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ON NONSTATIONARY VON KÁRMÁN VARIATIONAL INEQUALITIES

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ABSTRACT. We deal with systems consisting of a nonlinear evolution variational inequality for the deflection and a nonlinear quasistationary equation for the Airy stress function. The systems describe moderately large deflections of thin viscoelastic plates with an inner obstacle. We distinguish two kinds of problems. Pseudoparabolic variational inequality for the quasistationary deflections and the hyperbolic inequality for the dynamic case. In both cases we transform the original problem to one canonical inequality in a Hilbert space of deflections. The pseudoparabolic problem is solved using a semidiscrete approximation transforming the problem into the sequence of stationary variational inequalities. The hyperbolic problem is solved by the penalization method.

1. Introduction

Theodor von Kármán (1910) stated the nonlinear system of partial differential equations for large deflections and the Airy stress function of a thin elastic plate. This system has been treated systematically starting in the sixties by Berger and Fife [1]. Ciarlet [5] has justified the von Kármán system as the plate model derived from the equations for a 3-dimensional body.

Simultaneusly the unilateral boundary value problems for the von Kármán equations were investigated by Naumann [15] and John [11] with unilateral conditions on the boundary and the inner obstacle, respectively.

The nonstationary von Kármán systems are studied nowadays mainly in the dynamic form. In the case of viscoelastic plates the short or long memory term is appearing mostly in the linear part of the equation for a deflection ([8], [14]).

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We deal here with the isotropic long memory material of the Zener type. Originally integro-differential stress-strain relations can be transformed to the first-order differential relations. The differential term appears both in the variational inequality for the deflection as well as in the right-hand side of the equation for the Airy stress function. The nonlinear system and the corresponding canonical equation can then be derived in a similar way as the traditional elastic von Kármán system. The inner obstacle for the plate will be considered. The resulting pseudoparabolic system will be transformed into the nonstationary first-order variational inequality in the Hilbert space of admissible deflections. We obtain the sequence of stationary von Kármán variational inequalities after discretizing the time variable. A similar approach is used in [3], where the viscoelastic von Kármán variational inequality for the long memory material is considered. The existence of a solution for every discrete time value will be verified using a variational approach. The convergence of the sequence of continuous segment-line functions will be investigated.

Another approach will be used in the dynamic case. Applying the penalization term we transform the system consisting of a hyperbolic variational inequality for the perpendicular vibrations and the equation with a biharmonic left-hand side for the Airy stress function to one canonical nonlinear hyperbolic equation in a Hilbert space of deflections. The viscosity term enables to derive the convergence of the sequence of penalized deflections to a solution of the canonical variational inequality in similar way as for a broader set of of problems: for the string, membrane [10], for bodies [9], [7]. There is only limited results available for the case without viscosity cf. [16], [13].

2. The formulation of the pseudoparabolic problem

We assume a thin isotropic plate occupying the domain

$$Q = \big\{(x,z) \in R^3;\ x = (x_1,x_2) \in \Omega,\ -h/2 < z < h/2\big\},$$

where Ω is a bounded simply connected domain in R^2 with a Lipschitz boundary Γ. The plate is clamped on its boundary and subjected to the perpendicular load $f(t,x),\ t > 0,\ x \in \Omega$. The stress-strain relations for the isotropic viscoelastic long memory material are of the form

$$\sigma_{ij} = \frac{E(0)}{1-\mu^2}\big[(1-\mu)\varepsilon_{ij} + \mu\delta_{ij}\varepsilon_{kk}\big] + \frac{E'}{1-\mu^2} * \big[(1-\mu)\varepsilon_{ij} + \mu\delta_{ij}\varepsilon_{kk}\big](t),$$
$$i,j \in \{1,2\}, \quad \varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22}$$

with a Poisson ratio $\mu \in \big(0, \frac{1}{2}\big)$, a positive decreasing relaxation function

$E \in C^1(R^+)$ and a convolution product $f * g(t) = \int_0^t f(t-s)g(s)\,\mathrm{d}s$.

In the exponential case $E(t) = E_0 + E_1 e^{-\beta t},\ E_0 > 0,\ E_1 > 0,\ \beta > 0$ we can transform the previous integral stress-strain relation into the differential form corresponding to the Zener material [4]

$$\begin{aligned}\sigma'_{ij} + \beta\sigma_{ij} =& \frac{E_0+E_1}{1-\mu^2}\big[(1-\mu)\varepsilon_{ij} + \mu\delta_{ij}\varepsilon_{kk}\big]' \\ &+ \frac{\beta E_0}{1-\mu^2}\big[(1-\mu)\varepsilon_{ij} + \mu\delta_{ij}\varepsilon_{kk}\big], \\ \sigma_{ij}(0) =& \frac{E_0+E_1}{1-\mu^2}\big[(1-\mu)\varepsilon_{ij}+\mu\delta_{ij}\varepsilon_{kk}\big](0).\end{aligned}$$

Let us define the material constants $D_i = \frac{h^3}{12(1-\mu^2)}E_i,\ i = 0, 1,$ and the form

$$[v, w] = \partial_{11}v\partial_{22}w + \partial_{22}v\partial_{11}w - 2\partial_{12}v\partial_{12}w, \quad v,\ w \in H^2(\Omega). \tag{1}$$

Applying the similar approach as in the elastic case, e.g., [6] the following pseudoparabolic von Kármán system for the deflection w and the Airy stress function Φ can be derived:

$$(D_0 + D_1)\Delta^2 w' + \beta D_0\Delta^2 w - [\Phi, w] = f'(t) + \beta f(t),\ t > 0,\ x \in \Omega;$$

$$w = \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma, \tag{2}$$

$$\Delta^2\Phi = -\frac{h}{2}\big(\big(\ (E_0 + E_1)[w, w]' + \beta E_0[w, w]\big)\big),\ t > 0,\ x \in \Omega,$$

$$\Phi \ = \ \frac{\partial \Phi}{\partial \nu} \ = 0 \text{ on } \Gamma, \tag{3}$$

$$w(0, x) = w_0(x),\ x \in \Omega. \tag{4}$$

The initial deflection fulfils the stationary von Kármán system

$$(D_0 + D_1)\Delta^2 w_0 - [\Phi_0, w_0] = f(0), \quad w_0 = \frac{\partial w_0}{\partial \nu} = 0 \text{ on } \Gamma, \tag{5}$$

$$\Delta^2\Phi_0 = -\frac{h}{2}(E_0 + E_1)[w_0, w_0], \qquad \Phi_0 = \frac{\partial \Phi_0}{\partial \nu} = 0 \text{ on } \Gamma. \tag{6}$$

We denote $V = H_0^2(\Omega)$ the Sobolev space with the scalar product and the norm

$$((u, v)) = \int_\Omega \Delta u \Delta v\,\mathrm{d}x \quad \|u\| = ((u, u))^{1/2}.$$

The scalar product and the norm in the space $L^2(\Omega)$ will be denoted by $(\cdot, \cdot)$ and $\|\cdot\|_0$.

