(t_1) = F(x)(t_1) > 0$ and moreover from equation (3) it follows that x is decreasing on $[t_1, \infty)$. Let $t_2 \in (t_1, \infty)$ be the first point in which $x(t_2) = 0$. Then there exists a $t_3 \in [t_2, \infty)$ such that

$$\dot{x}(t_3) = -q(t_3)\big[x(t_3 - \sigma(t_3))\big]^{\alpha} < 0.$$

By decreasing character of x we always have $x = 0$ on $[t_2, \infty)$, which gives $\dot{x}(t_3) = 0$. This contradiction proves that x has no zeros on $[t_2, \infty)$ and so x is positive on $[t_1, \infty)$. The proof is complete. □

Theorem 1. *Suppose that $\alpha > 1$, (2) holds, $0 \le \dot{\sigma}(t) < 1$,*

$$\liminf_{t\to\infty} \sigma(t) \ge \sigma > 0,$$

there exists $\lambda > 0$ such that

$$\alpha e^{-\lambda\sigma} < 1 \tag{6}$$

and

$$\liminf_{t\to\infty} \left[\int\limits_{\sigma(t)}^{\tau(t)} p(s)\,\mathrm{d}s \, \exp\left(e^{\varepsilon t} - e^{\lambda t}\right)\right] > 0, \tag{7}$$

where $0 < \varepsilon < \lambda$. Then every solution of Eq. (1) oscillates.

P r o o f. According to conditions (6), (7) we may choose $\varepsilon < \lambda_1 < \lambda$ and $T > t_0$ such that

$$\alpha e^{-\lambda_1\sigma} < 1 \tag{8}$$

and due to condition (7) and the next inequality

$$e^{\lambda t} - e^{\varepsilon t} \ge b + \lambda t + (\alpha - 1)e^{\lambda_1 t}, \qquad b > 0,\ t \ge T,$$

we obtain

$$\int\limits_{\sigma(t)}^{\tau(t)} p(s)\,\mathrm{d}s \ge \lambda e^{\lambda t} \exp\left[(\alpha-1)e^{\lambda_1 t}\right], \qquad t \ge T.$$

Define

$$q(t) = \lambda e^{\lambda t} \exp\left[(\alpha-1)e^{\lambda_1 t}\right], \qquad t \ge T.$$

According to Lemma 1, if every solution of the equation

$$\dot{x}(t) + q(t)\big|x\big(t - \sigma(t)\big)\big|^{\alpha} \operatorname{sgn}\big[x(t - \sigma(t))\big] = 0, \qquad t \ge T, \tag{9}$$

oscillates, then also every solution of Eq. (1) oscillates.

Assume to the contrary that Eq. (9) has a nonoscillatory solution $x(t)$. Without loss of generality we may suppose that $x(t)$ is an eventually positive solution of (9). Then with regard to definition of $q(t)$ we obtain that

$$0 < x(t-\sigma(t)) < 1 \quad \text{and} \quad \dot{x}(t) < 0 \qquad \text{for} \quad t \geq T_1 > T.$$

Set

$$y(t) = -\ln x(t), \quad t \geq T_1.$$

Then

$$y(t) > 0 \qquad \text{for} \quad t \geq T_1$$

and has a increasing character. From Eq. (9) it follows that

$$\dot{y}(t) = q(t)\exp\bigl[y(t) - \alpha y(t-\sigma(t))\bigr], \qquad t \geq T_1. \tag{10}$$

With regard to (8) we may choose $0 < \lambda_2 < \lambda_1$ and $0 < \sigma_1 < \sigma$ such that

$$\alpha e^{-\alpha_2\sigma_1} < 1.$$

We will consider three possible cases.
Case 1. $y(t) \leq \alpha e^{(\lambda_1-\lambda_2)\sigma_1}\, y\bigl(t-\sigma(t)\bigr)$ eventually holds.
Let $T_2 > T_1$ be such that

$$y(t) \leq \alpha e^{(\lambda_1-\lambda_2)\sigma_1}\, y\bigl(t-\sigma(t)\bigr), \qquad t \geq T_2.$$