The plane $z = 0$ is the inner obstacle for the deflection. The convex closed cone

$$K = \big\{ w \in V \mid w(x) \geq 0 \qquad \text{for all} \quad x \in \Omega \big\}$$

is then an admissible set of deflections. Instead of the problem (2)–(6) we will then investigate a unilateral problem in a following variational form:

To find a couple $\{w, \Phi\} : [0, \infty) \to K \times V$ fulfilling

$$\big(\big((D_0 + D_1) w'(t) + \beta D_0 w(t), v - w(t) \big)\big)$$
$$\geq \big([\Phi(t), w(t)], v - w(t) \big) + \big(f'(t) + \beta f(t), v - w(t) \big) \qquad \text{for all} \quad v \in K, \tag{7}$$

$$\big(\big(\Phi(t), \varphi\big)\big) = -\frac{h}{2} \big(\big((E_0 + E_1)[w(t), w(t)]' + \beta E_0 [w(t), w(t)], \varphi \big)\big) \qquad \text{for all} \quad \varphi \in V, \tag{8}$$

$$\big(\big((D_0 + D_1) w(0), v - w(0) \big)\big)$$
$$\geq \big([\Phi_0, w(0)], v - w(0) \big) + \big(f(0), v - w(0) \big) \qquad \text{for all} \quad v \in K, \tag{9}$$

$$((\Phi_0, \varphi)) = -\frac{h}{2} \big((E_0 + E_1)[w(0), w(0)], \varphi \big) \qquad \text{for all} \quad \varphi \in V. \tag{10}$$

In order to convert a system (7)–(10) into an initial value problem for a nonlinear pseudoparabolic variational inequality we introduce the bilinear operator $B : H^2(\Omega)^2 \to V$ defined by the uniquely solved equation

$$\big(\big(B(u, v), \varphi\big)\big) = \int_\Omega [u, v] \, \varphi \, \mathrm{d}x \qquad \text{for all} \quad \varphi \in V, \tag{11}$$

We recall [6] that the operator B is symmetric and compact,

$$\big(\big(B(u, v), w\big)\big) = \big(\big(u, B(v, w)\big)\big) \qquad \text{for all} \quad u, v, w \in V \tag{12}$$

and due to Lemma 1 from [12] $B : H^2(\Omega)^2 \to W_p^2(\Omega)$ and

$$\|\Phi(u, v)\|_{W_p^2(\Omega)} \leq c \, \|u\|_{H^2(\Omega)} \|v\|_{W_p^1(\Omega)} \qquad \text{for all} \quad u \in H^2(\Omega), \, v \in W_p^1(\Omega). \tag{13}$$

Let us define the function $q : [0, \infty) \to V$ uniquely defined as a solution of the identity

$$\big(\big(q(t), v\big)\big) = \frac{1}{D_0 + D_1} \int_\Omega f(t) \, v \, \mathrm{d}x \qquad \text{for all} \quad v \in V.$$

Using the expression (11) for the determining of $\Phi(t), \Phi_0$ and setting

$$a = \frac{\beta D_0}{D_0 + D_1} = \frac{\beta E_0}{E_0 + E_1},$$

$$b = \frac{h(E_0 + E_1)}{2(D_0 + D_1)} = \frac{6(1 - \mu^2)}{h^2}$$

we obtain the initial value problem equivalent with (7)–(10):

To find a function $w : [0, \infty) \to K$ fulfilling

$$\begin{aligned} &\big((w'(t) + aw(t), v - w(t)\ \big)\big) \\ &\quad + b\big(\big(B(w, w)' + aB(w, w), B(w(t), v - w(t)\ \big)\big) \\ &\geq \big(\big(q'(t) + \beta q(t), v - w(t)\ \big)\big) \qquad \text{for all} \quad v \in K, \end{aligned} \tag{14}$$

$$\begin{aligned} &\big(\big(w(0), v - w(0)\ \big)\big) \\ &\quad + b\big(\big(B(w, w)(0), B(w(0), v - w(0)\ \big)\big) \\ &\geq \big(\big(\ q(0), v - w(0)\ \big)\big) \qquad \text{for all} \quad v \in K. \end{aligned} \tag{15}$$

3. Approximation by the method of Rothe

We find a solution to the initial value problem (14), (15) by converting it into the sequence of stationary problems having a form of the canonical von Kármán variational inequalities. We apply the Rothe's method or the method of lines in a similar way as in [2], where the nonlinear parabolic variational inequalities have been solved. For each $n \in N$ we set

$$\tau = \frac{T}{n},\ t_i = i\tau,\ w_i = w(t_i),\ i = 0, 1, \ldots, n;\ \delta w_j = \frac{1}{\tau}(w_j - w_{j-1}),\ j = 1, \ldots, n\,.$$

We substitute the initial value problem (14), (15) by the finite sequence of stationary variational inequalities

$$\begin{aligned} &((w_0, v - w_0)) + b\big(\big(B(w_0, w_0), B(w_0, v - w_0)\ \big)\big) \\ &\geq ((q_0, v - w_0)) \qquad \text{for all} \qquad v \in K, \end{aligned} \tag{16}$$

$$\begin{aligned} &((\delta w_i + aw_i, v - w_i)) + b\big(\big(\delta B(w_i, w_i) + aB(w_i, w_i), B(w_i, v - w_i)\big)\big) \\ &\geq ((\delta q_i + \beta q_i, v - w_i)) \qquad \text{for all} \qquad v \in K,\ i = 1, \ldots n\,. \end{aligned} \tag{17}$$

Both variational inequalities have solutions w_0 and $w_i,\ i = 1, \ldots, n$, respectively. They are solutions of the minimizing problems

$$\begin{aligned}
&J_i(w_i) = \min_{v\in K} J_i(v), \qquad i = 0, 1, \ldots, n\,,\\
&J_0(v) = \frac{1}{2}\,\|v\|^2 + \frac{b}{4}\,\|B(v,v)\|^2 - ((q_0, v)),\\
&J_i(v) = \frac{1}{2}\,(1+\tau a)\Big[\|v\|^2 + \frac{b}{2}\,\|B(v,v)\|^2\Big] - \frac{b}{2}\,\big((B(w_{i-1}, w_{i-1}), B(v,v)\big)\\
&\qquad\qquad -((w_{i-1} + q_i, v)), \qquad i = 1, \ldots n\,.
\end{aligned}$$

In order to obtain a priori estimates not depending on T we express the values w_i in a form

$$w_i = e^{-\alpha\tau i}\, u_i,\ \ \alpha > 0,\ \ i = 0, 1, \ldots, n\,. \tag{18}$$

We have the following expression of the difference δw_i:

$$\delta w_i = \delta e^{-\alpha\tau i}\, u_i + e^{-\alpha\tau(i-1)}\,\delta u_i,\ \ i = 1, \ldots, n\,. \tag{19}$$

After setting $i = j,\ \delta q_j + \beta q_j = f_j,\ v = 0$ in (17) and multiplying with $e^{\alpha\tau j}$ we obtain

$$\begin{aligned}
&e^{\alpha\tau}((\delta u_j, u_j)) + \left(a - \frac{e^{\alpha\tau}-1}{\tau}\right)\|u_j\|^2\\
&\quad + be^{-2\alpha\tau(j-1)}\,\big(\big(\ \delta B(u_j,u_j), B(u_j,u_j)\ \big)\big)\\
&\quad + be^{-2\alpha\tau j}\left(a - \frac{e^{2\alpha\tau}-1}{\tau}\right)\|B(u_j,u_j)\|^2\\
&\le\ e^{\alpha\tau j}\,((f_j, u_j)), \qquad j = 1, \ldots, n\,.
\end{aligned}$$

Summing up and using the relation

$$2\tau\sum_{j=1}^{i}((\delta u_j, u_j)) = \|u_i\|^2 - \|u_0\|^2 + \sum_{j=1}^{i}\tau^2\|\delta u_j\|^2$$

we arrive at

$$\begin{aligned}
&e^{\alpha\tau}\big(\|u_i\|^2 + be^{-2i\alpha\tau}\,\|B(u_i,u_i)\|^2\big)\\
&\quad +2(a - \alpha e^{\alpha\tau})\sum_{j=1}^{i}\tau\big(\|u_j\|^2 + b\,\|B(u_j,u_j)\|^2\big)\\
&\le\ e^{\alpha\tau}\big(\|u_0\|^2 + b\,\|B(u_0,u_0)\|^2\big) + 2\sum_{j=1}^{i}\tau e^{\alpha\tau j}\,((f_j,u_j)), \quad i = 1, \ldots, n.
\end{aligned}$$

We obtain directly from the inequality (16) the estimate

$$\|w_0\|^2 + b\,\|B(w_0, w_0)\|^2 \le \|q_0\|^2. \tag{20}$$

Setting $\alpha > 0,\ \tau_0 > 0$ such that $\alpha e^{\alpha\tau} \le \frac{a}{2}$ we achieve considering $w_0 = u_0$ the estimate

$$\|u_i\|^2 \le \|q_0\|^2 + \frac{1}{a}\sum_{j=1}^{i} \tau e^{2\alpha\tau j}\,\|f_j\|^2.$$

Implying the expression (18) we obtain the uniform a priori estimate

$$\|w_i\|^2 \le \|q_0\|^2 + \frac{1}{a}\sum_{j=1}^{i} \tau e^{-2\alpha\tau(1+i-j)}\,\|\delta q_j + \beta q_j\|^2, \qquad i = 1, \dots, n\,, \quad \tau \le \tau_0\,. \tag{21}$$

We continue with the estimates for the sum of differences. Setting $v = w_{i-1}$ in (16) we obtain in a similar way as above

$$\begin{aligned} & a\,\|w_i\|^2 + \frac{1}{2}\,ab\,\|B(w_i, w_i)\|^2 \\ &\quad + \sum_{j=1}^{i} \tau\big[(2-\varepsilon)\|\delta w_j\|^2 + b\,\|\delta B(w_j, w_j)\|^2\big] \\ &\quad + b\sum_{j=1}^{i} \tau^2\big(\big(\,\delta B(w_j, w_j) + aB(w_j, w_j), B(\delta w_j, \delta w_j)\,\big)\big) \\ & \le \frac{1}{\varepsilon}\sum_{j=1}^{i} \tau\|\delta q_j + \beta q_j\|^2. \end{aligned} \tag{22}$$

We have the inequality

$$\tau\|\delta B(w_j, w_j)\| \le 2\,\|B\| \max_{j=0,1,\dots,n} \|w_j\|^2. \tag{23}$$

Let us assume that

$$2b\,\|B\|\|w_j\|^2 < 1, \quad j = 1, \dots, n\,. \tag{24}$$

Comparing with the a priori estimate (21) we can see that the condition

$$\|q_0\|^2 + \frac{1}{a}\sum_{j=1}^{i} \tau e^{-2\alpha\tau(1+i-j)}\|\delta q_j + \beta q_j\|^2 < \frac{1}{b\,\|B\|^2}, \quad i = 1, \dots, n \tag{25}$$

is sufficient for fulfilling the estimate (24). Then we obtain from (22) the a priori estimate

$$\sum_{j=1}^{i} \tau\big[\|\delta w_j\|^2 + \|\delta\, B(w_j,w_j)\|^2\big] \le C_1, \quad i=1,\dots,n\,. \tag{26}$$

Let us assume that a function $q \in W^{2,1}(0,T;V)$ fulfils the bound

$$\|q(0)\|^2 + \frac{1}{a}\int_0^t e^{-2\alpha(t-s)}\,\|q'(s)+\beta q(s)\|^2\,\mathrm{ds} < \frac{1}{b\,\|B\|^2}, \quad t\in[0,T]. \tag{27}$$

Then there exists $\tau_0 > 0$ such that for $\tau < \tau_0$ the bounds (25) and hence the a priori estimate (26) hold.