We have

$$\begin{aligned}
\frac{y(t)}{e^{\lambda_1 t}} &\leq \frac{\alpha e^{(\lambda_1-\lambda_2)\sigma_1}\, y\bigl(t-\sigma(t)\bigr)}{e^{\lambda_1 t}} \\
&\leq \frac{\alpha e^{(\lambda_1-\lambda_2)\sigma(t)}\, y\bigl(t-\sigma(t)\bigr)}{e^{\lambda_1 t}} = \frac{\alpha e^{(\lambda_1-\lambda_2)(t-t+\sigma(t))}\, y\bigl(t-\sigma(t)\bigr)}{e^{\lambda_1 t}} \\
&= \alpha e^{-\lambda_2\sigma(t)}\, \frac{y\bigl(t-\sigma(t)\bigr)}{e^{\lambda_1(t-\sigma(t))}} < \alpha e^{-\lambda_2\sigma_1}\, \frac{y\bigl(t-\sigma(t)\bigr)}{e^{\lambda_1(t-\sigma(t))}}, \qquad t \geq T_2.
\end{aligned}$$

Define $z(t) = y(t)e^{-\lambda_1 t}$. Then we have

$$z(t) < \alpha e^{-\lambda_2\sigma_1}\, z\bigl(t-\sigma(t)\bigr), \qquad t \geq T_2.$$

In view of above inequality, applying the result of [4, Lemma 2.1] we obtain

$$\lim_{t\to\infty} z(t) = 0.$$

Then there exists a $T_3 > T_2$ such that

$$y(t) < e^{\lambda_1 t}, \qquad t \geq T_3. \tag{11}$$

Equation (10) implies that

$$\dot{y}(t) \geq q(t)\exp\bigl[(1-\alpha)y(t)\bigr]$$

and using (11) we get

$$\dot{y}(t) \ge q(t)\exp\left[(1-\alpha)e^{\lambda_1 t}\right] = \lambda e^{\lambda t}, \qquad t \ge T_3.$$

Integrating the last inequality we obtain

$$y(t) \ge y(T_3) + e^{\lambda t} - e^{\lambda T_3}, \qquad t \ge T_3,$$

which contradicts (11).

Case 2. $y(t) > \alpha e^{(\lambda_1-\lambda_2)\sigma_1}\, y\bigl(t-\sigma(t)\bigr)$ eventually holds. Choose $T_4 > T_3$ such that

$$y(t) > \alpha e^{(\lambda_1-\lambda_2)\sigma_1}\, y\bigl(t-\sigma(t)\bigr), q \qquad t \ge T_4.$$

Applying this inequality in (10) we get

$$\dot{y}(t) > q(t)\exp\left[\left(1-e^{(\lambda_2-\lambda_1)\sigma_1}\right)y(t)\right], \qquad t \ge T_4.$$

Set $c = 1-e^{(\lambda_2-\lambda_1)\sigma_1}$. Then $c>0$ and

$$\dot{y}(t)e^{-cy(t)} > q(t), \qquad t \ge T_4.$$

Integrating the above inequality we obtain

$$\int_{T_4}^{\infty} q(t)\,\mathrm{d}t < \int_{T_4}^{\infty} \dot{y}(t)e^{-cy(t)}\,\mathrm{d}t \le \frac{1}{c}e^{-cy(T_4)} < \infty,$$

which is a contradiction with definition of $q(t)$.