Let us further define the following functions determined by values $w_i,\ \delta w_i$

$$\begin{aligned} &w_n:[0,T]\to K,\ w_n(t)=w_{i-1}+(t-t_i)\delta w_i,\ t_{i-1}\le t\le t_i, \quad i=1,\dots,n\,,\\ &\bar w_n:[0,T]\to K,\ \bar w_n(0)=w_0,\ \bar w_n(t)=w_i,\ t_{i-1}< t\le t_i, \quad i=1,\dots,n\,.\end{aligned}$$

The sequence of functions $\{w_n\}$ is bounded in the space $W^{2,1}(0,T;V)$:

$$\|w_n\|_{W^{2,1}(0,T;V)} \le C_2, \quad n\in N. \tag{28}$$

Then there exists its subsequence (again denoted by $\{w_n\}$) and a function $w \in W^{2,1}(0,T;V)$ such that $w(t)\in K$ for every $t\in[0,T]$ and

$$w_n \rightharpoonup w \quad \text{in} \quad W^{2,1}(0,T;V), \tag{29}$$

$$w_n(t)\rightharpoonup w(t),\ \bar w_n(t)\rightharpoonup w(t) \quad \text{in} \quad V \qquad \text{for every } t\in[0,T], \tag{30}$$

$$w_n \rightharpoonup^* w,\ \bar w_n \rightharpoonup^* w \quad \text{in} \quad L^\infty(0,T;V), \tag{31}$$

$$w_n \to w,\ \bar w_n \to w \quad \text{in} \quad L^p\big(0,T;W^{r,1}(\Omega)\big), \quad p>1,\ r>1. \tag{32}$$

If we set $B(w_i,w_i)=U_i,\ i=0,1,\dots,n$ we obtain $U\in W^{2,1}(0,T;V)$ fulfilling

$$U_n \rightharpoonup U \quad \text{in} \quad W^{2,1}(0,T;V), \tag{33}$$

$$U_n(t)\rightharpoonup U(t),\ \bar U_n(t)\rightharpoonup U(t) \quad \text{in} \quad V \qquad \text{for every } t\in[0,T], \tag{34}$$

$$U_n \rightharpoonup^* U,\ \bar U_n \rightharpoonup^* U \text{ in } L^\infty(0,T;V). \tag{35}$$

Using the properties of the bilinear operator $B: V\times V\to V$ we obtain that

$$U(t)=B(w(t),w(t)),\ B(w_n,w_n)\rightharpoonup B(w,w) \quad \text{in} \quad W^{2,1}(0,T;V). \tag{36}$$

Applying the weak semicontinuity of the function $\|.\|^2$ in the spaces V and $L^2(0,T;V)$ and the relation $\big((B(u,v),B(w,\varphi))\big)=\big([B(u,v),w],\varphi\big), u,v,w,\varphi\in V$

we arrive at the variational inequality

$$\int_0^T \big((w'(t)+aw(t), v(t)-w(t))\big)\mathrm{d}t$$
$$+ b\int_0^T \big((B(w,w)'+aB(w,w), B(w(t), v(t)-w(t))\big)\mathrm{d}t$$
$$\geq \int_0^T \big((q'(t)+\beta q(t), v-w(t))\big)\mathrm{d}t \qquad \text{for all} \quad v\in\mathcal{K}\,,$$
$$\mathcal{K}=\big\{v\in L^2(0,T;V)|\ v(t)\in K \qquad \text{for a.e.} \quad t\in[0,T]\big\}.$$

Using the Lebesgue mean value limit theorem we obtain that the variational inequality (14) is fulfilled. The inequality (15) is identical with (16). Then the following existence theorems holds.

Theorem 3.1. *Let $q \in W^{2,1}\big([0,T];V\big)$ fulfil the condition* (27)*. Then there exists a solution $w\in W^{2,1}(0,T;V)$ of the variational inequalities* (14), (15)*.*

There exists a subsequence of a sequence $\{w_n\}$ of segment line functions defined by discrete values w_i fulfilling the inequalities (16), (17*) such that a weak convergence* (29) *holds.*

Theorem 3.2. *Let $f\in W^{2,1}\big([0,T];V^*\big)$ fulfil the bound*

$$\|f(0)\|_0^2+\frac{1}{a}\int_0^t e^{-2\alpha(t-s)}\,\|f'(s)+\beta f(s)\|_0^2\,\mathrm{d}s<\frac{(D_0+D_1)^2}{b\,\|B\|^2},\quad t\in[0,T]. \quad (37)$$

Then there exists a triple $\{w,\Phi,\Phi_0\}\in W^{2,1}(0,T;V)\times L^2(0,T;V)\times V$ solving the problem (7)–(10)*.*

Remark 3.3. The exponential character of the conditions (27) and (37) implies that the bounds for the right hand sides q, q' or f, f' do not depend on the length T of the time interval. The bounds are closely related to the questions of the uniqueness of a solution.

4. Dynamic contact problem

We deal in this section with a dynamic contact problem for a von Kármán plate made of a short memory material. For simplicity the boundary conditions for a free boundary of the plate are considered. The case of Dirichlet conditions is more complicated. The approach used in [7] for the membrane case can be applied. The constants $E_0, E_1 > 0$ denote the Young modulus of elasticity and the modulus of viscosity, respectively. We introduce further the constants $a = \frac{h^2}{12}$, $b = \frac{h^2}{12\varrho(1-\mu^2)}$, where h is the the plate thickness and ϱ is the density of the material. We set $Q = (0,T) \times \Omega$, $S = (0,T) \times \Gamma$ and $H^{1,2}(Q)$ the anisotropic Sobolev space of functions from $L^2(\Omega)$ with first and the second generalized derivatives with respect to time and space variables respectively. Classical formulation is composed of the system

$$\left.\begin{aligned} \ddot{u} - a\triangle\ddot{u} + b(E_1\triangle^2\dot{u} + E_0\triangle^2 u) - [u,v] &= f + g, \\ u \ge 0,\ g \ge 0,\ ug &= 0, \\ \triangle^2 v + E_1\partial_t[u,u] + E_0[u,u] &= 0 \end{aligned}\right\} \text{ on } Q\,, \tag{38}$$

the boundary conditions

$$u \ge 0,\ \Sigma(u) \ge 0,\ u\Sigma(u) = 0,\ \mathcal{M}(u) = 0,\ v = 0 \text{ and } \partial_\nu v = 0 \text{ on } S,$$

$$\begin{aligned} &M(u) = b\big[E_1 M(\dot{u}) + E_0 M(u)\big], \\ &M(u) = \triangle u + (1-\mu)(2\nu_1\nu_2\partial_{12}u - \nu_1^2\partial_{22}u - \nu_2^2\partial_{11}u), \\ &\Sigma(u) = b\big[E_1 V(\dot{u}) + E_0 V(u)\big] - a\partial_\nu\ddot{u}, \\ &V(u) = \partial_\nu\triangle u + (1-\mu)\partial_\tau\big[(\nu_1^2 - \nu_2^2)\partial_{12}u + \nu_1\nu_2(\partial_{22}u - \partial_{11}u)\big] \end{aligned} \tag{39}$$

and the initial conditions

$$u(0,\cdot) = u_0 \ge 0,\ \dot{u}(0,\cdot) = u_1 \text{ on } \Omega. \tag{40}$$

For $u, y \in L_2\big(I; H^2(\Omega)\big)$ we define the following bilinear form

$$A : (u,y) \to b\big(\partial_{kk}\, u\partial_{kk}\, y + \mu(\partial_{11}\, u\partial_{22}\, y + \partial_{22}\, u\partial_{11}\, y) + 2(1-\mu)\partial_{12}\, u\partial_{12}\, y\big)$$

and introduce a cone $\mathcal{K}$ as

$$\mathcal{K} := \Big\{y \in H^{1,2}(Q);\ \dot{y} \in L_2\big(I; H^1(\Omega)\big),\ y \ge 0\Big\}. \tag{41}$$

Then the variational formulation of the dynamic contact problem has the form:

Look for $\{u, \Phi\} \in \mathcal{K} \times L_2\big(I; \mathring{H}^2(\Omega)\big)$, *such that* $\dot{u} \in L_2(I; H^2(\Omega))$, $u(0,\cdot) = u_0$

and

$$\int_Q \big(A(E_1\dot u + E_0 u, y_1 - u) - a\nabla\dot u\cdot\nabla(\dot y - \dot u) - \dot u(\dot y - \dot u) - [u,v](y_1 - u)\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_\Omega \big(a\nabla\dot u\cdot\nabla(y_1 - u) + \dot u(y_1 - u)\big)(T,\cdot)\,\mathrm{d}x$$
$$\geq \int_\Omega \big(a\nabla u_1\cdot\nabla(y_1(0,\cdot) - u_0) + u_1(y_1(0,\cdot) - u_0)\big)\,\mathrm{d}x + \int_Q f(y_1 - u)\,\mathrm{d}x\,\mathrm{d}t, \tag{42}$$

$$\int_\Omega \big(\triangle\Phi\triangle y_2 + (E_1\partial_t[u,u] + E_0[u,u])y_2\big)\,\mathrm{d}x = 0 \quad \text{for all} \quad (y_1,y_2)\in\mathcal{K}\times\mathring{H}^2(\Omega). \tag{43}$$

Using the operator $B : V\times V\to V$ we reformulate the system (42), (43) into:

Problem $\mathcal{P}$. *We look for $u\in\mathcal{K}$ such that $\dot u\in L_2\big(I;H^2(\Omega)\big)$, $u(0,.) = u_0$ and*

$$\begin{aligned}\int_Q &\big(E_1 A(\dot u, y-u) + E_0 A(u, y-u) - a\nabla\dot u\cdot\nabla(\dot y - \dot u) - \dot u(\dot y - \dot u)\big)\,\mathrm{d}x\,\mathrm{d}t\\ &+ \int_\Omega \big[u, E_1\partial_t B(u,u) + E_0 B(u,u)\big](y-u)\,\mathrm{d}x\,\mathrm{d}t\\ &+ \int_\Omega \big(a\nabla\dot u\cdot\nabla(y-u) + \dot u(y-u)\big)(T,\cdot)\,\mathrm{d}x\\ \geq &\int_\Omega \big(a\nabla u_1\cdot(\nabla y(0,\cdot) - \nabla u_0) + u_1(y(0,\cdot) - u_0)\big)\,\mathrm{d}x + \int_Q f(y-u)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{44}$$

holds for any $y\in\mathcal{K}$.

For any $\eta > 0$ we define the penalized problem. Its classical formulation is the initial-boundary value problem

$$\begin{aligned}&\ddot u - a\triangle\ddot u + b(E_1\triangle^2\dot u + E_0\triangle^2 u) - [u,v] = f + \eta^{-1}u^-,\\ &\triangle^2 v + E_1\partial_t[u,u] + E_0[u,u] = 0,\\ &\Sigma(u) = 0,\ \mathcal{M}(u) = 0,\ v = 0 \quad \text{and} \quad \partial_\nu v = 0 \ \text{ on } S,\\ &u(0,\cdot) = u_0 \geq 0,\ \dot u(0,\cdot) = u_1 \ \text{ on } \Omega.\end{aligned} \tag{45}$$

Applying the operator B we obtain in the same way as above the variational formulation for the unknown deflection u:

Problem $\mathcal{P}_\eta$. *We look for* $u \in H^{1,2}(Q)$ *such that* $\dot{u} \in L_2\big(I; H^2(\Omega)\big)$, $\ddot{u} \in L_2\big(I; H^1(\Omega)\big)$, and the equation

$$\int_Q \big(a\nabla\ddot{u}\cdot\nabla z + \ddot{u}z + E_1 A(\dot{u}, z) + E_0 A(u, z) + [u, E_1\partial_t B(u, u) + E_0 B(u, u)]z - \eta^{-1}u^- z\big)\,\mathrm{d}x\,\mathrm{d}t = \int_Q fz\,\mathrm{d}x\,\mathrm{d}t, \tag{46}$$

holds for any $z \in L_2\big(I; H^2(\Omega)\big)$ and the conditions (40) remain valid.