Case 3. $y(t) - \alpha e^{(\lambda_1-\lambda_2)\sigma_1}\, y\bigl(t-\sigma(t)\bigr)$ is oscillatory. Set

$$u(t) = y(t) - \alpha e^{(\lambda_1-\lambda_2)\sigma_1}\, y\bigl(t-\sigma(t)\bigr).$$

Then $u(t)$ is oscillatory and there exists an increasing infinite sequence $\{t_n\}$ of real numbers with $T_4 < t_1 < t_2 \dots$ such that

$$u(t_n) = 0, \qquad n = 1,2,\dots,$$

and

$$u(t) > 0 \qquad \text{for} \quad t \in (t_{2n-1}, t_{2n}), \qquad n = 1,2,\dots\ .$$

There exists also an increasing infinite sequence $\{\xi_n\}$, $\xi_n \in (t_{2n-1}, t_{2n})$ such that

$$u(\xi_n) = \max\bigl\{u(t) : t_{2n-1} \le t \le t_{2n}\bigr\}$$

and

$$\dot{u}(\xi_n) = 0,\ n = 1,2,\dots .$$

Then

$$\begin{aligned} \dot{y}(t) &= q(t)\exp\left[u(t) + \alpha\left(e^{(\lambda_1-\lambda_2)\sigma_1} - 1\right)y(t-\sigma(t))\right], \quad t \ge T_4, \quad (12)\\ \dot{u}(\xi_n) &= \dot{y}(\xi_n) - \alpha e^{(\lambda_1-\lambda_2)\sigma_1}\bigl(1-\dot{\sigma}(\xi_n)\bigr)\dot{y}\bigl(\xi_n-\sigma(\xi_n)\bigr), \end{aligned}$$

and

$$\dot y(\xi_n) = \alpha e^{(\lambda_1-\lambda_2)\sigma_1}\big(1-\dot\sigma(\xi_n)\big)\dot y\big(\xi_n-\sigma(\xi_n)\big). \tag{13}$$

Combining (12) and (13) we obtain

$$\begin{aligned}
&q(\xi_n)\exp\Big[u(\xi_n)+\alpha\Big(e^{(\lambda_1-\lambda_2)\sigma_1}-1\Big)y(\xi-\sigma(\xi_n))\Big]\\
&= \alpha e^{(\lambda_1-\lambda_2)\sigma_1}\big(1-\dot\sigma(\xi_n)\big)\dot y\big(\xi_n-\sigma(\xi_n)\big)\\
&= \alpha e^{(\lambda_1-\lambda_2)\sigma_1}\big(1-\dot\sigma(\xi_n)\big)q\big(\xi_n-\sigma(\xi_n)\big)\exp\Big[u\big(\xi_n-\sigma(\xi_n)\big)\\
&\quad+\alpha(e^{(\lambda_1-\lambda_2)\sigma_1}-1)y\big(\xi_n-\sigma(\xi_n)-\sigma(\xi_n-\sigma(\xi_n))\big)\Big].
\end{aligned}$$

Using the definition of $q(t)$ and the inequality $-\lambda\sigma(\xi_n) < -\lambda_1\sigma_1$, we obtain

$$\begin{aligned}
&q(\xi_n)\exp\Big[u(\xi_n)+\alpha\Big(e^{(\lambda_1-\lambda_2)\sigma_1}-1\Big)y\big(\xi_n-\sigma(\xi_n)\big)\Big]\\
&< \lambda e^{\lambda_1\sigma_1}e^{\lambda(\xi_n-\sigma(\xi_n))}\exp\Big[(\alpha-1)e^{\lambda_1(\xi_n-\sigma(\xi_n))}+u\big(\xi_n-\sigma(\xi_n)\big)\\
&\quad+\alpha(e^{(\lambda_1-\lambda_2)\sigma_1}-1)y\big(\xi_n-\sigma(\xi_n)-\sigma(\xi_n-\sigma(\xi_n))\big)\Big]\\
&< \lambda e^{\lambda\xi_n}\exp\Big[(\alpha-1)e^{\lambda_1(\xi_n-\sigma(\xi_n))}+u\big(\xi_n-\sigma(\xi_n)\big)\\
&\quad+\alpha(e^{(\lambda_1-\lambda_2)\sigma_1}-1)y\big(\xi_n-\sigma(\xi_n)-\sigma(\xi_n-\sigma(\xi_n))\big)\Big].
\end{aligned}$$