To derive the a priori estimates for solutions of (46) we put

$$z = \begin{cases} \dot{u} & \text{for} \quad t \le s, \\ 0 & \text{for} \quad t > s \end{cases}$$

in (46) with an arbitrary $s \in I$. We denote $Q_s \equiv (0, s) \times \Omega$. Taking in mind

$$\int_\Omega [u, v]y\,\mathrm{d}x = \int_\Omega [u, y]v\,\mathrm{d}x \tag{47}$$

if at least one element of $\{u, v, y\}$ belongs to $\mathring{H}^2(\Omega)$, cf. [6], we get

$$\int_{Q_s} \left(\frac{1}{2}\partial_t\left(a|\nabla\dot{u}|^2 + \dot{u}^2 + E_0 A(u, u) + \frac{E_0}{2}\big(\triangle B(u, u)\big)^2 + \eta^{-1}(u^-)^2\right) + E_1 A(\dot{u}, \dot{u}) + \frac{E_1}{2}\big(\triangle\partial_t B(u, u)\big)^2\right)\mathrm{d}x\,\mathrm{d}t = \int_Q f\dot{u}\,\mathrm{d}x\,\mathrm{d}t. \tag{48}$$

This leads to the a priori estimate

$$\|\dot{u}\|^2_{L_2(I;H^2(\Omega))} + \|\dot{u}\|^2_{L_\infty(I;H^1(\Omega))} + \|u\|^2_{L_\infty(I;H^2(\Omega))} + \|\partial_t B(u, u)\|^2_{L_2(I;H^2(\Omega))} \le c \equiv c(f, u_0, u_1). \tag{49}$$

Moreover, the estimate (13) implies

$$\|\partial_t B(u, u)\|_{L_2(I;W^2_p(\Omega))} \le c_p \equiv c_p(f, u_0, u_1) \qquad \text{for all} \quad p > 2. \tag{50}$$

The estimates are η independent. Since for a fixed $\eta > 0$ the penalty term $\eta^{-1}u^-$ belongs, e.g., to $H^1(Q)$, this together with (49) and (45) yields an estimate of $\ddot{u} - a\triangle\ddot{u}$ in $L_2(I; H^2(\Omega)^*)$. We verify the existence and the uniqueness of a solution to the penalized problem. The following generalization of the Aubin's compactness lemma verified in [10] Theorem 3.1 will be essentially used:

Lemma 4.1. *Let $B_0 \hookrightarrow\hookrightarrow B \hookrightarrow B_1$ be Banach spaces, the first reflexive and separable. Let $1 < p < \infty,\ 1 \le q < \infty$. Then*

$$W \equiv \{v;\ v \in L_p(I;B_0),\ \dot v \in L_q(I,B_1)\} \hookrightarrow\hookrightarrow L_p(I;B).$$

Theorem 4.2. *Let $f \in L_2(Q),\ u_i \in H^2(\Omega),\ i = 0,\,1$. Then there exists a unique solution $u \in H^{1,2}(Q)$ of the problem $\mathcal{P}_\eta$.*

Proof. (*i*) *The existence.* Let us denote by $\{w_i \in H^2(\Omega);\ i \in \mathbb{N}\}$ an orthonormal basis of $H^2(\Omega)$. We construct the Galerkin approximation u_m of a solution in a form

$$\begin{aligned}
&u_m(t) = \sum_{i=1}^m \alpha_i(t) w_i,\ \alpha_i(t) \in \mathbb{R}, \quad i = 1,\dots,m,\ m \in \mathbb{N},\\
&\int_\Omega \Big(a\nabla \ddot u_m(t)\cdot\nabla w_i + \ddot u_m(t) w_i + E_1 A\big(\dot u_m(t), w_i\big) + E_0 A\big(u_m(t), w_i\big)\\
&\quad + \big[u_m(t), w_i\big]\big(E_1\partial_t B(u_m,u_m)(t) + E_0 B(u_m,u_m)(t)\big) - \eta^{-1} u_m(t)^- w_i\Big)\,\mathrm{d}x\\
&= \int_\Omega f(t) w_i\,\mathrm{d}x, \quad i = 1,\dots,m,
\end{aligned} \tag{51}$$

$$u_m(0) = u_{0m},\ \dot u_m(0) = u_{1m},\ u_{im} \to u_i \text{ in } H^2(\Omega), \quad i = 0,1. \tag{52}$$

The system (51) can then be expressed in the form

$$\ddot\alpha_i = F_i(t, \dot\alpha_1,\dots,\dot\alpha_m, \alpha_1,\dots,\alpha_m), \quad i = 1,\dots,m.$$

Its right-hand side satisfies the conditions for the local existence of a solution fulfilling the initial conditions corresponding to the functions $u_{0m},\ u_{1m}$. Hence there exists a Galerkin approximation $u_m(t)$ defined on some interval $[0,t_m]$, $0 < t_m < T$. Applying similar a priori estimates as above we receive the conditions for the prolongation of a solution to the whole interval $[0,T]$. After multiplying the equation (51) by $\dot\alpha_i(t)$, summing up with respect to i and integrating we obtain the estimates analogous to (49), (50):

$$\begin{aligned}
&\|\dot u_m\|^2_{L_2(I;H^2(\Omega))} + \|\dot u_m\|^2_{L_\infty(I;H^1(\Omega))} + \|u_m\|^2_{L_\infty(I;H^2(\Omega))}\\
&+ \|\partial_t B(u_m,u_m)\|^2_{L_2(I;H^2(\Omega))} \le c \equiv c(f,u_0,u_1),
\end{aligned} \tag{53}$$

$$\|\partial_t B(u_m,u_m)\|_{L_2(I;W_p^2(\Omega))} \le c_p \equiv c_p(f,u_0,u_1), \qquad \text{for all} \quad p > 2. \tag{54}$$

The estimate (54) further implies

$$\begin{aligned}&\big[u_m, E_1\partial_t\Phi(u_m,u_m) + E_0\Phi(u_m,u_m)\big] \in L_2\big(I; L_r(\Omega)\big), \quad r = \frac{2p}{p+2},\\ &\big\|[u_m, E_1\partial_t B(u_m,u_m) + E_0 B(u_m,u_m)]\big\|_{L_2(I;L_r(\Omega))} \le c_r \equiv c_r(f,u_0,u_1). \qquad (55)\end{aligned}$$

After multiplying the equation (51) by $\ddot{\alpha}_i(t)$, summing up with respect to i integrating and using the imbedding $L_q(\Omega) \hookrightarrow H^1(\Omega)$, $q = 2p/(p-2)$ we obtain the estimate

$$\|\ddot{u}_m\|^2_{L_2(I;H^1(\Omega))} \le c_\eta, \quad m \in \mathbb{N}. \qquad (56)$$

We proceed with the convergence of the Galerkin approximation. Applying the estimates (53)–(56) and the Aubin's compactness lemma we obtain for any $p \in [1,\infty]$ a subsequence of $\{u_m\}$ (denoted again by $\{u_m\}$), and a function u the convergences

$$\begin{aligned}
\dot{u}_m &\rightharpoonup^* \dot{u} && \text{in } L_\infty\big(I; H^1(\Omega)\big),\\
\dot{u}_m &\rightharpoonup \dot{u} && \text{in } L_2\big(I; H^2(\Omega)\big),\\
\ddot{u}_m &\rightharpoonup \ddot{u} && \text{in } L_2\big(I; H^1(\Omega)\big),\\
\dot{u}_m &\to \dot{u} && \text{in } L_p\big(I; H^1(\Omega)\big) \cap L_2\big(I; H^{2-\varepsilon}(\Omega)\big) \qquad \forall\, \varepsilon \in (0,1),\\
u_m &\to u && \text{in } C_0\big(I; W^1_p(\Omega)\big),\\
\partial_t B(u_m,u_m) &\rightharpoonup \partial_t B(u,u) && \text{in } L_2\big(I; W^2_p(\Omega)\big). \qquad (57)
\end{aligned}$$

The fourth convergence follows for $p \le 2$ from the second and third one via the compact imbedding theorem, for $p > 2$ by interpolation of the previous result with the first convergence. The fifth convergence $u_m \to u$ in (57) follows from the second and the fourth one by a standard interpolation and imbedding technique (cf. [7], Chapter 2). The last convergence is a consequence of (13) and the second and fifth convergence.

Let $\mu \in \mathbb{N}$ and $z_\mu = \sum_{i=1}^m \phi_i(t) w_i,\ \phi_i \in \mathcal{D}(0,T),\ i = 1,\dots,\mu$. The convergence process (57) and the property (47) imply that a function u fulfils

$$\int_Q \Big(a\nabla\ddot{u}\cdot\nabla z_\mu + \ddot{u} z_\mu + E_1 A(\dot{u}, z_\mu) + E_0 A(u, z_\mu) \\ + \big[u, E_1\partial_t\Phi(u,u) + E_0\Phi(u,u)\big] z_\mu - \eta^{-1} u^- z_\mu\Big)\,\mathrm{d}x\,\mathrm{d}t = \int_Q f z\,\mathrm{d}x\,\mathrm{d}t.$$

Functions $\{z_\mu\}$ form a dense subset of the set $L_2\big(I; H^2(\Omega)\big)$, hence a function u fulfils the identity (46). The initial conditions (40) follow due to (52) and the proof of the existence of a solution is complete. □

(ii) The uniqueness. Let $u, \hat{u}$ be two solutions of Problem $\mathcal{P}_\eta$ and let $w = u - \hat{u}$. We have for arbitrary $s \in I$ the relations

$$\int_{Q_s} \Big(a\nabla \ddot{w} \cdot \nabla z + \ddot{w} z + E_1 A(\dot{w}, z) + E_0 A(w, z) \\ + \big([u, E_1 \partial_t \Phi(u,u) + E_0 \Phi(u,u)] - [\hat{u}, E_1 \partial_t \Phi(\hat{u},\hat{u}) + E_0 \Phi(\hat{u},\hat{u})] \big) z \\ - \eta^{-1}(u^- - \hat{u}^-) z \Big) \,\mathrm{d}x\,\mathrm{d}t = 0 \qquad \text{for all} \quad z \in L_2(I; H^2(\Omega)),$$

$$w(0,.) = \dot{w}(0,.) = 0 \quad \text{on} \quad \Omega.$$

After setting $z = \dot{w}$ we get

$$\frac{1}{2}\Big(a\|\nabla \dot{w}\|_{L_2(\Omega)^2} + \|\dot{w}\|_{L_2(\Omega)} + E_0 \int_\Omega A(w,w)\mathrm{d}x \Big)(s) + E_1 \int_{Q_s} A(\dot{w},\dot{w})\,\mathrm{d}x\,\mathrm{d}t$$

$$= \int_Q s\big([\hat{u}, E_1 \partial_t \Phi(\hat{u},\hat{u}) + E_0 \Phi(\hat{u},\hat{u})] - [u, E_1 \partial_t \Phi(u,u) + E_0 \Phi(u,u)] \\ + \eta^{-1}(u^- - \hat{u}^-) \big) \dot{w}\,\mathrm{d}x\,\mathrm{d}t.$$

Using the estimate (55) with u and $\hat{u}$ instead of u_m and the imbedding $L_q(\Omega) \hookrightarrow H^1(\Omega)$, $q = \frac{2p}{p-2}$ we obtain the inequality

$$\|\dot{w}\|^2_{H^1(\Omega)}(s) \le \frac{c}{\eta} \int_0^s \|\dot{w}\|^2_{H^1(\Omega)}(t)\,\mathrm{d}t, \qquad \text{for every} \quad s \in I.$$

The Gronwall lemma implies $\|\dot{w}\|_{H^1(\Omega)}(s) = 0$ for every $s \in I$ and the uniqueness of a solution follows due to zero initial conditions for $w \equiv u - \hat{u}$.