The above inequality implies

$$\begin{aligned}
&\exp\Big[u(\xi_n)+\alpha\Big(e^{(\lambda_1-\lambda_2)\sigma_1}-1\Big)y\big(\xi_n-\sigma(\xi_n)\big)\Big]\\
&< \exp\Big[-(\alpha-1)e^{\lambda_1\xi_n}+(\alpha-1)e^{\lambda_1(\xi_n-\sigma(\xi_n))}+u\big(\xi_n-\sigma(\xi_n)\big)\\
&\quad+\alpha(e^{(\lambda_1-\lambda_2)\sigma_1}-1)y\big(\xi_n-\sigma(\xi_n)-\sigma(\xi_n-\sigma(\xi_n))\big)\Big].
\end{aligned}$$

Thus we get

$$\begin{aligned}
&u(\xi_n)+\alpha\Big(e^{(\lambda_1-\lambda_2)\sigma_1}-1\Big)y\big(\xi_n-\sigma(\xi_n)\big)\\
&< u\big(\xi_n-\sigma(\xi_n)\big)+\alpha\Big(e^{(\lambda_1-\lambda_2)\sigma_1}-1\Big)y\big(\xi_n-\sigma(\xi_n)-\sigma(\xi_n-\sigma(\xi_n))\big)\\
&\quad-(\alpha-1)(1-e^{-\lambda_1\sigma_1})e^{\lambda_1\xi_n}, \qquad n=1,2,\dots\ .
\end{aligned} \tag{14}$$

If $\limsup\limits_{t\to\infty} u(t) = \limsup\limits_{n\to\infty} u(\xi_n) = \infty$, then there exists a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ such that $u(\xi_{n_k}) = \max\{u(t) : T_4 \le t \le \xi_{n_k}\}$, $k = 1, 2, \dots$. Then from (14) it follows

$$0 < \alpha(e^{(\lambda_1-\lambda_2)\sigma_1} - 1)\Big[y\big(\xi_{n_k} - \sigma(\xi_{n_k})\big) - y\big(\xi_{n_k} - \sigma(\xi_{n_k}) - \sigma(\xi_{n_k} - \sigma(\xi_{n_k}))\big)\Big]$$
$$< -(\alpha-1)(1-e^{-\lambda_1\sigma_1})e^{\lambda_1\xi_{n_k}} < 0, \qquad k = 1,2,\dots\,,$$

which is a contradiction.

If $\limsup\limits_{t\to\infty} u(t) = \limsup\limits_{n\to\infty} u(\xi_n) < \infty$, then (14) implies that

$$\begin{aligned} 0 &< \limsup_{n\to\infty}\Big\{u(\xi_n) + \alpha\left(e^{(\lambda_1-\lambda_2)\sigma_1} - 1\right)\Big[y\big(\xi_n - \sigma(\xi_n)\big) \\ &\quad - y\big(\xi_n - \sigma(\xi_n) - \sigma(\xi_n - \sigma(\xi_n))\big)\Big]\Big\} \\ &\le \limsup_{n\to\infty}\Big[u\big(\xi_n - \sigma(\xi_n)\big) - (\alpha-1)(1-e^{-\lambda_1\sigma_1})e^{\lambda_1\xi_n}\Big] = -\infty. \end{aligned}$$

This is a contradiction. The proof is complete. □

Corollary 1. *Suppose that $\alpha > 1$, (2) holds, $0 \le \dot\sigma(t) < 1$,*

$$\liminf_{t\to\infty} \sigma(t) \ge \sigma > 0,$$

and there exists $\lambda > \sigma^{-1}\ln\alpha$ such that

$$\liminf_{t\to\infty}\left[\int\limits_{\sigma(t)}^{\tau(t)} p(s)\,\mathrm{d}s\,\exp\left(-e^{\lambda t}\right)\right] > 0. \tag{15}$$