We proceed with the existence theorem for the original problem.

Theorem 4.3. *Let $f \in L_2(Q)$, $u_i \in H^2(\Omega)$, $i = 0, 1.$, $u_0 \ge 0$. Then there exists a solution $u \in H^{1,2}(Q)$ of the problem $\mathcal{P}$.*

To prove this theorem we need to perform the limit process $\eta \searrow 0$. We shall write u_η for the solution of the problem $\mathcal{P}_\eta$. To get the crucial estimate for the penalty, we put $z = 1$ in (46). After an obvious calculation with the use of (47) we get

$$\int_Q \eta^{-1} u_\eta^- \,\mathrm{d}x\,\mathrm{d}t = \int_Q (\ddot{u}_\eta - f)\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega (\dot{u}_\eta(T,\cdot) - u_1)\,\mathrm{d}x - \int_Q f\,\mathrm{d}x\,\mathrm{d}t,$$

hence regarding the second norm in (49) we arrive at the estimate

$$\|\eta^{-1} u_\eta^-\|_{L_1(Q)} \le c(f, u_0, u_1). \tag{58}$$

which is independent of η. The standard imbedding $L_1(\Omega) \hookrightarrow H^\alpha(\Omega)$, valid for any $\alpha > 1$ and the a priori estimates (49) and (50) imply for the functional φ_η

$$\varphi_\eta : w \to \int_Q a\nabla \dot{u}_\eta \nabla w + \dot{u}_\eta w \,\mathrm{d}x\,\mathrm{d}t \tag{59}$$

the following estimate

$$\|\dot{\varphi}_\eta\|_{L_1(I;H^2(\Omega)^*)} \le c, \tag{60}$$

with η independent of c. The Lemma 4.1 yields the compact imbedding

$$\big\{v;\ v \in L_2\big(I; L_2(\Omega)+L_2(\Gamma)\big),\ \dot{v} \in L_1\big(I; H^2(\Omega)^*\big)\big\} \hookrightarrow\hookrightarrow L_2\big(I; H^1(\Omega)^*\big), \tag{61}$$

hence $\{\varphi_\eta; \eta > 0\}$ is relatively compact in $L_2\big(I; H^1(\Omega)^*\big)$.

The a priori estimates (49), (50), the relation (61) and the standard theory of linear elliptic equations yield the existence of a sequence $\eta_k \searrow 0$ such that for $u_k \equiv u_{\eta_k}$ the following convergences hold for any real $p \ge 1$

$$\dot{u}_k \rightharpoonup \dot{u} \qquad \text{in } L_2\big(I; H^2(\Omega)\big) \tag{62}$$

$$\dot{u}_k \to \dot{u} \qquad \text{in } L_2\big(I; W^1_p(\Omega)\big), \tag{63}$$

$$u_k \to u \qquad \text{in } C_0\big(I; W^1_p(\Omega)\big), \tag{64}$$

$$\partial_t B(u_k, u_k) \rightharpoonup \partial_t B(u,u) \text{ in } L_2\big(I; W^2_p(\Omega)\big),\ p \ge 1. \tag{65}$$

Indeed, the crucial strong convergence (63) is straightforward for $p = 2$ if we calculate $\|\dot{u}_k - \dot{u}\|^2_{L_2(I;H^1(\Omega);a)} \equiv \int_Q a\nabla(\dot{u}_k - \dot{u})\nabla(\dot{u}_k - \dot{u}) + (u_k - u)^2 \,\mathrm{d}x\,\mathrm{d}t$ $= \langle \varphi_k - \varphi, \dot{u}_k - \dot{u}\rangle \to 0$. Its interpolation with (62) and the appropriate imbedding theorem proves it for $p \in (2,\infty)$. The convergence (64) follows then immediately from the strong convergence $u_k \to u$ in $H^1(I; H^{2-\varepsilon}(\Omega))$ (valid for every $\varepsilon \in [0,1)$) via the standard imbedding theorem. Hence (65) can be derived as earlier.

We put the test function $z = y - u_k$ in (46) for $y \in \mathcal{K}$ and perform the integration by parts in the terms containing $\ddot{u}$. We arrive at the inequality

$$
\begin{aligned}
&\int\limits_Q \big(- a|\nabla \dot{u}_k|^2 - \dot{u}_k^2 + E_0(A(u_k,u_k) + (\triangle B(u_k,u_k))^2)\big)\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int\limits_\Omega (a\nabla \dot{u}_k \cdot \nabla u_k + \dot{u}_k u_k)(T,\cdot) + \frac{1}{2}E_1\big(A(u_k,u_k) + (\triangle B(u_k,u_k))^2\big)(T,\cdot) \\
&\le \int\limits_Q \Big(-a\nabla \dot{u}_k \cdot \nabla y - \dot{u}_k y + A(E_1\dot{u}_k + E_0 u_k, y) \\
&\quad + \big[u_k, E_1\partial_t B(u_k,u_k) + E_0 B(u_k,u_k)\big] y + f(u_k - y)\Big)\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int\limits_\Omega \Big((a\nabla \dot{u}_k \cdot \nabla y + \dot{u}_k y)(T,\cdot) - (a\nabla \dot{u}_k \cdot \nabla y + \dot{u}_k y)(0,\cdot) \\
&\quad + \frac{1}{2}E_1\big(A(u_k,u_k) + (\triangle B(u_k,u_k))^2\big)(0,\cdot)\Big)\mathrm{d}x, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for all} \quad k = 1,2\dots,\ y \in \mathcal{K}.
\end{aligned}
$$

Applying the convergences (62)–(65) and the weak lower semicontinuity verify that the limit u is a solution of the original problem $\mathcal{P}$.

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