Then every solution of Eq. (1) *oscillates.*

Proof. If the condition (15) holds, then also (7) is satisfied and we can apply Theorem 1. □

Theorem 2. *Suppose that $\alpha > 1$, $\sigma > 0$,*

$$\limsup_{t\to\infty} \sigma(t) < \sigma,$$

there exists $\mu > 0$ such that

$$\alpha e^{-\mu\sigma} > 1$$

and

$$\limsup_{t\to\infty}\left[\int_0^{\tau(t)} p(s)\exp\left(-\alpha e^{\mu(t-s)}\right)\,\mathrm{d}s\,\exp\left(\alpha e^{\mu(t-\sigma(t))}\right) \times \exp\left(\left(1-\alpha e^{-\mu\sigma}\right)e^{\mu t}\right)\right] < \infty. \tag{16}$$

Then Eq. (1) *has an eventually positive solution.*

P r o o f. According to condition (16) there exists a $T > t_0$ such that

$$\begin{aligned}
&\int_0^{\tau(t)} p(s)\exp\bigl(-\alpha e^{\mu(t-s)}\bigr)\,\mathrm{d}s \\
&\le \exp\bigl(-\alpha e^{\mu(t-\sigma(t))}\bigr)\exp\bigl((\alpha e^{-\mu\sigma}-1)e^{\mu t}\bigr)\mu e^{\mu t} \\
&\le \exp\bigl(-\alpha e^{\mu(t-\sigma(t))}\bigr)\exp\bigl((\alpha e^{-\mu\sigma(t)}-1)e^{\mu t}\bigr)\mu e^{\mu t} \\
&\le \mu e^{\mu t}\exp\bigl(-e^{\mu t}\bigr), \qquad t \ge T.
\end{aligned} \tag{17}$$

We define the set $S \subset C_{loc}\bigl([T,\infty),R\bigr)$ of functions x which satisfy the inequalities

$$0 \le x(t) \le v(t) \qquad \text{for} \quad t \ge T,$$

where

$$v(t) = \exp\bigl(-e^{\mu t}\bigr), \qquad t \ge T.$$

The operator $F : S \to C_{loc}\bigl([T,\infty),R\bigr)$ is defined by

$$F(x)(t) = \begin{cases} \int_t^{\infty}\int_0^{\tau(s)} p(\xi)\bigl[x(s-\xi)\bigr]^{\alpha}\,\mathrm{d}\xi\,\mathrm{d}s & \text{for} \quad t \ge T_1, \\ v(t) - v(T_1) + F(x)(T_1) & \text{for} \quad t \in [T,T_1), \end{cases}$$

where $T_1 > T$ is such that $t - \tau(t) \ge T$ for $t \ge T_1$.

If $x \in S$, then by virtue of (17) we get

$$\begin{aligned}
0 \le F(x)(t) &= \int_t^{\infty}\int_0^{\tau(s)} p(\xi)\bigl[x(s-\xi)\bigr]^{\alpha}\,\mathrm{d}\xi\,\mathrm{d}s \\
&\le \int_t^{\infty}\int_0^{\tau(s)} p(\xi)\bigl[v(s-\xi)\bigr]^{\alpha}\,\mathrm{d}\xi\,\mathrm{d}s = \int_t^{\infty}\int_0^{\tau(s)} p(\xi)\exp\bigl(-\alpha e^{\mu(s-\xi)}\bigr)\,\mathrm{d}\xi\,\mathrm{d}s \\
&\le \int_t^{\infty} \mu e^{\mu s}\exp\bigl(-e^{\mu s}\bigr)\,\mathrm{d}s = \exp\bigl(-e^{\mu t}\bigr) = v(t), \qquad t \ge T_1.
\end{aligned}$$

Thus $F(S) \subset S$ and we can proceed as in the proof of Lemma 1. So Eq. (1) has a nonoscillatory solution. The proof is complete. □

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